Descent in algebraic K-theory

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LiveTeX-ed notes. Comments/corrections are welcome. cwlin416@gmail.com. Note $\subset$ does *not* mean strict inclusion. I use [?] to denote stuffs I have potentially missed.

Contents

1 Overview

1.1. Reminder. Let $A \in \mathrm{CRng}, X \in \mathrm{Sch}$ we have the following abelian groups.

$$K_0(A) := K_0(\mathrm{Proj}_A) \simeq K_0(\mathrm{Perf}_A)$$

$$K_0(X) := K_0(\mathrm{Vect}(X))$$

We also have their enhancement as E_∞-spaces.

$$K(A) := \text{group completion of}\, \mathrm{Proj}_A^{\simeq}$$

$$K^{\mathrm{naive}}(X) := K(\mathrm{Vect}(X)), \quad \text{Quillen } K\text{-theory of Vector Bundle}$$

$$K(X) := K(\mathrm{Perf}(X)), \quad \text{Waldhausen } K\text{-theory of perfect complexes}$$

1.2. If X has resolution property, then

$$K(X) \simeq K^{\mathrm{naive}}(X)$$

$$K(\mathrm{Spec}(A)) \simeq K(A)$$

Theorem 1.3. (Thomason). Algebraic K-theory satisfies descent. That is: $X \mapsto K(X)$ is a "sheaf" in some sense.

1.4. Warning. The presheaf $X \mapsto K_0(X)$ of sets/abelian groups is *not* a sheaf.

1.5. Reminder. (Sheaf of sets). X a topological space.

- Let $U(X)$ be poset (viewed as a category) of open subsets of X ordered by inclusion.
- A *presheaf* on X is a presheaf on $U(X)$.
- A presheaf $F : U(X)^{\mathrm{op}} \to \mathrm{Set}$ is a *sheaf* if for all open cover $X = \bigcup_i U_i$, the diagram

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j}(U_i \cap U_j)$$

  is a limit diagram.

Example 1.6. $X \in \mathrm{Sch}$, $E \xrightarrow{\pi} X$ a vector bundle. $\Gamma(-, E)$ the sheaf of sections of E.

$$\Gamma(U, E) := \{ s : U \to E \,:\, s \text{ is a section } \}, \quad U \hookrightarrow X \text{ open}$$

Example 1.7.

$$X \mapsto \mathrm{Vect}(X)/\simeq = \{\text{ iso. classes of vect. bundles}\}$$

is *not* a sheaf.

Problem: vector bundles have nontrivial automorphisms.

Solution: $X \mapsto \mathrm{Vect}^{\simeq}$ (groupoid) is a *sheaf* in the 2-categorical sense. This means

$$\mathrm{Vect}(X)^{\simeq} \longrightarrow \prod_i \mathrm{Vect}(U_i)^{\simeq} \rightrightarrows \prod_{i,j} \mathrm{Vect}(U_i \cap U_j)^{\simeq} \Rrightarrow \prod_{i,j,k} \mathrm{Vect}(U_i \cap U_j \cap U_k)^{\simeq}$$

The third arrows is the cocycle condition.

Example 1.8. $X \mapsto \mathrm{Perf}(X)^{\simeq}$.

Problem. Now we have derived/higher automorphisms. This comes from the fact that we have Ext groups.

Solution. Consider the ∞-groupoid of perfect complexes.

$$\mathrm{grpd} \longrightarrow 2\,\mathrm{grpd} \longrightarrow \cdots \longrightarrow \infty - \mathrm{grpd} \simeq \mathrm{Spc}$$

Theorem 1.9. $X \mapsto \mathrm{Perf}(X)^{\simeq}$ is a sheaf of $\infty - \mathrm{grpd}$. That is, we have a cosimplicial diagram indexed on Δ.

$$\mathrm{Perf}(X)^{\simeq} \longrightarrow \prod_i \mathrm{Perf}(U_i)^{\simeq} \rightrightarrows \prod_{i,j} \mathrm{Perf}(U_i \cap U_j)^{\simeq} \Rrightarrow \prod_{i,j,k} \mathrm{Perf}(U_i \cap U_j \cap U_k)^{\simeq} \qquad \cdots$$

Theorem 1.10. (Thomason). $X \mapsto K(X)$ is a sheaf of $\infty - \mathrm{grpd}$ (over qcqs schemes).

Corollary 1.11. (Mayer-Vietoris). $X = U \cup V$, Zariski cover.

$$\cdots \xrightarrow{\partial} K_n(X) \longrightarrow K_n(U) \oplus K_n(V) \longrightarrow K_n(U \cap V) \xrightarrow{\partial} K_{n-1}(X) \longrightarrow \cdots$$

Remark 1.12. Also work for algebraic spaces - we have Nisnevich coverings in this case. This is referred to as *Scallop decompositions* in Lurie.

2 ∞-categories

[Cis20], [Lura, 1].

Definition 2.1. (Simplicial sets). We have the following category.

$$\Delta \coloneqq \{[n] : n \in \mathbb{N}\} \quad [n] = \{0, 1, \dots, n\}$$

$$\mathrm{Hom}_{\Delta}([m], [n]) = \text{ order-preserving maps}$$

A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. This is equivalent to the data

$$(X_n)_{n \in \mathbb{N}}$$

$$\forall \alpha : [m] \to [n] \text{ in } \Delta$$

$$\alpha^* : X_n \to X_m \text{ functorially }.$$

2.2. Notation. $n \in \mathbb{N}$. $0 \leq i \leq n$.

- $\delta_n^i : [n-1] \hookrightarrow [n]$. The injective map which "skips" i.
- $\sigma_n^i : [n+1] \to [n]$. The surjective map which "doubles" i.

2.3. If $X \in \mathrm{Set}_\Delta$ (the category of simplicial sets). We have induced maps

- *face maps.* $d_n^i : X_n \to X_{n-1}$ induced by δ_n^i.
- *degenracy maps.* $s_n^i : X_n \to X_{n+1}$ induced by σ_n^i.

Example 2.4. If X is a set, $c(X) \in \mathrm{Set}_\Delta$ is *constant simplicial set.* The functor

$$\mathrm{Set} \xrightarrow{c} \mathrm{Set}_\Delta$$

is fully faithful.

Example 2.5. The standard simplex. $\Delta^n \in \mathrm{Set}_\Delta$. The simplicial set represented by $[n]$.

$$\Delta_i^n := \mathrm{Hom}_\Delta([i],[n])$$

Example 2.6. $\partial\Delta^n$.

$$\partial^k\Delta^n \subset \Delta^n \text{ image of } \Delta^{n-1} \to \Delta^n, k\text{th } \textit{face} \text{ of } \Delta^n$$

$$\partial\Delta^n := \bigcup_k \partial^k\Delta^n \ \textit{boundary} \text{ of } \Delta^n$$

Example 2.7. $\Lambda_k^n \subset \Delta^n$ is union of $\partial^i\Delta^n$, $j \neq k$. ($\partial\Delta^n$ minus kth face) *horns* of Δ^n.

2.1 Categoires as simplicial sets

2.8. C a category. $N(C) \in \mathrm{Set}_\Delta$, *nerve of* C.

$$N(C)_n := \mathrm{Fun}_{\mathrm{Cat}}([n],C) = \{c_0 \to \cdots \to c_n \text{ strings of morphisms }\}$$

- $N(C)_0$=objects of C.
- $N(C)_1$=morphisms of C.
- $N(C)_2$=diagrams $C_0 \to C_1 \to c_2$ in C.
- $\cdots$
- $N(C)$ has all info about C.

2.9. $C \mapsto N(C)$ defines a *fully faithful* functor

$$N : \mathrm{Cat} \to \mathrm{Set}_\Delta$$

Moreover there is a left adjoint $\tau : \mathrm{Set}_\Delta \to \mathrm{Cat}$.

2.10. Let $X \in \mathrm{Set}_\Delta$.

- *objects* of X: 0-simplices. $\Delta^0 \to X$
- *morphisms* of X: 1- simplices. $\Delta^1 \to X$.

- *source/target* of $f \in X_1$.

$$X_1 \underset{d_1^1}{\overset{d_1^0}{\rightrightarrows}} X_0$$

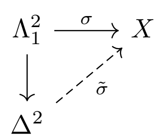
$$s(f) := d_1^1(f), t(f) = d_1^0(f)$$

- *identity* of $x \in X_0$. $s_0^0 : X_0 \to X_1$, $x \mapsto \mathrm{id}_x$.

Definition 2.11. $X \in \mathrm{Set}_\Delta$ is a *compsable par* in X, is a map

$$\Lambda_1^2 \to X$$

a *composition* of a composable pair is a lift



2.12. $X \in \mathrm{Set}_\Delta$ is in the essential image of $N : \mathrm{Cat} \to \mathrm{Set}_\Delta$ iff

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda_k^n, X)$$

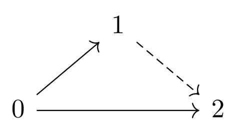
is bijective for all $n \geq 2$, $0 < k < n$. In particular composition exists and is unique.

2.2 Groupoids and Kan complexes

Remark 2.13. C is a groupoid iff moprhisms in C are invertible iff $N(C) \in \mathrm{Set}_\Delta$ satifies the following:

$$\mathrm{Hom}(\Delta^n, N(C)) \xrightarrow{\simeq} \mathrm{Hom}(\Lambda_k^n)$$

for all $n \geq 2$, $0 \leq k \leq n$. The corner cases allow us to construct inverses:



Definition 2.14. (Kan complex). $X \in \mathrm{Set}_\Delta$ is a *Kan complex* iff

$$\mathrm{Hom}(\Delta^n, X) \xrightarrow{\mathrm{res}} \mathrm{Hom}(\Lambda_k^n, X)$$

is *surjective* $\forall 0 \leq k \leq n$. "Kan complex= generalized groupoids where compositions and inverses exist but not uniquely".

Theorem 2.15. (Milnor). There is an equivalence

$$\{ \text{ homotopy cat. of CW cplx. } \} \xrightarrow{\simeq} \{ \text{ homotopy cat. of Kan cplx. } \}$$

which is given by

$$X \mapsto \mathrm{Sing}(X)_\bullet \in \mathrm{Set}_\Delta$$

$$\mathrm{Sing}(X)_n := \{ \Delta^n_{\mathrm{Top}} \to X \text{ continuous maps } \}$$

Composition in a Kan complex corresponds to composition of paths in a space.

2.3 ∞-categories as weak Kan complex

2.16. groupoid :: category.
Kan cplx:: ???

Definition 2.17. (Boardman-Vogt) is a *weak Kan complex* a.k.a *quasi-category* iff

$$\mathrm{Hom}(\Delta^n, X) \xrightarrow{\mathrm{res}} \mathrm{Hom}(\Lambda^n_k, X)$$

is surjective for all $n \geq 2$, $0 < k < n$.

2.18. Construction. X be a weak Kan complex. $f, g : x \to y$. A *homotopy* $f \simeq g$ is a 2-simplex $\Delta^2 \xrightarrow{\sigma} X$.

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{id}{\searrow} & \\ x & & \xrightarrow{\quad g \quad} & & y \end{array}$$

The *homotopy category* hX has obbjects 0-simplices. Morphisms homotopy classes of x to y.

Definition 2.19. X wkc. $f : X \to y$ is an *isomoprhism* iff following equivalent holds

- f invertible.
- f isomoprhism hX.

X is an ∞ − grpd iff every moprhism in X is an iso.

Remark 2.20. There is unit map $X \to N\tau X$.

Theorem 2.21. (Joyal). X wkc. Tfae

- X is a Kan complex.
- X is an ∞ − grpd.

Example 2.22. (Cat. theory of wkc).

- $\mathrm{Fun}(X, Y) = \underline{\mathrm{Hom}}(X, Y)$ internal hom of Set_Δ.
- $\mathrm{Map}_X(s, y)$. X is a wkc. $x, y \in X_0$, we can construct an ∞ − grpd of maps

$$\begin{array}{ccc} \mathrm{Map}_X(x, y) & \longrightarrow & \mathrm{Fun}(\Delta^1, X) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ \Delta^0 & \xrightarrow[(x,y)]{} & X \times X \end{array}$$

- adjunctions.

Definition 2.23. An ∞-*category*(platonic concept) is a weak Kan complex (shadow).

2.4 ∞-category of Kan complexes

2.24. Construction. Kan (large) simplicial sets. ($\neq N$(Cat .of Kan. complexes).

- $\mathrm{Kan}_0 = \{\text{small Kan complexes}\}$
- $\mathrm{Kan}_1 = \{\text{maps of Kan complexes}\}$.
- $\mathrm{Kan}_2 = \{(X, Y, Z, f, g, h, \sigma) : f : X \to Y, g : Y \to Z, h : X \to Z, \sigma \in \mathrm{Fun}(X, Z), d_0^1\sigma = g \circ f, d_0^2 = h\}$
- $\cdots$
- Kan_n = tuples of Kan complexes $X_0 \to X_n, X_i \to X_j$ "compatible up to coherent homotopy".

Definition 2.25. There is a fully faithful functor of ∞-categories

$$\mathrm{Set} \hookrightarrow \mathrm{Kan}$$

An object $X \in \mathrm{Kan}$ is in the ess. image iff X is homotopy equivlaent to a constnat simplicial set iff $\pi_i(X) \simeq 0$ for all $i > 0$.

3 Animated modules

[ARV10], [ČS19, 5], [Lura, 5.5.8].

3.1 Algebraic categories

Definition 3.1. $\mathcal{C}$ category. $\mathcal{C}$ is *algebraic* if there exists an essentially small full subcategory $F_{\mathcal{C}} \subset \mathcal{C}$, admitting finite coproducts, which extends to an equivalence

$$\mathrm{Fun}_\pi(F_{\mathcal{C}}^{op}, \mathrm{Set}) \to \mathcal{C}$$

where lhs denotes product preserving functors.

Example 3.2. The category Set is algebraic, with $F_{\mathcal{C}} = \mathrm{Fin}$. The category of finite sets.

$$\mathrm{Set} \simeq \mathrm{Fun}_\pi(\mathrm{Fin}^{op}, \mathrm{Set})$$

The forward map is the Yoneda embedding.

$$X \mapsto (Y \mapsto \mathrm{Hom}(Y, X))$$

Conversely, given F in rhs, we have the data of

- $F_0 \simeq \{*\}$
- $F_n := F(\{1, \ldots, n\}) \simeq F_1^n =: (F\{1\})^n$ for all n.
- $\{1, \ldots, n\} \to \{1, \ldots, m\}$ induces a map

$$X_m \simeq F_m \to F_n \simeq X_n$$

In other words, all the data is encoded by $X = F_1 \in \mathrm{Set}$. Hence, the inverse map is given by

$$F \mapsto F_1$$

Example 3.3. Ab is algebraic. $F_{\mathrm{Ab}} = \{\text{f.g. free ab groups}\} \subset \mathrm{Ab}$. F_{Ab} can be identified with the category

- Objects are $n \in \mathbb{N}$.
- Morphisms are
$$\mathrm{Hom}_{F_{\mathrm{Ab}}}(m,n) = \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}^{\oplus m}, \mathbb{Z}^{\oplus n}) \simeq \mathrm{Mat}_{n\times m}(\mathbb{Z})$$
- composition is identified with matrix multiplication.

