Lecture 7. Perfect complexes

7.1 Perfect complexes

Def: $\mathcal{C}$ stable ∞-category $\mathcal{C}_0 \subseteq \mathcal{C}$ full subcategory

$\mathcal{C}_0$ is stable if it contains the zero object $0 \in \mathcal{C}$ and is closed under co/fibres in $\mathcal{C}$.

$\mathcal{C}_0$ is thick if it is stable and moreover closed under direct summands (= retracts) in $\mathcal{C}$.

Def: A commutative ring, $M \in D(A)$ is perfect if it is contained in the thick subcategory generated by the object $A \in D(A)$.

$$D_{perf}(A) \subseteq D(A) \qquad \text{full subcat.}$$

Ex: $f \in A$ $\qquad \mathrm{Cofib}(A \xrightarrow{f} A) \in D_{perf}(A)_{\geq 0}$
(Koszul "complex" on the element f)

$f_1, \dots, f_n \in A \qquad K_{f_1,\dots,f_n} := \bigotimes_i^L \mathrm{Cofib}(A \xrightarrow{f_i} A) \in D_{perf}(A)$

Non-example. $A = k[x]/(x^2)$ $\qquad k \in D(A)$ is not perfect

Def: X scheme $\qquad \mathcal{F} \in D(X)$ is perfect if
$\forall\ U = \mathrm{Spec}(A) \subseteq X \qquad \mathcal{F}_U \in D(U) \simeq D(A)$ is perfect

Corollary: The presheaf of ∞-categories
$(\mathrm{Sch})^{op} \to \infty\text{-Cat}$
$D_{perf} : X \longmapsto D_{perf}(X)$ satisfies Zar descent

Proof: $D_{perf} \subseteq D$ subpresheaf

Since D is a sheaf, D_{perf} is a sheaf as long as it is defined by a "local" property ∎

Theorem: X quasi-compact quasi-separated scheme
Then every quasi-coherent complex $\mathcal{F} \in D(X)$ can be written as a filtered colimit of perfect cpx's.

$$\mathcal{F} \simeq \varinjlim_{\alpha} \mathcal{F}_\alpha \qquad \mathcal{F}_\alpha \in D_{perf}(X)$$

Variant 1: if $\mathcal{F} \in D(X)_{\geq 0}$ then $\mathcal{F}_\alpha$ can be taken connective.

Variant 2: if $\mathcal{F}$ is supported on a closed subset $Z \subseteq X$ (with complement $X \setminus Z$ quasi-compact) then also $\mathcal{F}_\alpha$ can be taken supported on Z

($\mathcal{F}$ supported on Z $\Leftrightarrow$ $j^*(\mathcal{F}) \simeq 0$ $\quad j: X \setminus Z \hookrightarrow X$)
$D(X \text{ on } Z) \subseteq D(X)$
$D_{perf}(X \text{ on } Z) \subseteq D_{perf}(X)$

7.2: Compactly generated ∞-categories

Def: $\mathcal{C}$ ∞-category with filtered colimits
An object $X \in \mathcal{C}$ is compact if
$$\mathrm{Maps}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathrm{Anim}$$
commutes w/ filtered colims

<u>Rmk</u>: $\mathcal{C}$ stable ∞-category
$\mathcal{C}_0 \subseteq \mathcal{C}$ full subcategory of compact objects is <u>thick</u>

(• filtered colims commute with finite limits in Anim
• a retract of an isomorphism is an isomorphism)

<u>Ex</u>: A ring. Every $M \in D(A)$ perfect is a compact object.

<u>Proof</u>: Suffices to show for $A \in D(A)$ since A generates $D_{perf}(A)$ as a thick subcategory.

$\Leftrightarrow$ $\mathrm{Maps}_{D(A)}(A, -) \simeq (-)^\circ : D(A) \to \mathrm{Anim}$ commutes w/ filtered colimits.

<u>Def</u>: $\mathcal{C}$ ∞-category
$\mathcal{C}$ is <u>compactly generated</u> if it admits (small) colimits and every object $X \in \mathcal{C}$ is a filtered colimit of compact objects $X_\alpha \in \mathcal{C}_0$ where $\mathcal{C}_0 \subseteq \mathcal{C}$ ess. small full subcategory.

