7.4 : Compact generation of $D(X)$

Theorem A: X qcqs scheme

- The inclusion $D_{perf}(X) \hookrightarrow D(X)$ induces an equivalence

$$\mathrm{Ind}(D_{perf}(X)) \xrightarrow{\sim} D(X)$$

- In particular: $D(X)$ is compactly generated and the compact objects are the perfect complexes.

- Variant: $Z \subseteq X$ closed subset with qc open complement $X \setminus Z$
$$\mathrm{Ind}(D_{perf}(X \text{ on } Z)) \xrightarrow{\sim} D(X \text{ on } Z).$$

Theorem B: X qcqs scheme $U \subseteq X$ qc open

- Every $\mathcal{F} \in D_{perf}(U)$ is a direct summand of $j^*(\mathcal{G})$ where $\mathcal{F}_X \in D_{perf}(X)$

$j^*: \mathrm{Perf}(X) \to \mathrm{Perf}(U)$

- Moreover, the essential image of j^* is closed under fibres:

$\forall\ \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ exact triangle in $D_{perf}(U)$

where $\mathcal{F} \simeq j^*(\mathcal{F}_X)$, $\mathcal{F}'' \simeq j^*(\mathcal{F}''_X)$ $\qquad \mathcal{F}_X, \mathcal{F}''_X \in D_{perf}(X)$

$\Rightarrow$ also $\mathcal{F}' \simeq j^*(\mathcal{F}'_X)$ $\qquad \mathcal{F}'_X \in D_{perf}(X)$

- Variant: $Z \subseteq X$ closed, $X \setminus Z \subseteq X$ qc complement
if $\mathcal{F}$ supported on $Z \cap U$ then $\mathcal{F}_X$ supported on U

Lemma: If Theorem A holds for a given X then Theorem B holds for X (and arbitrary $U \subseteq X$).

Proof.

Claim 1 $\quad \mathcal{F} \in D_{perf}(U) \qquad \mathbb{R}j_*(\mathcal{F}) \in D(X)$

$$\text{Thm A} \Rightarrow \mathbb{R}j_*(\mathcal{F}) \simeq \varinjlim_{\alpha} \mathcal{G}_\alpha \qquad \mathcal{G}_\alpha \in D_{perf}(X)$$

$$\text{Apply } j^* \longrightarrow \quad j^*\mathbb{R}j_*(\mathcal{F}) \simeq \varinjlim_{\alpha} j^*(\mathcal{G}_\alpha)$$

Recall: $\quad \mathcal{F} \simeq j^*\mathbb{R}j_*(\mathcal{F})$

$$\Rightarrow \quad \mathcal{F} \simeq \varinjlim_{\alpha} j^*(\mathcal{G}_\alpha)$$

$$\pi_0 \operatorname{Maps}(\mathcal{F}, \mathcal{F}) \simeq \pi_0 \operatorname{Maps}(\mathcal{F}, \varinjlim_{\alpha} j^*\mathcal{G}_\alpha)$$

$$\simeq \varinjlim_{\alpha} \pi_0 \operatorname{Maps}(\mathcal{F}, j^*(\mathcal{G}_\alpha)) \qquad \text{(compact)}$$

$\Rightarrow$ $id_{\mathcal{F}}$ factors through $j^*(\mathcal{G}_\alpha)$ for some α

$\Rightarrow$ $\mathcal{F}$ is a direct summand of $j^*(\mathcal{G}_\alpha)$

Claim 2: $\quad \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ in $D_{perf}(U)$

$\Rightarrow \mathbb{R}j_*(\mathcal{F}') \to \mathbb{R}j_*(\mathcal{F}) \to \mathbb{R}j_*(\mathcal{F}'')$ exact in $D(X)$

Claim: $\mathbb{R}j_*(\mathcal{F}'')$ is a filtered colimit of $\mathcal{F}''_\alpha \in D_{perf}(X)$ such that $j^*(\mathcal{F}''_\alpha) \simeq \mathcal{F}''$.

$$\mathcal{F} \simeq j^*(\mathcal{F}_X), \quad \mathcal{F}'' \simeq j^*(\mathcal{F}''_X)$$

$$\mathcal{C} := \mathrm{Fib}\left(\mathrm{unit} : \mathcal{F}''_X \to \mathbb{R}j_* j^*(\mathcal{F}''_X)\right)$$

$$\mathbb{R}j_* j^*(\mathcal{F}''_X) \simeq \mathbb{R}j_*(\mathcal{F}'')$$

Note: $\mathcal{C} \in D(X \text{ on } X \setminus U)$ supported away from U

$\boxed{\text{Thm A}} \Rightarrow \mathcal{C} \simeq \varinjlim_\alpha \mathcal{C}_\alpha \qquad \mathcal{C}_\alpha \in D_{perf}(X \text{ on } X\setminus U)$