We have
$$\mathrm{Ab} \simeq \mathrm{Fun}_{\pi}(F_{\mathrm{Ab}}^{\mathrm{op}}, \mathrm{Set})$$
Object of rhs is identified with data of

- $F_0, F_1, \ldots \in \mathrm{Set}$.
- $F_0 \simeq \{*\}$, $F_n \simeq F_n^{\times n}$.
- $\phi \in \mathrm{Mat}_{n\times m}(\mathbb{Z})$, $F_\phi : F_m \to F_n$.

This implies we can define the underlying set as $G = F_1 \in \mathrm{Set}$.

- Note the *operation* maps $G^{\times n} \to G$ corresponds to $n \times 1$ matrices.
$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$
corresponds with operation of forming a linear combination with coefficients a_i.
$$(x_1, \ldots, x_n) \mapsto \sum a_i \cdot x_i$$
- *addition*
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow (x_0, x_1) \mapsto x_1 + x_2$$
- *zero.* [] empty $(0 \times 1$-matrix yields $\{*\} \xrightarrow{0} G$.
- *additive inverse.* $[-1] \Leftrightarrow G \to G$

Remark 3.4. There is a canonical choice of $F_{\mathcal{C}}$, namely the full subcategory of *compact projective* objects.

- $X \in \mathcal{C}$ is *compact* iff $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves filtered colimits. (thought of as poset)
- $X \in \mathcal{C}$ is *projective* iff preservers reflexive coequalizers. (thought of as equivalence relation).

Example 3.5. In Set, $X \in \mathrm{Set}$ is cpt. proj. iff X is finite.
In Ab, $X \in \mathrm{Ab}$ is cpt. proj. iff X is f.g. free.

3.6. Moreover, if $\mathcal{C}$ is algebraic then $\mathrm{Fun}_{\pi}(F_{\mathcal{C}}^{op}, \mathrm{Set}) \simeq \mathcal{C}$ is the free completion of $F_{\mathcal{C}}$ by filtered colimits and reflexive coequalizers.

Free completion. For every category $\mathcal{D}$ with filtered colimits + reflexive coequalizers, there is an equivalence given by restriction
$$\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(F_{\mathcal{C}}, \mathcal{D})$$
where the lhs consists of functors $F : \mathcal{C} \to \mathcal{D}$ preserving filtered colimits+ reflexive coequalizers.

3.7. Reminder. A *reflixve pair* in$\mathcal{C}$ is a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} y \quad (s : y \to X)$$

such that $fs = gs$.

3.8. *Reflexive coequalizers* are colimits induced by *reflexive pairs.* These generalize quotients by equivalence relations. For example

$$R \subset X \times X$$

yields a reflexive pair

$$R \rightrightarrows X$$

Definition 3.9. $\mathcal{C}$ an ∞-category. $X : \Delta^{op} \to \mathcal{C}$ a simplicial diagram in $\mathcal{C}$. The colimit of X is denoted

$$|X_\bullet| := \varinjlim_{[n]\in\Delta^{op}} X_n$$

$$\cdots \qquad X_1 \rightleftarrows X_0 \longrightarrow |X_\bullet|$$

and is called the *geometric realization* of X. We should think of these as higher cat. version of equivalence relations.

Definition 3.10. $\mathcal{C}$ an algebraic category. An *animation* of $\mathcal{C}$ is an ∞-cat $\mathrm{Anim}(\mathcal{C})$ equipped with a f.f functor $F_\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ such that for all $\mathcal{D}$ ∞-cat (admitting fil. colim + geo. realization.)

$$\mathrm{Fun}'(\mathrm{Anim}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(F_\mathcal{C}, \mathcal{D})$$

is an equivalence. The lhs consists of $F : \mathrm{Anim}(\mathcal{C}) \to \mathcal{D}$ preserving filt. colimit + geo. realizations.

- The ∞*-category of anime* is an animation of the category of sets. We denote it by Anim.

Theorem 3.11. (Quillen, Lurie).

- The f.f. functor Fin $\hookrightarrow$ Kan exhibits the ∞-cat Kan as an ∞-category of anime.
- $\mathcal{C}$ an algebraic category. Then the Yoneda embedding

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathrm{Fun}(F_\mathcal{C}^{op}, \mathrm{Anim}) \\ & \searrow & \uparrow \\ & & \mathrm{Fun}_\pi(F_\mathcal{C}^{op}, \mathrm{Anim}) \end{array}$$

  induces a factorization as depicted, and exhibits the target as an animation of $\mathcal{C}$. i.e. $\mathrm{Fun}(F_\mathcal{C}^{op}, \mathrm{Anim})$ is a model of animation for any algebraic category $\mathcal{C}$. This is proven in [Lura, 5.5.8].

Definition 3.12. Recall. Set $\hookrightarrow$ Kan $\simeq$ Anim. An anima. $X \in$ Anim is *discrete* if it is isoo. to an object in the essential image.

If $\mathcal{C}$ is an algebraic category, an object $X \in \mathrm{Anim}(\mathcal{C})$ is *discrete* if

$$X : F_\mathcal{C}^{op} \to \mathrm{Anim}$$

factors through Set $\hookrightarrow$ Anim. iff the *underlying anima* of X is discrete. We denote this category

$$\mathrm{Anim}^\heartsuit \subset \mathrm{Anim}$$

3.13. Claim.

1. The assignment
$$(F_{\mathcal{C}}^{op} \to) \, \mathrm{Set} \mapsto (F_{\mathcal{C}}^{op} \to \mathrm{Set} \hookrightarrow \mathrm{Anim})$$
defines a ff functor
$$\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$$
with ess. image $\mathrm{Anim}(\mathcal{C})^{\heartsuit}$.

2. The functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ admits left adjoint
$$\pi_0 : \mathrm{Anim}(\mathcal{C}) \to \mathcal{C}$$
given by composition with
$$\pi_0 : \mathrm{Anim} \to \mathrm{Set}$$
the connected component functor.

3.14. Construction. (Underlying anima). Assume $F_{\mathcal{C}}$ is generated under finite coproducts by one object, 1. $X \in \mathrm{Anim}(\mathcal{C})$.
$$X^{\circ} := X(1) \in \mathrm{Anim}, \quad X(1) = \mathrm{Map}_{\mathrm{Anim}(\mathcal{C})}(1, X)$$

3.2 Animated modules

3.15. A a comm. ring. $F_A := \{A^{\oplus n} : n \in \mathbb{N}\} \subset \mathrm{Mod}_A$ consists of f.g. free A-module.

Definition 3.16. An *animated A-module* is a product preserving functor
$$M : (F_A)^{op} \to \mathrm{Anim}$$

Remark 3.17. Harry. There is an equivalence of categories between the animated A-modules defined above and those with domain the compact projective objects of Mod_A.

3.18. Notation. $\mathcal{D}(A)_{\geq 0} := \mathrm{Anim}(\mathrm{Mod}_A)$. This will be shown to be equivalent to the usual ∞-category of connective chain complex.

3.19. $M \in \mathcal{D}(A)_{\geq 0}$ consists of

- $M_n \in \mathrm{Anim}$ for all $N \in \mathbb{N}$.
- $M_m \to M_n$ for all $\varphi : \mathrm{Mat}_{m \times n}(A)$.
- $\varphi \in \mathrm{Mat}_{m \times n}(A), \psi \in \mathrm{Mat}_{l \times m}(A)$ a homotopy between $(M_{\psi\varphi} : M_l \to M_n)$ and $(M_l \xrightarrow{M_{\psi}} M_n \xrightarrow{M_{\varphi}} M_n)$
- + a homotopy coherent system of compatibilities between homotopies.

subject to the condition
$$M_n \xrightarrow{\simeq} M^{\times n}$$
for all n and
$$M_0 \simeq *$$
The data of relevance

- $M^{\circ} = M_1 \in \mathrm{Anim}$.

- Operations

$$(M^\circ)^{\times n} \to M^\circ$$

  corresponds to $\varphi \in \mathrm{Mat}_{n\times 1}(A)$.

- action of A on M°

$$A \to \mathrm{End}(M^\circ)$$

$$A \simeq \mathrm{Mat}_{1\times 1}(A) = \mathrm{Hom}_{F_A}(1,1) \xrightarrow{M} \mathrm{Hom}_{\mathrm{Anim}}(M_1, M_1) \simeq \mathrm{End}(M^\circ)$$

- Assoc + commutativity up to coherent homotopy. For all $x, y, z \in M$ ($\Leftrightarrow x, y, z : * \to M^0$). This follows from the diagram

$$\begin{array}{ccc} M^\circ \times M^\circ \times M^\circ & \xrightarrow{a\times \mathrm{id}} & M^\circ \times M^\circ \\ \downarrow{\scriptstyle \mathrm{id}\times a} & & \downarrow{\scriptstyle a} \\ M^\circ \times M^\circ & \longrightarrow & M^\circ \end{array}$$

3.3 Derived functors

3.20. The idea is to resolve by simplicial diagrams.

3.21. Construction. (Left derived functors). $F : \mathcal{C} \to \mathcal{D}$ functor between algebraic categories. If F preserves filt. colimits + refl coeq's, then this induces

$$\mathbb{L}F : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{D})$$

unique functor such that

- $\mathbb{L}F$ preserves filt colimit + geo. realization.
-

$$\begin{array}{ccccccc} F_{\mathcal{C}} & \hookrightarrow & \mathcal{C} & \xrightarrow{F} & \mathcal{D} & \longrightarrow & \mathrm{Anim}(\mathcal{D}) \\ \downarrow & & & \nearrow_{\exists ! \mathbb{L}F} & & & \\ \mathrm{Anim}(\mathcal{C}) & & & & & & \end{array}$$

- For all $X \in \mathrm{Anim}(\mathcal{C})$, $\pi_0 \mathbb{L}F(X) \simeq F(\pi_0 X) \in \mathcal{C}$.
- If F preserves fin. coproduct then $\mathbb{L}F$ does too.

Definition 3.22. $\mathbb{L}F$ is the *left derived* functor of F.

3.23. In the classcal theory left derived functors don't compose well.

Proposition 3.24. $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{E}$ functors between alg. cats. preserving filt. colimit+ reflex. coeq. Assume one of the following holds

1. F sends $F_{\mathcal{C}} \to F_{\mathcal{D}} \subset \mathcal{D}$. (More generally sends $F_{\mathcal{C}}$ to filt. colimits of objects in $F_{\mathcal{D}}$.)
2. $\mathbb{L}G : \mathrm{Anim}(\mathcal{D}) \to \mathrm{Anim}(\mathcal{E})$ preserves discrete objects. i.e. a diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathrm{Anim}(\mathcal{D}) & \xrightarrow{\mathbb{L}G} & \mathrm{Anim}(\mathcal{E}) \end{array}$$

   More generally for all $X \in F_{\mathcal{C}}$, $\mathbb{L}G(F(X))$ is discrete in $\mathrm{Anim}(\mathcal{D})$.

Then, there is a canonical equivalence

$$\mathbb{L}G \circ \mathbb{L}F \simeq \mathbb{L}(G \circ F) : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{E})$$

Example 3.25. $\varphi : A \to B$ ring homomoprhism

$$\varphi^* : \mathrm{Mod}_A \to \mathrm{Mod}_B, \quad M \mapsto M \otimes_A B$$

This implies the existence of the functor

$$\mathbb{L}\varphi^* : \mathcal{D}(A)_{\geq 0} \to \mathcal{D}(B)_{\geq 0}, \qquad \begin{array}{ccc} F_A & \xrightarrow{-\otimes_A B} & F_B \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}(A)_{\geq 0} & \xrightarrow{\mathbb{L}\varphi^*} & \mathcal{D}(B)_{\geq 0} \end{array}$$

which preservers colimits. Further

$$\pi_0 \mathbb{L}\varphi^* M \simeq \varphi^* \pi_0 M$$

next time we will identify this functor as

$$- \otimes^{\mathbb{L}}_A B = \mathbb{L}\varphi^*$$

4 Nonconnective animated modules

[Lurb, 1], [Lurc, c].

4.1 Suspensions and loops spaces

Definition 4.1. $\mathcal{C}$ an ∞-category with *terminal object* $\mathrm{pt} \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, \mathrm{pt})$ is a contractible anima. $f : X \to Y$ a morphism in $\mathcal{C}$.

- $\mathrm{cofib}(f)$=*cofiber* of f is the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

- $\mathrm{fib}_y(f)$=*fiber* of f at any "point" $y : \mathrm{pt} \to Y$, is the pullback square

$$\begin{array}{ccc} \mathrm{fib}_y(f) & \longrightarrow & X \\ \big\downarrow & \lrcorner & \big\downarrow f \\ \mathrm{pt} & \xrightarrow[y]{} & Y \end{array}$$

Example 4.2. Let $\mathcal{C}$ be an ordinary category (viewed as an ∞-category), then

$$\mathrm{cofib}(f) \simeq \mathrm{coker}(f)$$

$$\mathrm{fib}_y(f) \simeq \ker(f)$$

Example 4.3. $\mathcal{C}$ an ∞-cat. $X \in \mathcal{C}$, object. Then

- The *suspension* is

$$\Sigma X \simeq \operatorname{cofib}(X \to \mathrm{pt})$$

- The *loop space* $\Omega_x(X)$

$$\operatorname{fib}_x(\mathrm{pt} \xrightarrow{x} X)$$

In otherwords we have the cocartesian and cartesian squares respectively.

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{pt} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{pt} & \longrightarrow & \Sigma X \end{array} \qquad \begin{array}{ccc} \Omega_x(X) & \longrightarrow & \mathrm{pt} \\ \downarrow & \lrcorner & \downarrow x \\ \mathrm{pt} & \xrightarrow[x]{} & X \end{array}$$

Remark 4.4. In ordinary category $\Sigma X = \mathrm{pt}, \Omega_x(X) = \mathrm{pt}$ for all $x, x : \mathrm{pt} \to X$.

Example 4.5. In the Kan $\simeq$ Anim

$$\Sigma\varnothing \simeq S^0, \quad S^{n+1} \simeq \Sigma S^n, \quad \forall n \geq 0$$

$$\begin{array}{ccc} \varnothing & \longrightarrow & \mathrm{pt} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{pt} \sqcup \mathrm{pt} \end{array}$$

Remark 4.6. Point of a loop space $\Omega_x(X)$ is equivalent to

$$\begin{array}{cccc} \mathrm{pt} & & & \\ & \searrow & & \\ & \Omega_x(X) & \longrightarrow & \mathrm{pt} \\ & \downarrow & \lrcorner & \downarrow x \\ & \mathrm{pt} & \xrightarrow[x]{} & X \end{array}$$

corresponding to a commutative square

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow[\sigma_1]{} & \mathrm{pt} \\ \downarrow \sigma_2 & \searrow & \downarrow x \\ \mathrm{pt} & \xrightarrow[x]{} & X \end{array}$$

which requires specifying a *further data* of homotopy. i.e. have to specify 2 simplicies σ_1, σ_2 up to which the triangle commute

$$\Delta^1 \times \Delta^1 \to \mathcal{C}$$

This data is equivalent

$$(\mathrm{pt} \xrightarrow{x} X) \simeq (\mathrm{pt} \xrightarrow{x} X) \quad (\sigma_1)$$

$$(\mathrm{pt} \xrightarrow{x} X) \simeq (\mathrm{pt} \xrightarrow{x} X) \quad (\sigma_1)$$

which is equivalent to loop in X based at $x \in X$.