<u>Constr</u> (Ind-completion): $\mathcal{C}$ small stable ∞-category

$\mathrm{Ind}(\mathcal{C}) := \mathrm{Fun}_{lex}(\mathcal{C}^{op}, \mathrm{Anim}) = \{$<u>left-exact</u> functors$\}$
finite-limit-preserving

<u>Claims</u>: 1) $\mathrm{Ind}(\mathcal{C})$ stable.
2) Yoneda $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Anim})$ factors through $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ which exhibits $\mathrm{Ind}(\mathcal{C})$

as the free completion of $\mathcal{C}$ by filtered colimits.

3) Factors through $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})^{\omega} = \{$compact objects in $\mathrm{Ind}(\mathcal{C})\}$

and this functor is an idempotent completion (every object in the target is a direct summand of an object in $\mathcal{C}$).

Rmk: $\mathcal{C}$ is compactly generated

$\Leftrightarrow$ $\exists$ full subcat $\mathcal{C}_0 \subseteq \mathcal{C}$ (ess. small admitting finite colimits)

s.th $\mathrm{Ind}(\mathcal{C}_0) \xrightarrow{\sim} \mathcal{C}$ is an equivalence

$\Leftrightarrow$ $\mathrm{Ind}(\mathcal{C}^{\omega}) \xrightarrow{\sim} \mathcal{C}$ is an equivalence

$\mathcal{C}^{\omega}$ = full subcat of compact objs

Prop: $\mathcal{C}$ algebraic category

1) $\mathrm{Anim}(\mathcal{C})$ is compactly generated

2) If $\mathcal{C}$ additive, then $\mathrm{Anim}^{ac}(\mathcal{C})$ is compactly gen

Proof: 1) $\mathrm{Anim}(\mathcal{C}) \subseteq \mathrm{Fun}(F_{\mathcal{C}}^{op}, \mathrm{Anim}) =: \mathcal{D}$

$L : \mathcal{D} \to \mathrm{Anim}(\mathcal{C})$ localization

preserves filtered colimits

$L(\mathcal{D}^{\omega}) \subseteq \mathrm{Anim}(\mathcal{C})$ generates under filt. colims.

and L preserves compact objects

0) General fact: $\mathcal{A}$ compactly generated w/ zero obj.

$$\Rightarrow \quad \mathrm{Stab}(\mathcal{A}) := \left(\varprojlim \cdots \xrightarrow{\Omega} \mathcal{A} \xrightarrow{\Omega} \mathcal{A} \right)$$

is compactly generated.

<u>Corollary</u>: A ring $\quad D(A)$, $D(A)_{\geq 0}$ are compactly gen

Moreover $M \in D(A)$ is compact $\Leftrightarrow$ $M \in D_{perf}(A)$.

<u>Proof</u>: $\quad \mathrm{Ind}(D_{perf}(A)) \xrightarrow{\sim} D(A)$

$$\Rightarrow \quad D_{perf}(A) \xrightarrow{\sim} D(A)^{\omega}. \qquad \blacksquare$$

<u>Corollary</u>: X scheme $\quad \mathcal{F} \in D(X)$ compact
$\Rightarrow$ $\mathcal{F}$ is perfect

<u>Proof</u>: Suffices to show $\mathcal{F}_U \in D(U)$ is perfect
$\forall\, U \subseteq X$ affine open $\qquad j: U \hookrightarrow X$

$\mathcal{F}_U = j^*(\mathcal{F})$ $\qquad j^*$ preserves compact objects
since $\mathbb{R}j_*$ preserves (filtered) colimits
(§6)

U affine $\Rightarrow$ $\mathcal{F}_U$ perfect $\quad \blacksquare$

7.3 Grothendieck prestable ∞-categories

Def: $\mathcal{C}$ ∞-category is called **presentable** if

- κ-compactly generated for some regular card. κ
  (admits colimits and every obj is a κ filt. colimit of κ-compact objects $\in \mathcal{C}_0$, $\mathcal{C}_0 \subseteq \mathcal{C}$ full subcat ess small, κ small colimits)

A prestable ∞-category $\mathcal{C}$ is **Grothendieck** if

- $\mathcal{C}$ presentable
- filtered colimits are left-exact
  ($\Leftrightarrow$ Ω $\mathcal{C} \to \mathcal{C}$ comm. w/ filt. colims)

Rmk: ab cat $\leadsto$ prestable ∞-categories

Groth ab $\leftrightarrow$ Groth prestable

$$\mathcal{C}^{\heartsuit} \longleftarrow \mathcal{C}$$

Prop: Limits of Groth prestable ∞-categories and left-exact colim-pres functors are Groth. prestable.