$$\mathcal{F}''_\alpha := \mathrm{Cofib}(\mathcal{C}_\alpha \to \mathcal{C} \to \mathcal{F}''_X) \in D_{perf}(X)$$

$$j^*(\mathcal{F}''_\alpha) \simeq \mathrm{Cofib}(0 \to \mathcal{F}'') \simeq \mathcal{F}''$$

$$\varinjlim_\alpha \mathcal{F}''_\alpha \simeq \mathrm{Cofib}(\mathcal{C} \to \mathcal{F}''_X) \simeq \mathbb{R}j_*(\mathcal{F}'')$$

$$\mathcal{F}_X \xrightarrow{\mathrm{unit}} \mathbb{R}j_* j^*(\mathcal{F}_X) \simeq \mathbb{R}j_*(\mathcal{F}) \longrightarrow \mathbb{R}j_*(\mathcal{F}'') \simeq \varinjlim_\alpha \mathcal{F}''_\alpha$$

factors through $f_\alpha : \mathcal{F}_X \to \mathcal{F}''_\alpha$
(since $\mathcal{F}_X$ compact)

Finally let $\mathcal{F}'_X := \mathrm{Fib}(f_\alpha : \mathcal{F}_X \to \mathcal{F}''_\alpha)$

$$j^*(\mathcal{F}'_X) \simeq \mathrm{Fib}(\mathcal{F} \to \mathcal{F}'') \simeq \mathcal{F}'. \qquad \blacksquare$$

Proof of Thm A:

$\bigcup_i V_i = X$ affine open cover of X

$U_i := \bigcup_{j=1}^{i} V_j$ $\qquad \emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$

Induction on n. $\quad \mathcal{F} \in D(X)$, $\mathcal{F}_i := \mathcal{F}|_{U_i}$ nonzero

Claim: $\exists\, \beta_i : \mathcal{G}_i \to \mathcal{F}_i$ in $D(U_i)$ with $\mathcal{G}_i \in D_{perf}(U_i)$

$U_1 = V_1$ is affine $\Rightarrow$ have $\beta_1 : \mathcal{G}_1 \to \mathcal{F}_1$

$U_i = U_{i-1} \cup V_i \qquad V_i$ affine

$$\begin{array}{ccc} W & \subseteq & V_i \\ \cap & & \cap \\ U_{i-1} & \subseteq & U_i \end{array} \qquad W := U_{i-1} \cap V_i$$

$$\boxed{\begin{tikzcd} D(U_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D(U_{i-1}) \arrow[d] \\ D(V_i) \arrow[r] & D(W) \end{tikzcd}}$$

$W \subseteq V_i$ qc open

$\beta_{i-1}|_W \quad \mathcal{G}_{i-1}|_W \to \mathcal{F}_{i-1}|_W$

Apply Thm B (have this by Lemma + affine case)
to assume that $\mathcal{G}_{i-1}|_W$ lifts to $\mathcal{H} \in D_{perf}(V_i)$
(possibly adding a summand to $\mathcal{G}_{i-1}$).

Similarly lift $\quad v_0 := \beta_{i-1}|_W : \mathcal{H}/_{U \cap V_i} \to \mathcal{F}/_{U \cap V_i}$
to $v : \mathcal{H} \to \mathcal{F}|_{V_i}$ in $D_{perf}(V_i)$

Finally apply descent to glue $v : \mathcal{H} \to \mathcal{F}|_{V_i}$ on V_i
and $\beta_{i-1} : \mathcal{G}_{i-1} \to \mathcal{F}|_{U_{i-1}}$ on U_{i-1}

$\rightsquigarrow$ glue to $\beta_i : \mathcal{G}_i \to \mathcal{F}|_{U_i}$. $\blacksquare$

Lecture 8: Waldhausen K-theory

8.1: Waldhausen's $S_\bullet$-construction

Def: Waldhausen ∞-category is an ∞-category $\mathcal{C}$ with zero object, with a class of cofibrations

1) The class of cofibrations contains all isos, and is closed under composition.

2) $\forall X \in \mathcal{C}$ $\quad 0 \to X$ is a cofibration

3) Cofibrations are closed under co-base change along any morphism

Ex: If $\mathcal{C}$ (pre)stable then there is a canonical Waldhausen structure where all maps are cofibrations.