Remark 4.7. Harry. Can also think of loop space as $X(x,x)$ the endomorphism space of x, which makes sense even if X were an ∞-cat.

4.2 Infinite loop space

Definition 4.8. 1. A *pointed anime* pair (X, x_0) $X \in \text{Anim}$ $x_0 : \text{pt} \to X$ (i.e. pointed Kan complex).

2. $\text{Anim}_* = \{\infty\text{-cat of pointed anime}\}$.

3. (X, x_0) is *n-connective* if $\pi_i(X, x_0) \simeq *$ for all $i < n$.

Example 4.9. Every $X \in \text{Anim}_*$ is 0-connective. $X \in \text{Anim}_*$ is 1-connective iff X is connected iff $\pi_0 X \simeq 0$.

Theorem 4.10. (1-fold loop spaces).

- A 1-*fold loop space* is a pair (X_0, X_1) of pointed anime together with an isomoprhism,

$$X_0 \simeq \Omega(X_1)$$

- X_0 is the underlying anima, and X_1 is the delooping of X_0.
- (X_0, X_1) is *connective* if X_i is i-connective for all i, iff, X_1 is 1-connective.

Claim.

1. For every $X \in \text{Anim}_*$, ΩX admits an $\mathbb{E}_1$-group structure. In particular the functor

$$\{\text{1-fold loop spaces}\} \to \text{Anim}_*$$

factors through $\{\mathbb{E}_1\text{-groups}\} \xrightarrow{\text{forget}} \text{Anim}_*$.

2. Restricted to the full subcategory of *connective* loop space, this induces an equivalence of categories

$$\{\text{connec. 1-fold loops spc}\} \xrightarrow{\simeq} \{\mathbb{E}_1\text{groups}\}$$

This implies any $\mathbb{E}_1$-group on $X \in \text{Anim}_*$ gives rise to a unique delooping BX, BX is 1-*connective pointed anime* $\Omega BX \simeq X$.

Remark 4.11.

$$\begin{array}{ccc} \{\text{1-fold loop spaces}\} & \xrightarrow{\simeq} & \text{Anim}_* \\ \cup\uparrow & & \cup\uparrow \\ \{\text{connec. 1-fold loops spc}\} & \xrightarrow{\simeq} & \text{ptd. connected anima} \end{array}$$

$$(X_0, X_1) \longmapsto X_1$$

$$(\Omega(X), X) \longleftarrow X$$

Definition 4.12. A *spectrum* X is a sequence of pointed anime $X = (X_0, X_1, \dots)$ together with isomoprhisms $X_n \simeq \Omega_{n+1}$ for all $n \geq 0$.

- X is an *infinite delooping* of X_0.

Definition 4.13. A spectrum X is *connective* if X_n is *n-connective* pointed anima. (for all $n \geq 0$.)

1. Spt = ∞-cat of spectra.
2. $\mathrm{Spt}_{\geq 0}$ full subccategory of connective spectra.
3. $$\mathrm{Spt} \simeq \varprojlim(\cdots \xrightarrow{\Omega} \mathrm{Anim}_* \xrightarrow{\Omega} \mathrm{Anim}_*)$$
4. $$\mathrm{Spt}_{\geq 0} \simeq \varprojlim(\cdots \xrightarrow{\Omega} (\mathrm{Anim}_*)_{\geq 1} \xrightarrow{\Omega} (\mathrm{Anim}_*)_{\geq 0})$$

Remark 4.14. Projections

$$\mathrm{Spt} \xrightarrow{\Omega^{\infty-n}} \mathrm{Anim}_*$$

$$(X_0, X_1, \dots) \mapsto (X_n)$$

Theorem 4.15. (infinite loop space machine). Boardman-Vogt, Segal, Peter may, Lurie.

1. $X \in \mathrm{Spt}$, $\Omega^\infty X \in \mathrm{Anim}_*$, admits an $\mathbb{E}_\infty - \mathrm{grp}$ structure.
$$\Omega^\infty : \mathrm{Spt} \to \mathbb{E}_\infty - \mathrm{grp}$$
2. When restricted to $\mathrm{Spt}_{\geq 0}$, we have an equivalence
$$\mathrm{Spt}_{\geq 0} \hookrightarrow \mathrm{Spt} \xrightarrow{\Omega^\infty} \mathbb{E}_\infty - \mathrm{grp}$$

4.3 Stable ∞-categories

Definition 4.16. $\mathcal{C}$ ∞-category is *stable* if

- admits finite limits and a zero object (terminal + initial).
- $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence.
$$\Sigma\Omega \simeq \mathrm{id}, \quad \Omega\Sigma \simeq \mathrm{id}$$
These behave like shift functors in a triangulated category.

Theorem 4.17. Spt is a stable ∞-category.

Proof. (Sketch). Looking at

$$\begin{array}{ccccccccc}
\mathrm{Spt} & \longrightarrow & \cdots & \xrightarrow{\Omega} & \mathrm{Anim}_* & \xrightarrow{\Omega} & \mathrm{Anim}_* & \xrightarrow{\Omega} & \mathrm{Anim}_* \\
\downarrow{\scriptstyle\Omega} & & & & \downarrow{\scriptstyle\Omega} & & \downarrow{\scriptstyle\Omega} & \nwarrow & \downarrow{\scriptstyle\Omega} \\
\mathrm{Spt} & \longrightarrow & \cdots & \xrightarrow{\Omega} & \mathrm{Anim}_* & \xrightarrow{\Omega} & \mathrm{Anim}_* & \xrightarrow{\Omega} & \mathrm{Anim}_*
\end{array}$$

The inverse is given by a collection of compatible map

$$\begin{array}{ccc}
\mathrm{Spt} & \xrightarrow{\theta} & \mathrm{Spt} \\
& {\scriptstyle\theta_n}\searrow & \downarrow{\scriptstyle\Omega^{\infty-n}} \\
& & \mathrm{Anim}_*
\end{array}$$

[1]

$$\Omega \circ \theta_n \simeq \theta_{n+1}, \quad \theta_n := \Omega^{\infty-n}, \Omega \circ \Omega^{\infty-n-1} \simeq \Omega^{\infty-n}$$

This implies $\theta \simeq \Sigma$. □

[1] Check the maps here

Proposition 4.18. $\mathcal{C}$ an ∞-cat, tfae.

- $\mathcal{C}$ is stable.
- $\mathcal{C}$ admits finite colimits + zero objt, $\Sigma : \mathcal{C} \to \mathcal{C}$ is equivalence.
- $\mathcal{C}$ admits finite colimits + finite limits + zero object and any commutative square is cartesian iff it is cocartesian.

Definition 4.19. An *exact triangle* in a stable category $\mathcal{C}$ is a co/cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

0 is a zero object. This is denoted

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Warning. Have to specify a null homotopy $g \circ f \simeq 0$.

Remark 4.20. The homotopy category $h(\mathcal{C})$ is triangulated.

- $h(\mathcal{C}$ is additive.
- $\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y)$ are abelian groups. Hence

  $$\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y) \simeq \Omega \operatorname{Map}_{\mathcal{C}}(X, Y) \simeq \pi_1 \operatorname{Map}_{\mathcal{C}}(X, Y)$$

  Similarly

  $$\pi_0 \operatorname{Map}_{\mathcal{C}}(\Sigma^2 X, Y) \simeq \pi_2(\operatorname{Map}_{\mathcal{C}}(X, Y)$$

  is an abelian group. But we have $X \simeq \Sigma^2 \Omega^2 X$ since $\mathcal{C}$ is stable.
- $[n] := \Sigma^n, \Omega^n$ if $n > 0$, and < 0 respectively.
- exact triangles coming from $\mathcal{C} \to h\mathcal{C}$.

4.4 Animations of additive categories

4.21. $\mathcal{C}$ is algebraic category. If $\mathcal{C}$ is additive, then

$$\mathcal{C} \simeq \operatorname{Fun}_\pi(F_{\mathcal{C}}^{op}, \mathrm{Ab})$$

Idea. $X \in \mathcal{C}$ iff $X : F_{\mathcal{C}} \to \mathrm{Set}$ automatically takes value in Ab.

4.22. In particular $X \in \mathcal{C}$ if it is reprentable, In general, X is built out of filt. colimits and reflexive coequalizers of representable objects.

Proposition 4.23. $\mathcal{C}$ additive algebraic ategory.

$$\operatorname{Anim}(\mathcal{C}) \simeq \operatorname{Fun}_\pi(F_{\mathcal{C}}^{op}, \mathrm{Spt}_{\geq 0})$$

In particular this is ff. embedding

$$\operatorname{Anim}(\mathcal{C}) \hookrightarrow \operatorname{Fun}_\pi(F_{\mathcal{C}}^{op}, \mathrm{Spt}) =: \operatorname{Anim}^{\mathrm{nc}}(\mathcal{C})$$

with *stable* target and with ess. image closed under finite colimits and extensions.

Proof. (Sketch). Use the infinite loop space machine.

- Every $X \in \mathrm{Anim}(\mathcal{C})$ iff $X : F_{\mathcal{C}}^{op} \to \mathrm{Anim}$ actually takes values in $\mathbb{E}_\infty - \mathrm{grp}$. This is done by reducing to the representable objects.
- $\mathbb{E}_\infty - \mathrm{grp} \simeq \mathrm{Spt}_{\geq 0} \hookrightarrow \mathrm{Spt}$.

□

Remark 4.24. $\mathrm{Anim}^{\mathrm{nc}}(\mathcal{C}) \simeq \varprojlim(\cdots \xrightarrow{\Omega} \mathrm{Anim}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{Anim}(\mathcal{C}))$

Definition 4.25. A *nonconnective animated A-module* is an object in $\mathrm{Anim}^{\mathrm{nc}}(\mathcal{C})$, which we denote

$$\begin{array}{cccc} \mathcal{D}(A) & := & \mathrm{Anim}^{\mathrm{nc}}(\mathrm{Mod}_A) & \text{stable} \\ \subset & & & \\ \mathcal{D}(A)_{\geq 0} & = & \mathrm{Anim}(\mathrm{Mod}_A) & \text{prestable} \\ \subset & & & \\ \mathcal{D}(A)_{\geq 0}^{\heartsuit} & = & \mathrm{Mod}_A & \text{abelian} \end{array}$$

5 Sheaves

5.1 Addendum to Derived functors

5.1. Recall $\phi : A \to B$ a ring homo.

$$\mathrm{Mod}_A \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathrm{Mod}_B$$

5.2. ϕ_* admits a right adjoint, coextension of scalars.

$$M \in \mathrm{Mod}_A \mapsto \mathrm{Hom}_A(B, M) \in \mathrm{Mod}_B$$

This implies ϕ_* preserves small colimits. Inducing left derived functor,

$$\mathbb{L}\phi_* : \mathcal{D}(B)_{\geq 0} \to \mathcal{D}(A)_{\geq 0}$$

which commutes with sifted colimits and (co)finite products hence all colimits. Extends

$$F_B \hookrightarrow \mathrm{Mod}_B \longrightarrow \mathrm{Mod}_A \longrightarrow \mathcal{D}(A)$$

Proposition 5.3. The derived functor satisfies

1. $\mathbb{L}\phi_*$ preserves limits and colimits.
2. $\mathbb{L}\phi_*$ preserves underlying anima
3. $\mathbb{L}\phi_*$ conservative (detects isos).

Remark 5.4. 2 implies $\mathbb{L}\phi_*$ preserves and detects discreteness.

Proof. (of 2). Let $N \in \mathcal{D}(B)_{\geq 0}$.

$$\begin{aligned}\mathbb{L}\phi_*(N)^\circ &=\simeq \mathrm{Map}_{\mathcal{D}(A)_{\geq 0}}(A, \mathbb{L}\phi_*(N)) \\ &\simeq \mathrm{Map}_{\mathcal{D}(B)_{\geq 0}}(\mathbb{L}\phi^*(A), N) \\ &\simeq \mathrm{Map}_{\mathcal{D}(B)_{\geq 0}}(B, N) \\ &\simeq N^\circ\end{aligned}$$

Note that for $A \in F_A$,

$$\mathbb{L}\phi^*(A) \simeq \phi^*(A) \simeq B$$

(of 3). Follows from 2. and the fact

$$\mathcal{D}(A)_{\geq 0} \to \mathrm{Anim}$$

$$M \mapsto M^\circ$$

is conservative. This is because F_A is generated by A under finite coproducts, see 5.5. ☐

5.5. Conservativity of $M \mapsto M^\circ$. Let $\alpha : M \to N$ in $\mathcal{D}(A)_{\geq 0}$. Assume that $M^\circ \xrightarrow{\simeq} N^\circ$ iso. in Anim. We are given $M : F_A^{op} \to \mathrm{Anim}$. It suffices to show that

$$M(A^{\oplus n}) \xrightarrow{\alpha} N(A^{\oplus n})$$

iso for all n. But this is equivalent to

$$M(A)^{\times n} \xrightarrow{\simeq} N(A)^{\times n}$$

5.6. Notation. $\phi_* = \mathbb{L}\phi_* : \mathcal{D}(B)_{\geq 0} \to \mathcal{D}(A)_{\geq 0}$.

5.7. Construction. $- \otimes^{\mathbb{L}} -$. $\mathbb{R}\underline{\mathrm{Hom}}$ right adjoint bifunctor to

$$- \otimes^{\mathbb{L}} - : \mathcal{D}(A)_{\geq 0} \times \mathcal{D}(A)_{\geq 0} \to \mathcal{D}(A)_{\geq 0}$$

which is the left derived functor of

$$- \otimes - : \mathrm{Mod}_A \times \mathrm{Mod}_A \to \mathrm{Mod}_A$$

Remark 5.8. $\phi_*\phi^* : (-) \otimes_A B$ as endofunctor of Mod_A This implies

$$\mathbb{L}\phi_*\mathbb{L}\phi^* \simeq \mathbb{L}(\phi_*\phi^*) \simeq (-) \otimes_A^{\mathbb{L}} B$$

Lemma 5.9. $\phi : A \to B$ a *flat* ring homomoprhism then

$$\mathbb{L}\phi^* : \mathcal{D}(A)_{\geq 0} \to \mathcal{D}(B)_{\geq 0}$$

Write $\mathbb{L}\phi^*$ in this case. We have this diagram

$$\begin{array}{ccc} \mathcal{D}(A)_{\geq 0} & \xrightarrow{\phi^*} & \mathcal{D}(B)_{\geq 0} \\ \subset & & \subset \\ \mathrm{Mod}_B & \xrightarrow{\phi^*} & \mathrm{Mod}_B \end{array}$$

Proof. Recall that ϕ_* detects discreteness. It suffices to show

$$\phi_* \mathbb{L}\phi^* : \mathcal{D}(A)_{\geq 0} \to \mathcal{D}(A)_{\geq 0}$$

preserves discreteness. But the above map is identifed with

$$(-) \otimes_A^{\mathbb{L}} B$$

B flat A-module, implies that B is a filtered colimit of fg free A-modules (Lazard).