Ex: $D(X)$ is Groth. prestable $\forall X \in Sch$
$D(X)_{\geq 0}$

Theorem (Lurie): $\mathcal{C}$ Groth. prestable ∞-category
$\mathcal{C}$ is compactly generated iff for every nonzero $X \in \mathcal{C}$ there exists a compact obj X_0 and a nonzero map $X_0 \to X$.

[SAG, Cor C.6.3.3]

<u>Ex</u>: $D(A) \ni M$ nonzero $\Rightarrow$ $\exists\, x \in \pi_n(M)$ $n \geq 0$ nonzero

$\Leftrightarrow$ $A[n] \to M$ nonzero map in $D(A)$
$\uparrow$
compact. ∎

<u>Lemma</u>: $X = \mathrm{Spec}(A)$ affine $Z \subseteq \mathrm{Spec}(A)$ closed subset
$U := X \setminus Z$

$D(X \text{ on } Z) = \{ \mathcal{F} \in D(X) \mid \mathcal{F}|_U = 0 \}$ is compactly generated.
$D(X \text{ on } Z)_{\geq 0}$.

<u>Proof</u>: Suff to show for $D(X \text{ on } Z)_{\geq 0}$

Note $D(X \text{ on } Z)_{\geq 0} = \mathrm{Fib}(D(X)_{\geq 0} \xrightarrow{j^*} D(U)_{\geq 0})$

$\Rightarrow$ Grothendieck prestable.

Note: $M \in D(X)_{\geq 0}$ belongs to $D(X \text{ on } Z)_{\geq 0}$

$\Leftrightarrow$ $j^*(M) \simeq 0$ in $D(U)$

$\Leftrightarrow$ $j^*(\pi_n(M)) \simeq 0$ in $D(U)^\heartsuit$ $\forall n \geq 0$

$\Leftrightarrow$ $j_\alpha^*(\pi_n(M)) \simeq 0$ $U = \bigcup_\alpha U(f_\alpha)$ $f_\alpha \in A$
$\quad ''$ $j_\alpha : U(f_\alpha) \to X$

$\Leftrightarrow$ $\pi_n(M)[f_\alpha^{-1}] \simeq 0$

$\Leftrightarrow$ $\pi_n(M)$ is f_α^∞-torsion $\forall \alpha, \forall n$

$\Rightarrow$ $\exists\, k \gg 0$ $\underline{f_\alpha^k \cdot \pi_n(M)} = 0$ $\forall \alpha$

Check criterion: $M \in D(X \text{ on } Z)_{\geq 0}$ nonzero

$x \in \pi_n(M)$ nonzero

Choose $k \gg 0$ as above

"Koszul complex"

$\Rightarrow$ $\boxed{K_{(f_\alpha^k)_\alpha}[n]} \xrightarrow{\tilde{x}} M$ nonzero map on $(f_\alpha^k)_\alpha$ ∎

$x : A[n] \to M$

Assume the sequence has only one element f $(U = U(f))$

$$\begin{tikzcd} A[n] \arrow[r, "f^k"] \arrow[dr, "0"'] & A[n] \arrow[r] \arrow[d, "x"] & K_{f^k}[n] \arrow[dl, dashed, "\tilde{x}"] \\ & M & \end{tikzcd} \quad \text{cofibre}$$

$\tilde{x}$ exists $\iff$ $x \circ f$ is null-homotopic

$(f^k x = 0)$

<u>Note</u>. $K_{f^k} \in D(A)_{\geq 0}$ is also supported on $V(f)$

$\Rightarrow$ $K_{f^k} \in D_{perf}(X \text{ on } Z)$.