Constr. ($S_\bullet$-Construction): $\mathcal{C}$ Waldhausen ∞-cat

$\mathrm{Gap}_{[n]}(\mathcal{C}) := \infty$-cat of diagrams $X\ \ I_n \to \mathcal{C}$

$$I_n := \{(i,j) \in [n]\times[n] \mid i \leq j\}$$

Satisfying. 1) $X_{i,i}$ zero obj $\forall i$

2) $\forall\ i \leq j \leq k$ $\quad X_{i,j} \to X_{i,k}$ cofibration

$$\begin{tikzcd} X_{i,j} \arrow[r] \arrow[d] & X_{j,j} \simeq 0 \arrow[d] \\ X_{i,k} \arrow[r] & X_{j,k} \end{tikzcd}$$

cocartesian square (cofibre seq.)

underlying ∞-groupoid

$$S_n(\mathcal{C}) := \left(\mathrm{Gap}_{[n]}(\mathcal{C})\right)^{\simeq} \in \infty\text{-}\mathrm{Grpd} \simeq \mathrm{Anim}$$

As n varies $\Rightarrow$ $S_\bullet(\mathcal{C}) : \Delta^{op} \to \mathrm{Anim}$ simplicial diagram

Def: $$K(\mathcal{C}) := \Omega \underbrace{|S_\bullet(\mathcal{C})|}_{\text{geometric realization}} \in \mathrm{Anim}$$

Ex: X scheme $K(X) := K(\mathrm{Perf}(X))$

$$(K(\mathrm{Perf}(X)_{\geq 0}) \simeq K(X))$$

$$K(X \text{ on } Z) := K(\mathrm{Perf}(X \text{ on } Z))$$

§2. The Fibration Theorem

Theorem (Waldhausen fibration theorem; Barwick)

$L : \mathcal{C} \to \mathcal{D}$ functor of compactly generated stable ∞-categories admitting a fully faithful right adjoint $i : \mathcal{D} \hookrightarrow \mathcal{C}$

Assume that L preserves compact objects ($\Leftrightarrow$ i preserves colimits)

$\mathcal{C}_0 := \mathrm{Ker}(\mathcal{C}^\omega \xrightarrow{L^\omega} \mathcal{D}^\omega)$ (induced functor on compact objs.)

There exists a fibre sequence $K(\mathcal{C}_0) \to K(\mathcal{C}^\omega) \to K(\mathcal{D}^\omega)$ in the ∞-cat of Anim.

Hoyois: K-theory of dualizable categories

8.3. Localization and descent theorems

Theorem: X qcqs scheme $U \subseteq X$ qc open
$j : U \hookrightarrow X$

Then. $K(X \text{ on } Z) \to K(X) \xrightarrow{j^*} K(U)$ is a fibre sequence of anima.

Proof: $j^* : D(X) \rightleftarrows D(U)$ with right adjoint $\mathbb{R}j_*$, $\mathbb{R}j_*$ preserves colimits

$$\ker(j^* : \mathrm{Perf}(X) \to \mathrm{Perf}(U)) =: \mathrm{Perf}(X \text{ on } Z)$$

cpt gen. by Thm A ■

Γ $\mathrm{Perf}(X)/\mathrm{Perf}(X \text{ on } Z) \longrightarrow \mathrm{Perf}(U)$ idempotent compl.

That is: fully faithful and ess. surj. up to direct summands.

Theorem: (Zariski descent):

- The presheaf of anima $(\mathrm{Sch}_{qcqs})^{op} \xrightarrow{K} \mathrm{Anim}$ satisfies Zariski descent

- In particular: for every $X \in \mathrm{Sch}_{qcqs}$

$$K : (X_{zar})^{op} \to \mathrm{Anim} \quad \text{also satisfies descent.}$$

- [Mayer-Vietoris] $X = U \cup V$ open covering by qc opens

$$\begin{tikzcd} K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & K(U) \arrow[d] \\ K(V) \arrow[r] & K(U \cap V) \end{tikzcd}$$

cartesian square of anima

Proof: Suffices to show Mayer-Vietoris claim.

$$\begin{tikzcd} K(X \text{ on } Z) \arrow[r] \arrow[d, "\circledast"'] & K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & K(U) \arrow[d] \\ K(V \text{ on } Z') \arrow[r] & K(V) \arrow[r] & K(U \cap V) \end{tikzcd}$$

$Z = X \setminus U$ (as a subset)

$Z' = V \setminus U \cap V$

Note: The square is cartesian if and only if it induces an isomorphism on homotopy fibres.

The horizontal rows are fibre sequences by localization theorem

But $K(X \text{ on } Z) \longrightarrow K(V \text{ on } Z')$ is induced by

$$D_{perf}(X \text{ on } Z) \longrightarrow D_{perf}(V \text{ on } Z')$$

Zariski descent for D_{perf}:

$$\begin{tikzcd} D_{perf}(X \text{ on } Z) \arrow[r] \arrow[d, "\wr"] & D_{perf}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D_{perf}(U) \arrow[d] \\ D_{perf}(V \text{ on } Z') \arrow[r] & D_{perf}(V) \arrow[r] & D_{perf}(U \cap V) \end{tikzcd}$$

Cartesian

$$\Longrightarrow K(X \text{ on } Z) \xrightarrow{\sim} K(V \text{ on } Z') \text{ iso} \qquad \blacksquare$$