Remark 5.10. FIltered colimits are preserved by the embedding

$$\mathrm{Mod}_A \hookrightarrow \mathcal{D}(A)_{\geq 0}$$

This implies

$$M \otimes_A^{\mathbb{L}} B \simeq M \otimes_A^{\mathbb{L}} (\varinjlim_{\alpha} N_\alpha) \simeq \varinjlim_{\alpha} M \otimes N_\alpha$$

where the last object is discrete by above remark. □

5.2 Addendum to Additive categories

Definition 5.11. $\mathcal{C}$ an ∞-category with finite colimits and zero object. $\mathcal{C}$ is *prestable* if the following equivalent conditions hold:

- $\mathcal{C}$ admits a ff embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ where $\mathcal{D}$ is *stable*, such that the essential image is closed under finite colimits and extensions.
- $\Sigma : \mathcal{C} \to \mathcal{C}$ is f.f. ($\Leftrightarrow \Omega\Sigma \simeq \mathrm{id}$).

Example 5.12. $\mathrm{Anim}(\mathcal{C})$ is prestable if $\mathcal{C}$ is additive.

Example 5.13. $\mathcal{D}(A)_{\geq 0}$ is prestable for all $A \in \mathrm{CRng}$.

Remark 5.14. Can use universal property of the construction

$$\mathrm{Anim}(\mathcal{C}) \rightsquigarrow \mathrm{Anim}^{\mathrm{nc}}(\mathcal{C}) = \varprojlim(\cdots \xrightarrow{\Omega} \mathrm{Anim}(\mathcal{C}))$$

to extend derived functors to nonconnective objects.

5.3 Reminder on sheaves of sets

5.15. If $X \in \mathrm{Top}$, $\mathcal{B}$ a basis. Assume $\mathcal{B}$ is *intersection closed*: for all $U \subset X$ open, $\exists U = \bigcup U_i$, $U_i \in \mathcal{B}$ such that

$$U_{i_0} \cap \cdots \cap U_{i_n} \in \mathcal{B} \text{ for all finite } \{i_0, \ldots, i_n\} \subseteq I$$

For any presheaf $\mathcal{F}$ on X tfae.

1. $\mathcal{F}$ is a sheaf.
2. for all $U = \bigcup U_i$ covering with $U_i \in \mathcal{B}$ we have the limit diagram

$$\mathcal{F}(U) \longrightarrow \textstyle\prod_i \mathcal{F}(U_i) \rightrightarrows \textstyle\prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

Example 5.16. If X *coherent* top. space (the collection $U_c(X)$ of compact opens of X is closed under intersection and forms a basis of X). Then $U_c(X)$ is an intersection closed basis, so we get $\mathcal{F}$ is a sheaf on X iff $U = \bigcup_i U_i$ finite covering $U_i \in \mathcal{B}$,

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

iff by indiction $U = U_1 \cup U_2$, $U_i \in \mathcal{B}$, it suffices to check Mayer-vietoris type condition:

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U_1) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_2) & \longrightarrow & \mathcal{F}(U_1 \cap U_2) \end{array}$$

6 Sheaves with values in ∞-categoires

Definition 6.1. X top. space, $X = \bigcup U_i$. The Čech nerve of the famil $(U_i)_i$ is the simplicial diagram

$$\cdots \textstyle\bigsqcup_{i,j,k} U_{i,j,k} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \bigsqcup_{i,j} U_{i,j} \rightrightarrows \bigsqcup U_i \qquad \text{denoted } \check{C}(U,X)$$

note that I have omitted the left morphisms. This is coproduct is taken in category of presheaves.

Definition 6.2. $\mathcal{F} : \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{V}$, $\mathcal{V}$ is an ∞-category with limmits. $\mathcal{F}$ satisfies *descent* (i.e. a *sheaf*) if:

$$\mathcal{F}(U) \to \mathrm{Tot}(\mathcal{F}\check{C}(U_i/U)))$$

(where U is covered by this U_i) is an iso in $\mathcal{V}$. The coslimplicial diagram is defined via lke.

Example 6.3. If $\mathcal{V}$ is an ordinary category, this limit is the same as the equalizer previously defined.

Theorem 6.4. X is a coherent top. space. $\mathcal{V}$ is an ∞-cat with limmits, $\mathcal{F} : \mathcal{U}(X)^{op} \to \mathcal{V}$ a presheaf tfae.

1. $\mathcal{F}$ is a sheaf.
2. for all $U, V \subseteq X$ cmopact open subsets

$$\begin{array}{ccc} \mathcal{F}(U \cup V) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V) \end{array}$$

   is cartesian in $\mathcal{V}$.

Theorem 6.5. X top. space. $\mathcal{B} \subseteq \mathcal{U}(X)$ basis which is ∩-closed (intersection closed), such that every $U \in \mathcal{B}$ is compact. $\mathcal{F} : U(X)^{\mathrm{op}} \to \mathcal{V}$ a presheaf. tfae.

1. $\mathcal{F}$ satisfies descent. (is a sheaf).
2. for all $U_1, \ldots, U_n \in \mathcal{B}$ such that $U = \bigcup U_i \in \mathcal{B}$, then

$$\mathcal{F}(U) \xrightarrow{\simeq} \varprojlim_{S \neq \varnothing} \mathcal{F}(U_S)$$

   limit over nonempty subset $S \subseteq \{1, \ldots, n\}$ where $U_S = \bigcap_{i \in S} U_i$.

Example 6.6. (Animated modules) X affine scheme = $\operatorname{Spec} A$. There exists a unique Zariski sheaf on Z_{Zar} (small zariski site)

$$D : (X_{\mathrm{Zar}})^{op} \to \infty - \mathrm{Cat}$$

whose values on $U(f) = V(f)^c$ is

$$\mathcal{D}(U(f)) \simeq \mathcal{D}(A[f^{-1}])$$

Moreover $\mathcal{D}(U)^{\heartsuit} \simeq$ q.c. sheaves on U.

6.1 Sheaves on site

6.7. Site = category $\mathcal{C}$ with Grothendieck topology τ. Then can talk about sheaves on $\mathcal{C}$.

Example 6.8. X is top. space. $\mathcal{U}(X)$ the Grothendieck topology is generated by family $(U_i \to U)_{i\in I}$, $U_i \subset U$ opens, $U = \bigcup U_i$.

Example 6.9. X_{Zar}=Small zariski site, of a scheme X as a category: $\mathcal{U}(X)$ with induced topology as 6.8.

Example 6.10. (Big Zariski (affine) site). The underlying category is $\mathrm{Sch}^{\mathrm{aff}}$. The topology τ is the topology generated by finite Zariski covering families

$$(U_i \hookrightarrow X) \text{ open embedding}$$

which is jointly surjective, i.e.

$$(\bigsqcup U_i \twoheadrightarrow X)$$

(equivalently is faithfully flat.)

Definition 6.11. $\mathcal{C}$ a site, $\mathcal{B} \subset \mathcal{C}$ a full subcategory is a *basis* for $\mathcal{C}$ if for all $X \in \mathcal{C}$, exists a family $(Y_\alpha \to X)_\alpha$, $Y_\alpha \in \mathcal{B}$ which generates a covering sieve.

- $\mathcal{B} \subseteq \mathcal{C}$ is $\cap$-closed if $\mathcal{C}$ admits fibered products.
- $\mathcal{B} \subseteq \mathcal{C}$ is closed under finite products
- for all $X \in \mathcal{C}$ exists family $(Y_\alpha \to X)_{\alpha\in\Lambda}$, $Y_\alpha \in \mathcal{B}$ *and*

$$Y_{\alpha_0} \times_X \cdots \times_X Y_{\alpha_n} \in \mathcal{B}$$

$$\forall \{\alpha_0, \ldots, \alpha_n\} \subset \Lambda$$

  (which generates a covering sieve).

Remark 6.12. $\mathcal{B} \subseteq \mathcal{C}$ basis $\Rightarrow$ induced a topology ono $\mathcal{B}$ from $\mathcal{C}$. (a sieve is covering $\Leftrightarrow$ image in $\mathcal{C}$ is covering).

Theorem 6.13. $\mathcal{C}$ site, $\mathcal{B} \subseteq \mathcal{C}$, $\cap$- closed basis, $F : \mathcal{C}^{op} \to \mathcal{V}$ presheaf. tfae.

1. $\mathcal{F}$ is a sehaf on $\mathcal{C}$.
2. $\mathcal{F}|_{\mathcal{B}}$ is a sheaf on $\mathcal{B}$.

Remark 6.14. Under assumptions on the topoology on $\mathcal{C}$ can prove that descent can be checked using squares (Voevodsky) cd-structures.

6.15. Let $i : \mathcal{B} \hookrightarrow \mathcal{C}$ be inclusion.

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V}) & \xrightarrow{i^*} & \mathrm{Fun}(\mathcal{B}^{op}, \mathcal{V}) \\ \subset & & \subset \\ \mathrm{Shv}_{\mathcal{V}}(\mathcal{C}) & \xrightarrow{i^*} & \mathrm{Shv}_{\mathcal{V}}(\mathcal{B}) \end{array}$$

Fact. i^* admits a right adjoint which is rke. Given $\mathcal{F}_0 : \mathcal{B}^{op} \to \mathcal{V}$,

$$\mathcal{F} := rke(\mathcal{F}_0) : \mathcal{C}^{op} \to \mathcal{V}, \quad \mathcal{F}(X) \simeq \varprojlim \mathcal{F}(X_0)$$

where the limit s taken over (X_0, u), $X_0 \in \mathcal{B}$, $u : X_0 \to X$ morphism.

7 Bonus: Animated modules vs. connected chain complex

Sebastian.

7.1. $R \in \mathrm{CRng}$. Write $\mathrm{Ch}(R)$ for the cat of chain complex of R-modules.

Proposition 7.2. [Lurb, 1.3.5.3] $\mathrm{Ch}(R)$ admits a left proper combinatorial model structure where

- cofibs = level-wise monomorphisms.
- weak equiv. = quasi-isos.

7.3. localization. $\mathcal{C}$ is an ∞-category. $W \subseteq \mathcal{C}$ is a subcategory, a localization of $\mathcal{C}$ by W is a functor $\gamma : \mathcal{C} \to W^{-1}\mathcal{C}$ s.t.

- $W^{-1}\mathcal{C}$ is an ∞-cat.
- such that for all ∞-cat $\mathcal{D}$,

  $$\mathrm{Fun}(W^{-1}\mathcal{C}, \mathcal{D}) \xrightarrow{\gamma^*} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

  is ff. and a functor $F : \mathcal{C} \to \mathcal{D}$ is in ess. image iff F inverts morphisms in W.

Example 7.4.
1. $\mathrm{Anim} \simeq [we]^{-1} N(\mathrm{Set}_\Delta)$
2. Any ∞-cat is a loc. of a 1-cat.
3. $hW^{-1}\mathcal{C} \simeq hW^{-1}h\mathcal{C}$

Definition 7.5. $\mathcal{D}(R) := qi^{-1}\,\mathrm{Ch}(R)$.

Theorem 7.6. $\mathcal{D}(R)$ is a presentabble stable ∞-cat.

Proof. More generally, $(\mathcal{C}, W)$ any combinatorial model category. Then $W^{-1}\mathcal{C}$ is presentable (Dugger). [Cis20, 7.11?]. For stability, we show $\Sigma : \mathcal{D}(R) \to \mathcal{D}(R)$ is an equivalence. We 1-pushout in $\mathrm{Ch}(R)$. For any $C_* \in \mathrm{Ch}(R)$,

$$\begin{array}{ccc} C_* & \hookrightarrow & \mathrm{Cone}(C_*)\text{''} = \text{''} C_* \oplus C_*[1] \\ \downarrow & \ulcorner & \downarrow \\ 0_* & \longrightarrow & C_*[1] \end{array}$$

note the differential on the cone is slightly different. $C_* \hookrightarrow \mathrm{Cone}(C_*)$ is a cofibration in $\mathrm{Ch}(R)$, It follows that the square is a pushout in $\mathcal{D}(R)$.

$\mathrm{Cone}(C_*) \simeq 0$. This implies Σ coincides with $C_* \mapsto C_*[1]$ and this is an equivalence. $\square$

Definition 7.7. A t-structure on a stable ∞-cat is a t-structure on $h\mathcal{C}$. i.e. we have two full subcats $h\mathcal{C}_{\geq 0}, h\mathcal{C}_{\leq 0} \subset h\mathcal{C}$ such that

- For $X \in h\mathcal{C}_{\geq 0}, Y \in h\mathcal{C}_{\leq 0}$
$$\mathrm{Hom}_{h\mathcal{C}}(X, Y[-1]) \simeq 0$$
- $h\mathcal{C}_{\geq 0}[1] \subset h\mathcal{C}_{\geq 0}$ and dually.
- for all $X \in h\mathcal{C}$¡ there is a fiber sequence
$$X' \to X \to X''$$
with $X' \in h\mathcal{C}_{\geq 0}, X'' \in h\mathcal{C}_{\leq 0}$.

7.8. We obtain two full subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$.

Example 7.9. $\mathcal{D}(R)$ with
$$\mathcal{D}(R)_{\geq 0} = \{C_* : H_*(C_*) = 0 \forall i < 0\}$$
$$\mathcal{D}(R)_{\leq 0} = \{C_* : H_*(C_*) = 0 \forall i > 0\}$$

Theorem 7.10. The functor $\mathrm{Mod}_R \to \mathcal{D}(R)_{\geq 0}$ (connective. chain complex) $M \mapsto M[0]$, induces an equivalence on ∞-cat,
$$\mathrm{Anim}(\mathrm{Mod}_R) \xrightarrow{\simeq} \mathcal{D}(R)_{\geq 0}$$

Here Mod_R is a 1-category.

Proof. Step 1. i is ff. True more generally, if $\mathcal{C}$ has t-structure, $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\geq 0} \simeq Nh\mathcal{C}^{\heartsuit}$. This is true since for $X, Y \in \mathcal{C}^{\heartsuit}$, then
$$\pi_n \mathrm{Map}_{\mathcal{C}}(x, y) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y[-n]) \simeq 0$$

Step 2. for all $P \in \mathrm{Proj}_R$ category of fg. project R-mod, $i(P) \in \mathcal{D}(R)_{\geq 0}$ is compact project. This implies the functor correpresented by $i(P)$ preserves filtered colimits+ geom. realizations.

filtered colimits. Let $K \in \mathcal{D}(R)$ is compact iff $K \in h\mathcal{D}(R)$ is compact. [Lurb, 1.4.4.1]
$$\bigoplus_\alpha \mathrm{Hom}_{h\mathcal{D}(R)}(K, T_\alpha) \xrightarrow{\simeq} \mathrm{Hom}_{h\mathcal{D}(R)}(K, \bigoplus_\alpha T_\alpha)$$

And the compact objects in $h\mathcal{D}(R)$ are precisely the perfect complexes. [2]. In particular $P[0]$ is a perfect complex.

geometric realizations.

Lemma 7.11. Let $Q \in \mathcal{D}(R)_{\geq 0}$ s.t. for all $X \in \mathcal{D}(R)_{\geq 0}$,
$$\mathrm{Hom}_{h\mathcal{D}(R)}(Q[-i], X) \simeq \mathrm{Ext}^i(Q, X) = 0 \text{ for all } i > 0 \Rightarrow Q \text{ is proj}$$

(projective as in) $\mathrm{Map}(Q, -)$ commutes with geometric generalizations.

[2] Equivalent to bounded complex of fg. proj. modules

Step 3. By 1+2, $i : \mathrm{Proj} \hookrightarrow \mathcal{D}(R)_{\geq 0}$ induces a ff colimit preserving functor, inducing ff. colimit preserving functor [Lura, 5.5.8.22], $F : \mathrm{Anim}(\mathrm{Mod}_R) \to \mathcal{D}(R)_{\geq 0}$.

Presentability implies we get right adjoint $G : \mathcal{D}(R)_{\geq 0} \to \mathrm{Anim}(\mathrm{Mod}_R)$. Let $X \in \mathcal{D}(R)_{\geq 0}$, have a counit map

$$\varepsilon_x : FGX \to X$$

wts ε_x is an equivalence. To show this we observe

$$\mathrm{Map}_{\mathrm{Anim}(\mathrm{Mod}_R)}(R, GX) \simeq \mathrm{Map}_{\mathcal{D}(R)_{\geq 0}}(R[0], FGx) \xrightarrow{\varepsilon_x} \mathrm{Map}_{\mathcal{D}(R)_{\geq 0}}(R[0] = FR, X)$$

where the outer triangle is equivlanece by adjunction. This implies

$$\begin{array}{ccc}
\pi_n(\mathrm{Map}_{\mathcal{D}(R)_{\geq 0}}(R[0], FGx) & \xrightarrow{\simeq} & \pi_n(\mathrm{Map}_{\mathcal{D}(R)_{\geq 0}}(R[0], X) \\
\Big| \simeq & & \Big| \simeq \\
\pi_0(\mathrm{Map}_{\mathcal{D}(R)_{\geq 0}}(R[n], FG(X)) & & H_n(X) \\
\Big| \simeq & \nearrow \simeq & \\
\mathrm{Hom}_{h\mathcal{D}(R)_{\geq 0}}(R[n], FGX) & & \\
\Big| \simeq & & \\
H_n(FGX) & &
\end{array}$$

□

7.12. Have map

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{D}(R)}(P, -) : \mathcal{D}(R) & \longrightarrow & \mathrm{Anim} \\
& {\scriptstyle h(P)} \searrow \quad \nearrow {\scriptstyle \Omega^\infty} & \\
& \mathrm{Sp} &
\end{array}$$

Observe

$$\pi_{-n}\underline{\mathrm{Map}}(P, X) = \mathrm{Ext}^n(P, X[n])$$

This implies $h(P)$ restricts to a functor

$$F : \mathcal{D}(R)_{\geq 0} \to \mathrm{Sp}_{\geq 0}$$

Goal F preserves geo. real. Then it suffices too show $\mathcal{D}(R)_{\geq 0} \to \mathrm{Sp}_{\geq 0} \xrightarrow{\Omega^\infty} \mathrm{Anim}.$ commutes with geo. realization. The second map commutes with geo realization from [Lurb, 1.4.3.9].

7.13. [Lurb, 1.3.3.10]. Suffices to see that for all n,

$$\mathcal{D}(R)_{\geq 0} \xrightarrow{F} \mathrm{Sp}_{\geq 0} \xrightarrow{\tau_{\leq n}} \mathrm{Sp}_{\geq 0, \leq n}$$

for all n. Now observe

$$\begin{array}{ccc}
\mathcal{D}(R)_{\geq 0} & \xrightarrow{F} & \mathrm{Sp}_{\geq 0} \\
\downarrow {\scriptstyle \tau_{\leq n}} & & \downarrow \\
(\mathcal{D}(R)_{\geq 0})_{\leq n} & \xrightarrow{\tau_{\leq n} F} & \mathrm{Sp}_{\geq 0, \leq n}
\end{array}$$

$\tau_{\leq n} F$ is a right exact functor between n-categories. Such a functor always preserves geoemtric realization. Note the vertical maps commutes with all colimits.

8 Quasi-coherent Sheaves

8.1 Addendum

Proposition 8.1. (Universal properties of $\mathrm{Anim}^{\mathrm{nc}}(\mathcal{C})$. $\mathcal{C}, \mathcal{D}$ are *additive* algebraic categories. $F : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{D})$

1. If F commutes with Ω then it extends uniquely to a functor

$$F^{\mathrm{nc}} : \mathrm{Anim}^{\mathrm{nc}}(\mathcal{C}) \to \mathrm{Anim}^{\mathrm{nc}}(\mathcal{D})$$

   such that $\Omega^{\infty-n} \circ F^{\mathrm{nc}} \simeq F \circ \Omega^{\infty-n}$.

   Informally, $(X_0, S_1, \dots) \mapsto (F(X_0), F(X_1), \dots)$.

2. If F commutes with Σ then it extends uniquely to a functor

$$F^{\mathrm{nc}} : \mathrm{Anim}^{\mathrm{nc}}(\mathcal{C}) \to \mathrm{Anim}^{\mathrm{nc}}(\mathcal{D})$$

   $F^{\mathrm{nc}} \circ \Sigma^{\infty-n} \simeq \Sigma^{\infty-n} \circ F$ for all $n \geq 0$, where

$$\Sigma^{\infty-n} : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}^{\mathrm{nc}}(\mathcal{C})$$

   is left adjoint to $\Omega^{\infty-n}$.

8.2.

$$\Sigma^{\infty} : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}^{\mathrm{nc}}(\mathcal{C})$$

F^{nc} preserves connective objects.

Proof. 1. We have diagram

$$\begin{array}{ccccccc} \mathrm{Anim}^{\mathrm{nc}}(\mathcal{C}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Anim}(\mathcal{C}) & \xrightarrow{\Omega} & \mathrm{Anim}(\mathcal{C}) \\ \Big\downarrow{\scriptstyle F^{\mathrm{nc}}} & & & & \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle F} \\ \mathrm{Anim}^{\mathrm{nc}}(\mathcal{D}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Anim}(\mathcal{C}) & \xrightarrow{\Omega} & \mathrm{Anim}(\mathcal{C}) \end{array}$$

   the diagram commutes by hypothesis.

2. Dually can prove

$$\mathrm{Anim}(\mathcal{C}) \xrightarrow{\Sigma} \mathrm{Anim}(\mathcal{C}) \longrightarrow \cdots \qquad \cdots \longrightarrow \mathrm{Anim}^{\mathrm{nc}}(\mathcal{C})$$

   colimit in the ∞-cst of presentable ∞-categories of colimit preserving functors. This follows from $Pr^L \simeq (Pr^R)^{\mathrm{op}}$ and we have a limit prserving +reflecting functor $Pr^R \hookrightarrow \widehat{\mathrm{Cat}}_\infty$.

∎

Example 8.3. $\phi : A \to B$ ring homomorphism.

$$\mathbb{L}\phi^* : \mathcal{D}(A) \to \mathcal{D}(B)$$

which restricts to

$$\mathbb{L}\phi^* : \mathcal{D}(A)_{\geq 0} \to \mathcal{D}(B)_{\geq 0}$$

similarly

$$\phi_* : \mathcal{D}(B) \to \mathcal{D}(A)$$

which restricts to

$$\phi_* : \mathcal{D}(B)_{\geq 0} \to \mathcal{D}(A)_{\geq 0}$$

8.2 Grothendieck topologies

8.4. Zariski topology on $\mathrm{CRng}^{\mathrm{op}}$ is the Grothendieck topology gen. by families

$$(A \xrightarrow{\phi_i} A_i)_i$$

where ϕ_i flat epimorphisms ($A_i \otimes_A A_i \xrightarrow{\simeq} A_i$) of finite projective such that $A \to \prod A_i$ fiathfully flat.

- Equivalently, finite familes $(A \to A[f_i^{-1}])_i$ where $f_i \in A$ jointly generate a unit ideal of A.
- Under the equivalence $\mathrm{CRng}^{op} \simeq \mathrm{Sch}^{\mathrm{aff}}$ this is the big affine Zariski Site.

Theorem 8.5. The functors

$$\mathcal{D}_{\geq 0}, \mathcal{D} : \mathrm{CRng} \to \mathrm{Cat}_\infty$$

which are respectively described as

$$A \mapsto \mathcal{D}(A)_{\geq 0}, \varphi \mapsto \mathbb{L}\varphi^*$$

$$A \mapsto \mathcal{D}(A), \varphi \mapsto \mathbb{L}\varphi^*$$

satisfies Zariski descent.

- In particular, fr every famil $(A \to A_i)_i$ generating a Zariski covering sieve there is a limit diagram

$$\mathcal{D}(A) \longrightarrow \prod_i \mathcal{D}(A_i) \rightrightarrows \prod_{i,j} \mathcal{D}(A_i \otimes_A A_j) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

8.3 Quasicoherent sheaves on affine schemes

Theorem 8.6. $X = \mathrm{Spec}(A)$ an affine scheme. Then there exists a unique Zariski sheaf of ∞-cat on the small Zariski X_{Zar}.

$$\mathcal{D} : X_{\mathrm{Zar}}^{op} \to \mathrm{Cat}_\infty$$

whose values on elementary opens $U(f)$ are given by

$$\mathcal{D}(U(f)) \simeq \mathcal{D}(A[f^{-1}])$$

Moreover: any affine open $U = \mathrm{Spec}\, R \subseteq X$,

$$\mathcal{D}(U) \simeq \mathcal{D}(B)$$

Remark 8.7. This implies there exists unique $\mathcal{O}_X \in \mathcal{D}(X)$ whose restriction to any elementary open $U(f) \subseteq X$ is $A[f^{-1}] \in \mathcal{D}(A[f^{-1}]) \simeq \mathcal{D}(U(f))$. This is the structure sheaf of X. Happens to be discrete.

8.8 (?)**.** *Claim:.* $\mathcal{D} : (\mathrm{Sch}^{\mathrm{aff}})^{op} \to \mathrm{Cat}_\infty$, $\mathrm{Spec}\, A \mapsto \mathcal{D}(A)$ is a Zariski sheaf.

Let $X \in \mathrm{Sch}^{\mathrm{aff}}$, $(U_\alpha \to X)_\alpha$, Zariski covering, then

$$\mathcal{D}(X) \to \mathrm{Tot}(\mathcal{D}(\check{C}(U_\alpha/X)_\bullet)$$

is an equivalence. Note U_α are all afine, so this is special case of the previous theorem.

Remark 8.9. Descent for $\mathcal{D}$ (limits of $\mathcal{D}_{\geq 0}$) follows from $\mathcal{D}_{\geq 0}$. The sheaf condition only involve limits.

Lemma 8.10. (fpqc conservativity) $\phi_\alpha : (A \to B_\alpha)_\alpha$ finite fmaily of flat ring homomorphisms such that $A \to B := \prod B_\alpha$ is faithfully flat, then the family of fucntors

$$\mathbb{L}\phi_\alpha^* : \mathcal{D}(A)_{\geq 0} \to \mathcal{D}(B_\alpha)_{\geq 0}$$

is jointly conservative. Some omit the $\mathbb{L}$ as the rings are flat, hence, preserves discrete objects.

Proof.

- Note $\mathcal{D}(B)_{\geq 0} \simeq \prod_\alpha \mathcal{D}(B_\alpha)_{\geq 0}$. Therefore we may as well assume that the family consists of a single faithfully flat map.
- *Claim*: if $\phi^* M \simeq 0$ then $M \simeq 0$ ($M \in \mathcal{D}(A)_{\geq 0}$).
- *Note.* $M \simeq 0 \Leftrightarrow \pi_i M = 0$ for all $i \geq 0$.[3]

$$\begin{aligned} &\Leftrightarrow \pi_i M \otimes_A B \simeq 0 \text{ for all } i \\ &\Leftrightarrow \pi_i(M \otimes_A^{\mathbb{L}} B) \simeq 0 \text{ for all } i \text{ (flatness)} \\ &\Leftrightarrow M \otimes_A^{\mathbb{L}} B \simeq 0 \end{aligned}$$

□

Proof. of Thm 8.6. Apply last lecture. Sufficient to show.

- *Claim.* $U, V \subset X$ affine opens such that $U \cup V$ affine. Then

$$\begin{array}{ccc} \mathcal{D}(U \cup V) & \longrightarrow & \mathcal{D}(U) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{D}(V) & \longrightarrow & \mathcal{D}(U \cap V) \end{array}$$

  (Take $\mathcal{U}_{\text{aff}}(X) \subset \mathcal{U}(X)$ affine opens. This is a $\cap$-closed basis since X affine.)
- wlog: $X = U \cup V$.
- $A := \Gamma(X, \mathcal{O}_X), A_1 = \Gamma(U, \mathcal{O}_U), A_2 = \Gamma(V, \mathcal{O}_V), A_{12} = \Gamma(U \cap V, \mathcal{O}_{U \cap V})$. *Want.*

$$F : \mathcal{D}(A) \xrightarrow{\simeq} \mathcal{D}(A_1) \times_{\mathcal{D}(A_{12})} \mathcal{D}(A_2)$$

- $G : \mathcal{D}(A_1) \times_{\mathcal{D}(A_{12})} \mathcal{D}(A_2) \to \mathcal{D}(A)$ right adjoint. The lhs consists of

$$(M_1, M_2, M_1 \otimes_{A_1} A_{12} \simeq M_2 \otimes_{A_2} A_{12}) \mapsto M_1 \times M_{12} M_2$$

  where $M_i \in \mathcal{D}(A_i)$.
- *Check.* unit. id $\simeq GF$. For all $M \in \mathcal{D}(A)_{\geq 0}$,

$$M \xrightarrow{\simeq} M_1 \times_{M_{12}} M_2$$

$$M_? := M \otimes_A A_?$$

[3] This is because $M \mapsto M^\circ$ is conservative.

- Apply fpqc conservativity: (derived) extension of scalars along $A \to A_1, A \to A_2$ is jointly conservative. ($A \to A_1, A \to A_2$) is a fin. flat family which is jointly faithfully flat.)
- *Want* $M \otimes_A A_i \to (M_1 \times_{M_{12}} \otimes M_2) \otimes_A A_i$ iso. for all $i \in \{1,2\}$. The rhs (wlog $i = 1$) [?]

$$\simeq (M_1 \otimes_A A_i) \otimes_{M_{A_1 2} \otimes_A A_i} (M_2 \otimes A_i)$$

- Now it suffices to show that G is conservative. Exercise. This implies by rke along $\mathcal{U}_{\text{aff}}(X) \subset \mathcal{U}(X)$ gives you sheaf of X_{Zar}.

□

8.4 Quasicoherent sheaves on schemes

Theorem 8.11. 1. There is a unique Zar sheaf

$$\mathcal{D} : (\text{Sch})^{op} \to \text{Cat}_\infty$$

which extends $\mathcal{D} : (\text{Sch}^{\text{aff}})^{op} \to \text{Cat}_\infty$.

2. Moreover

$$\mathcal{D}(X) \xrightarrow{\simeq} \varprojlim_{S, S \to X} \mathcal{D}(A)$$

over pairs $(S = \operatorname{Spec} A, S \to X)$ where S affine and $S \to X$ is a morphism.

8.12. Recall. (Comment on last time) Sch by Zariski site, topology generated $(U_i \to X)_i$ open immersions $\bigsqcup U_i \to X$ is jointly surjective.

Proof.
- There is an equivalence, for any $\mathcal{V}$ an ∞-cat with limits,

$$\text{Shv}^{\mathcal{V}}_{\text{Zar}}(\text{Sch}) \xrightarrow{\simeq} \text{Shv}^{\mathcal{V}}_{\text{Zar}}(\text{Shv}^{\text{aff}})$$

  $\text{Sch}^{\text{aff}} \subset \text{Sch}^{\text{sep}}$ and $\text{Sch}^{\text{sep}} \subseteq \text{Sch}$ are bases which are $\cap$-closed.
- $\mathcal{D}$:=rke of $\mathcal{D} : (\text{Sch}^{\text{aff}})^{\text{op}} \to \text{Cat}_\infty$.

□

Remark 8.13. No analogue statement for triangulated categories.

$$h\mathcal{D} : (\text{Sch}^{op} \to \text{TriCat} \to \text{Cat}$$

$$X \mapsto h\mathcal{D}(X)$$

do not satisfy descent.

$$\mathcal{D}(\mathbb{P}^1) \xrightarrow{\simeq} \mathcal{D}(A^n) \times_{\mathcal{D}(\mathbb{A}^1 \setminus \{0\})} \mathcal{D}(BA^1)$$

fails at $h\mathcal{D}$ level.

Corollary 8.14. $X \in \text{Sch}.$ [4] $(X_{\text{Zar}})^{op} \xrightarrow{\mathcal{D}} \text{Cat}_\infty$ is a Zariski sheaf. Proof by restriction. [?]

[4] Different to 8.6 which is for *affine*.

Proposition 8.15. $X \in \mathrm{Sch}$. The zariski sheaf $\mathcal{D} : (X_{\mathrm{Zar}})^{op} \to \mathrm{Cat}_\infty$ is the right kan eztension of its restriction to $\mathcal{U}_{\mathrm{aff}}(X)$. In particular, for all $U \subseteq x$ open

$$\mathcal{D}(U) \xrightarrow{\simeq} \varprojlim \mathcal{D}(V)$$

This implies $\mathcal{D}$ is the unique Zariski sheaf on X_{Zar} whose values on affine opens $U = \operatorname{Spec} A \subset X$ are $\mathcal{D}(U) \subseteq \mathcal{D}(A)$.

Proof. $\mathcal{U}_{\mathrm{sep}}(X) \subseteq \mathcal{U}(X)$, $\mathcal{U}_{\mathrm{aff}}(X)$,$\mathcal{U}_{\mathrm{sep}}(X)$ forms $\cap$-closed basis. □

Definition 8.16.

- A *quasi-coherent complex* on a scheme X is an object $\mathcal{F} \in \mathcal{D}(X)$.
- A *qcoh. animated sheaf/connective qcoh. complex.* on X is an object $\mathcal{F} \in \mathcal{D}(X)_{\geq 0}$

Example 8.17.

- $X = \operatorname{Spec} A$ affine $\mathcal{D}(X) \simeq \mathcal{D}(A)$.
- X scheme, $\mathcal{D}(X) \xrightarrow{\simeq} \varprojlim_{U \subseteq X} \mathcal{D}(U)$. A qcho complex on X amounts to
  - For all $U = \operatorname{Spec} A \subseteq X$ an $\mathcal{F}_U \in \mathcal{D}(U) \subseteq \mathcal{D}(A)$.
  - $\forall U \subseteq V \subseteq X$ inclusion of affine opens
  $$\mathcal{F}_U|_V \simeq \mathcal{F}|_V$$
  - homotopy coherent system of compatibilities between these isos.
- $\mathcal{F} \in \mathcal{D}(X)$ is *connective* $\Leftrightarrow$ $\forall U = \operatorname{Spec}(A) \subseteq X$ affine open, $\Gamma(U, \mathcal{F}) \in \mathcal{D}(A)$ is a *connective* animated module ($\in \mathcal{D}(A)_{\geq 0}$.
- $\mathcal{F} \in \mathcal{D}(X)$ is *discrete* $\Leftrightarrow$ $\forall U = \operatorname{Spec}(A) \subseteq X$ affine open, $\Gamma(U, \mathcal{F}) \in \mathcal{D}(A)$ is discrete.

Corollary 8.18.

- $X \mapsto \mathcal{D}(X)^\heartsuit$ Zariski sheaf of caegories, detrmined $\mathcal{D}(\operatorname{Spec} A)^\heartsuit \simeq \mathrm{Mod}_A$ for all $A \in$ CRng.
- $\mathcal{D}(X)^\heartsuit \simeq \mathrm{QCoh}(X)$ is abelian cat. of qcho $\mathcal{O}_X$-modules.

Remark 8.19. $\mathcal{D}(X)$ is a stable ∞-cat for all $X \in \mathrm{Sch}$. $\mathcal{D}(X)_{\geq 0}$ is prestable.

9 Direct image functor

Definition 9.1. (Inverse image functor). $f : X \to Y$ be a moprhism of schemes.

$$\mathcal{D}(f) : \mathcal{D}(Y) \to \mathcal{D}(X)$$

inverse imamge denoted $\mathbb{L}f^* := \mathcal{D}(f)$. $\mathcal{D} : (\mathrm{Sch})^{op} \to \mathrm{Cat}_\infty$.

9.2. We have induced functor

$$\mathcal{D}(Y)_{\geq 0} \xrightarrow{\mathbb{L}f^*} \mathcal{D}(X)_{\geq 0}$$

Example 9.3. $f : X \to Y$ a morphism of affine schemes. $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$. Corresponds to ring homo. $\phi : A \to B$, then

$$\mathbb{L}f^* : \mathcal{D}(Y) \simeq \mathcal{D}(A) \to \mathcal{D}(X) \simeq \mathcal{D}(B)$$

Example 9.4. If $j : U \to X$ is open affine subscheme. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}(X) & \longrightarrow & \varprojlim \mathcal{D}(V) \\ & \searrow^{j^*} & \downarrow \\ & & \mathcal{D}(U) \end{array}$$

In otherwords, give $\mathcal{F} = (\mathcal{F}_V)_V \in \mathcal{D}(X)$, then j^* is given by projection onto the component.

Example 9.5. f flat. $f : X \to Y$ implies induces a functor on discrete objects.

$$\mathbb{L}f^* : \mathcal{D}(Y)^{\heartsuit} \to \mathcal{D}(X)^{\heartsuit}$$

This follows by descent from affine case.

Corollary 9.6. (”Internal descent”). If $X \in \mathrm{Sch}$, $\mathcal{F} \in \mathcal{D}(X)$.

Notation. $\Gamma(U, \mathcal{F}) := \mathrm{Map}_{\mathcal{D}(U)}\, \mathcal{O}_U, \mathcal{F}_U)$, $U \subseteq X$ open. [5]

Claim: (For simplicity, assume X is qcqs) $\Gamma(-, \mathcal{F})$ is a sheaf of anima on X_{Zar}. In particular, we want

$$\begin{array}{ccc} \Gamma(U \cup V, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \\ \downarrow & \lrcorner & \downarrow \\ \Gamma(V, \mathcal{F}) & \longrightarrow & \Gamma(U \cap V, \mathcal{F}) \end{array}$$

for all $U, V \subseteq X$ q.c open.

Remark 9.7. Harry. This is also a sheaf on the large Zariski site over X.

Proof. Note that for any $\mathcal{G}, \mathcal{G}' \in \mathcal{D}(U \cup V)$,

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}(U \cup V)}(\mathcal{G}, \mathcal{G}') & \longrightarrow & \mathrm{Map}_{\mathcal{D}(U)}(\mathcal{G}|_U, \mathcal{G}'|_U) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}_{\mathcal{D}(V)}(\mathcal{G}|_V, \mathcal{G}'|_V) & \longrightarrow & \mathrm{Map}_{\mathcal{D}(U \cap V)}(\mathcal{G}|_{U \cap V}, \mathcal{G}'|_{U \cap V}) \end{array}$$

This follows because the square in $\mathcal{D}(-)$ is cartesian. And formation of mapping spaces of ∞-categories commutes with ”limits”.

Apply this to $\mathcal{G} = \mathcal{O}_{U \cup V}, \mathcal{G}' = \mathcal{F}|_{U \cup V}$. □

9.1 Direct image

Definition 9.8. If $f : X \to Y$ in Sch, we have $\mathbb{R}f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$ *direct image*, is the right adjoint of $\mathbb{L}F^*$.

Example 9.9. $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$, $phi : A \to B$, we have

$$\mathbb{R}f_* = \phi_* : \mathcal{D}(A) \to \mathcal{D}(B)$$

in particular preserves connective and discrete objects.[6]

[5] Can also think $\mathcal{F}_U \simeq j^*(\mathcal{F})$

[6] Hence, we would write $f_* = \mathbb{R}f_*$.

9.10. $\mathbb{R}f_*$ typically does *not* preserve connectivity for non-affine morphisms.

Theorem 9.11. $f : X \to Y$ any qcqs morphism

1. $\mathbb{R}f_*$ commutes with colimits.
2. *Base change formula.*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow p & \lrcorner & \downarrow q \\ X & \xrightarrow[f]{} & Y \end{array}$$

$$\mathbb{L}q^*\mathbb{R}f_* \to \mathbb{R}g_*\mathbb{L}p^*$$

   if f, q are are flat (more generally tor-independent squares)

3. *Projection formula:* $\mathcal{F} \in \mathcal{D}(X), \mathcal{G} \in \mathcal{D}(Y)$,

$$\mathbb{R}f_*(\mathcal{F}) \otimes^{\mathbb{L}} \mathcal{G} \xrightarrow{\simeq} \mathbb{R}f_*(\mathcal{F} \otimes^{\mathbb{L}} \mathbb{L}f^*\mathcal{G}))$$

   canonical iso.

Remark 9.12. Given commutive square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{C}' \\ \downarrow p & & \downarrow q \\ \mathcal{D} & \xrightarrow[g]{} & \mathcal{D}' \end{array}$$

assume f, p, q, g have right adjoints f^R, g^R, p^R, q^R, then the sqaure - by flipping the vertical arrows -

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{g} & \mathcal{D}' \\ \downarrow p^R & & \downarrow q^R \\ \mathcal{C} & \xrightarrow[f]{} & \mathcal{C}' \end{array}$$

Then the square commtues upto natural transformation.

$$fp^R \to q^R q f p^R \to q^R g p p^R \to q^R g$$

We say that the (original) square is *vertically right adjointable* if the natrual transformoation is an iso.

9.13. The base change formual says that

$$\begin{array}{ccc} \mathcal{D}(Y) & \xrightarrow{\mathbb{L}q^*} & \mathcal{D}(Y') \\ \downarrow \mathbb{L}f^* & & \downarrow \mathbb{L}g^* \\ \mathcal{D}(X) & \xrightarrow{\mathbb{L}p^*} & \mathcal{D}(X') \end{array}$$

Remark 9.14. $\mathbb{L}f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ is symmetric monoidal. $\Rightarrow \mathcal{D}(X)$ has a canonical $\mathcal{D}(Y)$-module structure. Projection formula for $\mathbb{R}f_*$ says that $\mathbb{R}f_*$ is $\mathcal{D}(Y)$-linear.

Lemma 9.15. $(\Phi_i : \mathcal{D}_i \to \mathcal{C}_i)_{i \in I}$ diagram in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$, I is some (∞-cat). Suppose that each square are vertically right adjointable for $i \to j$ in I,

$$\begin{array}{ccc} \mathcal{D}_i & \longrightarrow & \mathcal{D}_j \\ \downarrow{\scriptstyle \Phi_i} & & \downarrow{\scriptstyle \Phi_j} \\ \mathcal{C}_i & \longrightarrow & \mathcal{C}_j \end{array}$$

Consider the induced functor

$$\begin{array}{ccc} \mathcal{D} & \overset{\simeq}{\longrightarrow} & \varprojlim_i \mathcal{D}_i \\ \downarrow{\scriptstyle \Phi} & & \\ \mathcal{C} & \overset{\simeq}{\longrightarrow} & \varprojlim_i \mathcal{C}_i \end{array}$$

The square

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{D}_j \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi_j} \\ \mathcal{C} & \longrightarrow & \mathcal{C}_j \end{array}$$

is vertically right adjointable for all $i \in I$.

Proof. Of base change in affine case. By the the lemma, reduce to *animated* modules (connective complexes).

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow{\scriptstyle \psi} & \ulcorner & \downarrow{\scriptstyle \psi'} \\ A' & \xrightarrow{\phi'} & B' \end{array}$$

$$B' \simeq B \otimes_A A' \simeq B \otimes_A^{\mathbb{L}} A'$$

Do we have

$$\mathbb{L}\psi^* \phi_* \xrightarrow{?} \phi'_* \mathbb{L}\psi'^*$$
$$N \otimes_A^{\mathbb{L}} A' \to N \otimes_B^{\mathbb{L}} B'$$

But

$$N \otimes_B^{\mathbb{L}} B' \simeq N \otimes_B^{\mathbb{L}} B \otimes_A^{\mathbb{L}} A' \simeq N \otimes_A^{\mathbb{L}} A'$$

□

9.2 Descent for the direct image functor

Remark 9.16. $X \in \mathrm{Sch}$, $U, V \subseteq X$ open. $\mathcal{F} \in \mathcal{D}(U \cup V)$, $j_U : U \hookrightarrow X$ inclusion $(V, U \cap V)$. Then

$$j_{U \cup V,*}(\mathcal{F}) \xrightarrow{\simeq} j_{U,*}(\mathcal{F}_U) \times_{j_{U\cap V,*}} (\mathcal{F}_{U \cap V}) j_{V_*} \mathcal{F}(V)$$

Indeed this follows from the equivalence

$$(j^* U, j_V^*) : \mathcal{D}(U \cup V) \xrightarrow{\simeq} \mathcal{D}(U) \times_{\mathcal{D}(U \cap V)} \mathcal{D}(V)$$

Right adjoint :

$$(\mathcal{F}, \mathcal{G}, \mathcal{F}|_{U \cap V} \simeq \mathcal{G}|_{U \cap V}) \mapsto j_{U,*}(\mathcal{F}) \times_{j_{U \cap V,*}} j_{V,*}(\mathcal{F})$$

The map in question is the unit map of the adjunction.

Corollary 9.17. $f : X \to Y$, $X = U \cup V$ open cover, $f_U = f|_U$, etc. If $\mathcal{F} \in \mathcal{D}(X)$, then

$$f_*(\mathcal{F}) \xrightarrow{\simeq} f_{U,*}(\mathcal{F}_U) \times_{f_{U\cap V,*}(\mathcal{F}_{U\cap V})} f_{V,*}(\mathcal{F}_V)$$

Proof. Apply f_* to previous isomorphism. □

9.3 Sketch of proof of bsae change formula

9.18. *Case 1:* $j : U \hookrightarrow X$ is open immersion. $U \subseteq X$ is *quasicompact.* $f : X' \to X$ morphism of *affine schemes.* [7]

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ \downarrow{\scriptstyle f_U} & & \downarrow{\scriptstyle f} \\ U & \xhookrightarrow{j} & X \end{array}$$

Claim:

$$f^* j_* \xrightarrow{\simeq} j'_* f_U^*$$

If U is affine, then we already know this.

- $U = V \cup W$ and the claim holds for V and W, then it holds for U.

This follows from "descent for j_*." From sec. 9.2.

- U qc. can write $U = \bigcup_i U(f_i)$ by finite union of elementary opens. We conclude by induction.

9.19. *Case 2.* [8].

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Y, Y' affines, $f : X \to Y$ qc. Can argue similarly by inducitn on an affine open cover of X, "descent for f_*".

9.20. *General case.* Use "adjointability of limits" lemma and descent to reduce to the case where Y, Y' are affine.

9.4 Direct image along along open immersions

Corollary 9.21. $U \subseteq X$ qc open. $j : U \hookrightarrow X$. Then the functor

$$\mathbb{R}j_* : \mathcal{D}(U) \to \mathcal{D}(X)$$

is fully faithful.

- Equivalently, $j^*\mathbb{R}j_* \xrightarrow{\text{counit}} \mathrm{id} : \mathcal{D}(U) \to \mathcal{D}(U)$ is an isomorphism.

[7] All the functors now will have $\mathbb{R}$ omitted for simplicity.
[8] Now we revert back to original notation

Proof. We have cartesian square, j flat,

$$\begin{array}{ccc} U & = & U \\ \| & & \downarrow j \\ U & \xrightarrow{j} & X \end{array}$$

Apply base change formula. □

9.22. $Z \subset X$ closed subsecheme $i : Z \to X$, $\mathbb{R}_*$ is typically not f.f. even though $i_* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$ is fully faithful. What fails is that

$$\begin{array}{ccc} Z & = & Z \\ \| & & \downarrow i \\ Z & \xrightarrow{i} & X \end{array}$$

is *not* Tor-independent.

10 Perfect complexes

10.1 Perfect complexes

Definition 10.1. $\mathcal{C}$ stable ∞-category. $\mathcal{C}_0 \subseteq \mathcal{C}$ full subcategory.

- $\mathcal{C}_0$ is *stable* if it contains the zero object $0 \in \mathcal{C}$ and is closed under (co)fibers in $\mathcal{C}$. Equivalently $\mathcal{C}_0$ is a stable ∞-category, and the inclusion is an exact functor.
- $\mathcal{C}_0$ is *thick* if it stable and moreover closed under direct summands (=retracts) in $\mathcal{C}$.

Definition 10.2. $M \in \mathcal{D}(A)$ is *perfect* if it is contained in the thick subcategory generated by the object $A \in \mathcal{D}(A)$. $\mathcal{D}_{\mathrm{perf}}(A) \subseteq \mathcal{D}(A)$ be full subcategory of perfect modules.

Example 10.3. $f \in A$. $\mathrm{cofib}(A \xrightarrow{f} A) \in \mathcal{D}_{\mathrm{perf}}(A)_{\geq 0}$. This is Koszul "complex" on the element f). More generally if $f_1, \ldots, f_n \in A$, then

$$K_{f_1,\ldots,f_n} := \otimes^{\mathbb{L}} \mathrm{cofib}(A \xrightarrow{f_i} A) \in \mathcal{D}_{\mathrm{perf}}(A)_{\geq 0}$$

Non example. $A = k[x]/(x^2)$. $k \in \mathcal{D}(A)$ is *not* perfect.

Definition 10.4. X scheme. $\mathcal{F} \in \mathcal{D}(X)$ is *perfect* if for all $U = \mathrm{Spec}\, A \subseteq X$, $\mathcal{F}_U \in \mathcal{D}(U) \simeq \mathcal{D}(A)$ is perfect.

Corollary 10.5. The presheaf of ∞-cat,

$$\mathcal{D}_{\mathrm{perf}} : X \mapsto \mathcal{D}_{\mathrm{perf}}(X)$$

satisfies Zariski descent.

Proof. $\mathcal{D}_{\mathrm{perf}} \subseteq \mathcal{D}$ is sub presheaf. Since $\mathcal{D}$ is a sheaf, $\mathcal{D}_{\mathrm{perf}}$ is a sheaf as long as it is deifned by a "local" property. □

Theorem 10.6. X qcqs. Then every $\mathcal{F} \in \mathcal{D}(X)$ can written as a filtered colimit of perfect complexes.

$$F \simeq \varinjlim \mathcal{F}_\alpha, \mathcal{F}_\alpha \in \mathcal{D}_{\mathrm{perf}}(X)$$

Variants.

1. $\mathcal{F} \in \mathcal{D}(X)_{\geq 0}$ then $\mathcal{F}$ can be taken connective.
2. If $\mathcal{F}$ is supported on a closed subset $Z \subset X$ (with complement $X \smallsetminus Z$ qc) then also $\mathcal{F}_\alpha$ can be taken supported on Z.

$\mathcal{F}$ is supported on Z iff $j^*\mathcal{F} \simeq 0$, $j : X \smallsetminus Z \hookrightarrow X$. We let

$$\mathcal{D}_Z(X) \subseteq \mathcal{D}(X)$$

$$\mathcal{D}_{\mathrm{perf},Z}(X) \subseteq \mathcal{D}_{\mathrm{perf}}(X)$$

10.2 Compactly generated ∞-categories

Definition 10.7. $\mathcal{C}$ an ∞-category with filtered colimits. An object $X \in \mathcal{C}$ is *compact* if

$$\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Anim}$$

commutes with filtered colimits.

Remark 10.8. If $\mathcal{C}$ is stable ∞-cat. Then $\mathcal{C}_0 \subset \mathcal{C}$ full subcategory of compact objects is *thick*. This follows from the following ingredients

- filtered colimits commutes with finite limits.
- A retract of an iso is an iso.

Example 10.9. If A is a commutative ring. Every $M \in \mathcal{D}(A)$ perfect is a compact object.

Proof. By remark, suffices to show for A as a module over itself since A generate $\mathcal{D}_{\mathrm{perf}}(A)$ as a thick subcategory by definition. Now

$$\mathrm{Map}_{\mathcal{D}}(A)(A, -) \simeq (-)^{\circ} : \mathcal{D}(A) \to \mathrm{Anim}$$

commutes with filtered colimits. □

Definition 10.10. $\mathcal{C}$ an ∞-category. $\mathcal{C}$ is *compactly generated* if admits (small) colimits and every object $X \in \mathcal{C}$ is a filtered colimit of compact objects $X_\alpha \in \mathcal{C}_0$ where $\mathcal{C}_0 \subset \mathcal{C}$ is ess. small full subcategory.

10.11. Ind-completion. If $\mathcal{C}$ is a small stable ∞-cat.

$$\mathrm{Ind}(\mathcal{C}) := \mathrm{Fun}_{\mathrm{lex}}(\mathcal{C}^{op}, \mathrm{Anim}) = \{\text{finite limit preserving}\}$$

Claim:

1. $\mathrm{Ind}\,\mathcal{C}$ is stable.
2. Yoneda $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Anim})$ factors through $\mathrm{Ind}(\mathcal{C})$, exhibiting $\mathrm{Ind}(\mathcal{C})$ as the free completion of $\mathcal{C}$ by filtered colimits.
3. Factors through $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})^{\omega} = \{\text{compact object in } \mathrm{Ind}(\mathcal{C})\}$. This functor is an *idempotent completion*. i.e. it is ff. every object in the target is a direct summand(or retract) of an object in $\mathcal{C}$.

Remark 10.12. $\mathcal{C}$ is compactly generated iff $\exists$ full subcategory $\mathcal{C}_0 \in \mathcal{C}$ (ess. small and admits finite colimits) s.t. $\mathrm{Ind}(\mathcal{C}_0) \simeq \mathcal{C}$ is an equivalence iff $\mathrm{Ind}(\mathcal{C}^{\omega}) \xrightarrow{\simeq} \mathcal{C}$ is an equivalence, where $\mathcal{C}^{\omega}$ is full subcategory of compact objects.

Proposition 10.13. $\mathcal{C}$ an algebraic category.

1. $\mathrm{Anim}(\mathcal{C})$ is compactly generated.
2. If $\mathcal{C}$ is additive, then $\mathrm{Anim}^{\mathrm{nc}}(\mathcal{C})$ is compactly generated.

Proof. 1. $\mathrm{Anim}(\mathcal{C}) \subseteq \mathrm{Fun}(F_{\mathcal{C}}^{op}, \mathrm{Anim}) =: \mathcal{D}$. There is a localzation functor

$$L : \mathcal{D} \to \mathrm{Anim}(\mathcal{C})$$

which is left adjoint to inclusion. Hence $L(\mathcal{D}^{\omega}) \subset \mathrm{Anim}(\mathcal{C})$ generates under filtered colimits under colimits and L preserves compact objects.

2. General fact: $\mathcal{A}$ is compactly generated. Then

$$\mathrm{Stab}(\mathcal{A}) \coloneqq \varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{A} \xrightarrow{\Omega} \mathcal{A} \right)$$

is compactly generated. □

Corollary 10.14. A a ring. $\mathcal{D}(A), \mathcal{D}(A)_{\geq 0}$ are compactly generated. Morever $M \in \mathcal{D}(A)$ is compact iff $M \in \mathcal{D}_{\mathrm{perf}}(A)$.

Proof. $\mathrm{Ind}(\mathcal{D}_{\mathrm{perf}}(A)) \simeq \mathcal{D}(A)$. Hence $\mathcal{D}_{\mathrm{perf}}(A) \xrightarrow{\simeq} \mathcal{D}(A)^{\omega}$. □

Corollary 10.15. X scheme. $\mathcal{F} \in \mathcal{D}(X)$ is compact. Then $\mathcal{F}$ is perfect.

Proof. Suffices to show that $\mathcal{F}_U \in \mathcal{D}(U)$ is perfect for all $U \subset X$ affine open $j : U \hookrightarrow X$. $\mathcal{F}_U \coloneqq j^*(\mathcal{F})$. j^* preserves compact objects since $\mathbb{R}j_*$ preserves (filtered) colimits. Since U affine, this implies $\mathcal{F}_U$ is perfect. □

10.3 Grothendieck prestable ∞-categories

Definition 10.16. $\mathcal{C}$ ∞-category is called *presentable* if

- *κ-compactly generated* for some regular cardinal κ. (admits colimits and every object is a κ-filtered colimits of κ-compact objects in $\mathcal{C}_0 \subseteq \mathcal{C}$, full subcat, which is ess. small, κ-small colimits.)

Definition 10.17. A prestable ∞-cat. $\mathcal{C}$ is *Grothendieck*

- $\mathcal{C}$ presentable.
- filtered colimits are left exact iff $\Omega : \mathcal{C} \to \mathcal{C}$ commtues with filtered colimits.

Remark 10.18.

$$\text{ab cat} \leftrightarrow \text{prestable } \infty$$
$$\text{Groth ab} \leftrightarrow \text{Groth prestable } \infty$$

The map from rhs to lhs is

$$\mathcal{C} \mapsto \mathcal{C}^{\heartsuit}$$

Proposition 10.19. Limits of a diagram Groth stable ∞-cat with left exact colimit preserving functors is Groth. prestable.

Example 10.20. $\mathcal{D}(X)$ is Groth. prestable for all $X \in \mathrm{Sch}$. $\mathcal{D}(X)_{\geq 0}$.

Theorem 10.21. [Lurc, C.6.3.3] $\mathcal{C}$ a Groth. prestable ∞-cat. $\mathcal{C}$ is compactly generated iff for every nonzero $X \in \mathcal{C}$ there exists a compact object X_0 and a nonzero map $X_0 \to X$.

Example 10.22. $M \in \mathcal{D}(A)$ nonzero. There exists $x \in \pi_n(M), n \geq 0$ nonzer. This is equivalent to a nonzero map

$$A[n] \to M$$

$A[n]$ is a compact object because it is a sheaf of A.

Lemma 10.23. $X = \operatorname{Spec} A$ affine. $Z \subset \operatorname{Spec}(A)$ closed subset. $U := X \smallsetminus Z$. Then

$$\mathcal{D}_Z(X) = \{F \in \mathcal{D}(X) : \mathcal{F}|_U = 0\}$$

is compactly generated. Similarly for $\mathcal{D}_Z(X)$

Proof. Suffices to show for the connective case. Note

$$\mathcal{D}_Z(X)_{\geq 0} \simeq \operatorname{fib}\left(\mathcal{D}(X)_{\geq 0} \xrightarrow{j_*} \mathcal{D}(U)_{\geq 0}\right)$$

hence it is a Grothendieck prestable. Note $M \in \mathcal{D}(X)_{\geq 0}$ belongs to $\mathcal{D}_Z(X)_{\geq 0}$

$$\begin{aligned}
&\Leftrightarrow j^*(M) \simeq 0 \text{in } \mathcal{D}(U) \\
&\Leftrightarrow j^*_*(\pi_n(M)) \simeq 0 \text{ in } \mathcal{D}(U)^\heartsuit,\ \forall n \geq 0 \\
&\Leftrightarrow j^*_\alpha(\pi_n(M)) \simeq 0, \text{ where } U = \bigcup_\alpha U(f_\alpha), f_\alpha \in A, j_\alpha : U(f_\alpha) \to X \\
&\Leftrightarrow \pi_n(M)[f_\alpha^{-1}] \simeq 0\ \forall \alpha \\
&\Leftrightarrow \pi_n(M) \text{ is } f_\alpha^\infty\text{-torsion} \forall \alpha, n \\
&\Leftrightarrow \exists k \text{ sufficiently large } f_\alpha^k \pi_n(M) = 0 \text{ for all } \alpha
\end{aligned}$$

Check criterion. $M \in \mathcal{D}_Z(X)_{\geq 0}$ nonzero. $\alpha \in \pi_n(M)$ nonzero. Which is a map

$$A[n] \to M$$

Assume for simplicity, the sequence has only one element f $(U = U(f))$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & A[n] \arrow[r] \arrow[d, "\alpha"] & K_f \arrow[dl, dashed, "\tilde{\alpha}"] \\ & M & \end{tikzcd}$$

The top map is a cofiber sequence. Here $\tilde{\alpha}$ exists iff $\alpha \circ f$ is null homotopic. $(f^k \circ \alpha = 0)$.

Note. $K_{f^k} \in \mathcal{D}(A)_{\geq 0}$ is also supported on $V(f)$. This implies $K_{f^k} \in \mathcal{D}_{\text{perf},Z}(X)$.

In general, choose $k >> 0$. This implies

$$K_{(f_\alpha^k)_\alpha}[n] \to M$$

is nonzero map. "Koszul complex on $(f_\alpha^k)_\alpha$." ∎

10.4 Compact generation of $\mathcal{D}(X)$

Theorem 10.24 (A)**.** X qcqs scheme. The inclusion $\mathcal{D}_{\text{perf}}(X) \hookrightarrow X$ induces an inequivalences

$$\text{Ind}(\mathcal{D}_{\text{perf}}(X)) \xrightarrow{\simeq} X$$

In particular, $\mathcal{D}(X)$ is compactly generated and the compact objects are the pefect complexes.

- Variant: $Z \subseteq X$ closed subset with qc open complement $X \smallsetminus Z$,

$$\text{Ind}(\mathcal{D}_{\text{perf},Z}(X)) \xrightarrow{\simeq} \mathcal{D}_Z(X)$$

Theorem 10.25 (B)**.** X qcqs scheme $U \subset X$ qc open

1. Every $\mathcal{F} \in \mathcal{D}_{\text{perf}}(U)$ is a direct summand of some $j^*\mathcal{G}$ where $\mathcal{G} \in \mathcal{D}_{\text{perf}}(X)$. [9]
2. Moreover the essential image of j^* is closed under extensions: for all

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

an exact triangle in $\mathcal{D}_{\text{perf}}(U)$, where

$$\mathcal{F} \simeq j^*\mathcal{F}_X, \mathcal{F}^{''} \simeq j^*\mathcal{F}_X^{''}$$

with $\mathcal{F}'_X, \mathcal{F}''_X \in \mathcal{D}_{\text{perf}}(X)$, then also $\mathcal{F}' \simeq j^*(\mathcal{F}'_X), \quad \mathcal{F}'_X \in \mathcal{D}_{\text{perf}}(X)$.

- Variant: $Z \subseteq X$ clsoed, $X \smallsetminus Z \subseteq X$ qc complement. If $\mathcal{F}$ supported on $Z \cap U$, then $\mathcal{F}_X$ is supported on U.

Lemma 10.26. If Theorem A holds for a scheme X, then theorem B. holds for X and arbitrary $U \subseteq X$.

Proof. Proof of 1. $\mathcal{F} \in \mathcal{D}_{\text{perf}}(U)$. $\mathbb{R}j_*\mathcal{F} \in \mathcal{D}(X)$. Theorem A impies

$$j_*\mathcal{F} \simeq \varinjlim_{\alpha} G_\alpha, \quad \mathcal{G}_\alpha \in \mathcal{D}_{\text{perf}}(X)$$

Apply j^* then

$$j^*\mathbb{R}j_*(\mathcal{F}) \simeq \varinjlim_{\alpha} j^*(\mathcal{G})$$

Recall $\mathcal{F} \simeq j^*\mathbb{R}_j(\mathcal{F})$, so $\mathcal{F} \simeq \varinjlim_\alpha j^*(\mathcal{G}_\alpha)$.

$$\begin{aligned}\pi_0 \operatorname{Map}(\mathcal{F}, \mathcal{F}) &\simeq \pi_0(\mathcal{F}, \varinjlim_{\alpha} j^*\mathcal{G}_\alpha) \\ &\simeq \varinjlim \pi_0 \operatorname{Map}(\mathcal{F}, j^*(\mathcal{G}\alpha)\end{aligned}$$

This implies $\text{id}_{\mathcal{F}}$ factors through $j^*\mathcal{G}_\alpha$. This shows that $\mathcal{F}$ is a direct summand of $j^*(\mathcal{G}_\alpha)$.
Proof of 2. $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ in $\mathcal{D}_{\text{perf}}(U)$. This implies we have exact sequence in $\mathcal{D}(U)$,

$$\mathbb{R}j_*(\mathcal{F}') \to \mathcal{R}j_*\mathcal{F} \to \mathcal{R}j_*(F'')$$

Claim: $\mathbb{R}j_*(\mathcal{F}'')$ is a filtered colimit of $\mathcal{F}''_\alpha$ such that $j^*(\mathcal{F}''_\alpha) \simeq \mathcal{F}''$.

[9] $j^* : \text{Perf}(X) \to \text{Perf}(U)$

Indeed, let C be fiber of unit map

$$\mathcal{C} = \text{fiber} \to \mathcal{F}_X'' \to \mathbb{R}j_*j^*(\mathcal{F}_X'') \simeq \mathbb{R}j_*(\mathcal{F}'')$$

Note $\mathcal{C} \in \mathcal{D}_{X\smallsetminus U}(X)$ supported away from U. Hence, theorem A shows

$$\mathcal{C} \simeq \varinjlim_{\alpha} C_\alpha$$

$C_\alpha \in \mathcal{D}_{\text{perf},X\smallsetminus U}(X)$.

$$F_\alpha'' := \text{cofib}(\mathcal{C}_\alpha \to \mathcal{C} \to \mathcal{F}_X'') \in \mathcal{D}_{\text{perf}}(X)$$
$$j^*\mathcal{F}_\alpha'' \simeq \text{cofib}(0 \to \mathcal{F}'') \simeq \mathcal{F}''$$
$$\varinjlim_{\alpha} \mathcal{F}_\alpha'' \simeq \text{cofib}(\mathcal{C} \to \mathcal{F}_\alpha'') \simeq \mathbb{R}j_*(\mathcal{F}'')$$

Return to proof.

$$\mathcal{F}_X \xrightarrow{unit} \mathbb{R}j_*j^*(\mathcal{F}_X) \simeq \mathbb{R}j_*(\mathcal{F}) \to \mathbb{R}j_*(\mathcal{F}'') \simeq \varinjlim_{\alpha} \mathcal{F}_\alpha''$$

this factors through some map

$$f_\alpha : \mathcal{F}_X \to \mathcal{F}_\alpha''$$

since $\mathcal{F}_X$ is compact. Finally let $\mathcal{F}_X' := \text{fib}(f_\alpha : \mathcal{F}_X \to \mathcal{F}_\alpha'')$.

$$j^*(\mathcal{F}_\alpha') \simeq \text{fib}(\mathcal{F} \to \mathcal{F}'') \simeq \mathcal{F}'$$

□

Proof. Of Theorem A. $\bigcup V_i = X$, affine open cover of X. $U_i = \bigcup_{j=1}^{i} V_j$.

$$\varnothing = U_0 \subset U_1 \subset \cdots \subset U_n = X$$

Induction on n. Let $\mathcal{F} \in \mathcal{D}(X), \mathcal{F}_i := \mathcal{F}|_{U_i}$.

Claim: $\exists \beta_i : \mathcal{G}_i \to \mathcal{F}_i$ in $\mathcal{D}(U_i)_{\geq 0}$ with $\mathcal{G}_i \in \mathcal{D}_{\text{perf}}(U_i)$.

$U_1 = V_1$ is affine. Exists $\beta_1 : \mathcal{G}_1 \to \mathcal{F}_1$. $U_i = U_{i-1} \cup V_i$, V_i affine. From, $W := U_{i-1} \cap V$,

$$\begin{array}{ccc} W & \subset & V_i \\ \subset & & \subset \\ U_{i-1} & \subset & U_i \end{array}$$

we obtain cartesian square

$$\begin{array}{ccc} \mathcal{D}(U_i) & \longrightarrow & \mathcal{D}(U_{i-1}) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathcal{D}(U_i) & \longrightarrow & \mathcal{D}(W) \end{array}$$

$W \subset V_i$ qc open. $\beta_{i-1}|_W, \mathcal{G}_{i-1}|_W \to \mathcal{F}_{i-1}|_W$.

Apply Theorem B (have this by lemma + affine case) to assume that $\mathcal{G}_{i-1}|_W$ lifts to $\mathcal{H} \in \mathcal{D}_{\text{perf}}(V_i)$. (possibly adding a summand to $\mathcal{G}_{i-1}$),

Similarly lift, $v_0 := \beta_{i-1}|_W : \mathcal{H}_{U\cap V} \to \mathcal{F}|_{U\cap V}$ to $v : \mathcal{H} \to \mathcal{F}|_V$ in $\mathcal{D}_{\text{perf}}(V)$.

Finally apply descennt to glue $j : \mathcal{H} \to \mathcal{F}|_{V_i}$ on V_i and $\beta_{i-1} : \mathcal{G}_{i-1} \to \mathcal{F}|_{U_{i-1}}$. □

11 Waldhausen K-Theory

11.1 Waldhausen's $S_\bullet$ construction

Definition 11.1. A *Waldhausen ∞-category* is an ∞-category $\mathcal{C}$ with zero object with a class of *cofibrations*.

1. The class of cofibrations cotains all isos and is closed under composition.
2. for all $X \in \mathcal{C}$, $0 \to X$ is a cofibration.
3. Cofibrations are closed under cobase change along any morphism.

Example 11.2. If $\mathcal{C}$ is (pre)stable then there is a canonical Waldhausen structure where all maps are cofibrations. [10]

11.3. $S_\bullet$ construction. $\mathcal{C}$ Waldhausen ∞-cat.

$$I_n := \{(i,j) \in [n] \times [n] : i \le j\}$$

$$\text{Gap}_{[n]}(\mathcal{C}) := \infty \text{ -cat diagrams } X : I_n \to \mathcal{C} \text{ satisfying:}$$

1. $X_{i,i}$ is zero object for all i.
2. for all $i \le j \le k$, $X_{i,j} \to X_{i,k}$ is cofibration

$$\begin{array}{ccc} X_{i,j} & \longrightarrow & X_{i,k} \\ \downarrow & & \downarrow \\ X_{j,j} & \longrightarrow & X_{j,k} \end{array}$$

   is cocartesian.

$$S_n(\mathcal{C}) := \left(\text{Gap}_{[n]}(\mathcal{C})\right)^{\simeq} \in \text{Anim}$$

as n varies, we obtain

$$S_\bullet(\mathcal{C}) : \Delta^{op} \to \text{Anim}$$

Definition 11.4. $K(\mathcal{C}) := \Omega|S_\bullet(\mathcal{C})| \in \text{Anim}$

Example 11.5. $X \in \text{Sch}$. $K(X) := K(\mathcal{D}_{\text{perf}}(X))$. $(K(\mathcal{D}_{\text{perf}}(X))_{\ge 0} \simeq K(X))$.

$$K_Z(X) := K(\mathcal{D}_{\text{perf},Z}(X))$$

[10] This is similar to the Waldhausen structure defined in the classical case of perfect complexes.

11.2 The fibration theorem

Theorem 11.6. (Waldhausen fibration theorem; Barwick). Given a functor $i : \mathcal{C} \to \mathcal{D}$ of compactly generated stable ∞-cat. $L : \mathcal{C} \to \mathcal{D}$ functor of compactly generated stable ∞-category admitting a ff right adjoint $i : \mathcal{D} \to \mathcal{C}$. Assume that L preserves compact objects (iff i preserves colimtis)

$$\mathcal{C}_0 := \ker(\mathcal{C}^\omega \xrightarrow{L^\omega} \mathcal{D}^\omega)$$

induced functors on comapct objects. There exists a fiber sequence of anima

$$K(\mathcal{C}_0) \to K(\mathcal{C}^\omega) \to K(\mathcal{D}^\omega)$$

Remark 11.7. There is an extension of this by a note of Marc Hoyois. *K-theory of dualizable categories.*

11.3 Localization and descent theorems

Theorem 11.8. X be qcqs. $U \subseteq X$ qc open. $j : U \hookrightarrow X$. We have a fiber sequence in Anim

$$K_Z(X) \to K(X) \xrightarrow{j^*} K(U)$$

where the first map is inclusion of the perfect objects.

Proof. Consider

$$\mathcal{D}(X) \underset{\mathbb{R}j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{D}(U)$$

Also, $\mathbb{R}j_*$ preserves colimits.

$$\ker(j^* : \mathcal{D}_{\text{perf}}(X) \to \mathcal{D}_{\text{perf}}(U)) =: \mathcal{D}_{\text{perf},Z}(U)$$

compactly generated by Theorem A. □

Remark 11.9.

$$\mathcal{D}_{\text{perf}}(X)/\mathcal{D}_{\text{perf},Z}(X) \to \mathcal{D}_{\text{perf}}(U)$$

is an idempotent completion. That is, fully faihtful and ess. surj. up to direct summands.

Remark 11.10. This is in fact a fiber sequence of connective spectra.

Theorem 11.11. (Zariski descent).

- The presheaf of anima

  $$K : \text{Sch}_{qcqs}^{op} \to \text{Anim}$$

  satisfies Zariski descent.
- In particular: for every $X \in \text{Sch}_{qcqs}, K : (X_{\text{Zar}})^{op} \to \text{Anim}$ also satisfies descent.
- Mayer Vietoris. $X = U \cup V$ a open covering of qc open then

$$\begin{array}{ccc} K(X) & \longrightarrow & K(U) \\ \downarrow & \lrcorner & \downarrow \\ K(V) & \longrightarrow & K(U \cap V) \end{array}$$

Proof. Suffices to show Mayer-Vietoris claim. $Z = X \setminus U$, $Z' = V \setminus U \cap V$. We have fiber sequence

$$\begin{array}{ccccc} K_Z(X) & \longrightarrow & K(X) & \longrightarrow & K(U) \\ \downarrow & & & & \downarrow \\ K_{Z'}(V) & \longrightarrow & K(V) & \longrightarrow & K(U \cap V) \end{array}$$

The square is cartesian if and only if it induces isomorphisms on homotopy fibers. The horizontal rows are fiber sequences by localization theorem.

The map

$$K_Z(X) \to K_{Z'}(V)$$

is induced by $\mathcal{D}_{\mathrm{perf},Z}(X) \to \mathcal{D}_{\mathrm{perf},Z'}(V)$ Zariski descent on $\mathcal{D}_{\mathrm{perf}}$ implies we have cartesian square on bottom right

$$\begin{array}{ccccc} \mathcal{D}_{\mathrm{perf},Z}(X) & \longrightarrow & \mathcal{D}_{\mathrm{perf}}(X) & \longrightarrow & \mathcal{D}_{\mathrm{perf}}(U) \\ \downarrow \simeq & & \downarrow & \lrcorner & \downarrow \\ \mathcal{D}_{\mathrm{perf},Z'}(X) & \longrightarrow & \mathcal{D}_{\mathrm{perf}}(V) & \longrightarrow & \mathcal{D}_{\mathrm{perf}}(U \cap V) \end{array}$$

the left most induced map is an equivalence (formal consequence[11]). In particular the induced map on K theory is an equivalence. □

Remark 11.12. Étale descent is true in rational K-theory. Integrally, this is the content of Bloch Kato conjecture.

[11]Harry. 3-3 square + interchangeable limits.

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