Lecture 2: Animated modules

2.1. Algebraic categories

Def: $\mathcal{C}$ category. $\mathcal{C}$ is algebraic if there exists an essentially small full subcategory $\mathcal{F}_\mathcal{C} \subseteq \mathcal{C}$ which extends to an equivalence

which admits finite coproducts

$$\mathrm{Fun}_\pi(\mathcal{F}_\mathcal{C}^{op}, \mathrm{Set}) \longrightarrow \mathcal{C}$$

finite product-preserving functors $\mathcal{F}_\mathcal{C}^{op} \to \mathrm{Set}$

Ex: The category Set is algebraic, with $\mathcal{F}_\mathcal{C} = \mathrm{Fin}$ the category of finite sets.

$$\mathrm{Set} \simeq \mathrm{Fun}_\pi(\mathrm{Fin}^{op}, \mathrm{Set}).$$

$X \in \mathrm{Set} \rightsquigarrow \mathrm{Fin}^{op} \to \mathrm{Set}$

$Y \mapsto \mathrm{Hom}(Y, X)$

$F : \mathrm{Fin}^{op} \to \mathrm{Set} \rightsquigarrow$ sets $F_0 = F(\emptyset)$, $F_1 = F(\{1\})$, $F_2 = F(\{1,2\})$, ...

isos. $F_0 \simeq \mathrm{pt} = \{*\}$

$F_n \simeq F_1^n \quad \forall n$

$\{1, \dots, n\} \to \{1, \dots, m\}$

$\Rightarrow F_m \longrightarrow F_n$

$\quad X^m \longrightarrow X^n$ projection

$X := F_1 \in \mathrm{Set}$

Ex (abelian groups) Ab is algebraic

$F_{Ab} = \{\text{f.g. free ab. groups}\} \subseteq Ab$

F_{Ab} can be identified w/ the category:

- objects: $n \in \mathbb{N}$
- Morphisms: $\mathrm{Hom}_{F_{Ab}}(m,n) = \mathrm{Hom}_{Ab}(\mathbb{Z}^{\oplus m}, \mathbb{Z}^{\oplus n})$
$\cong \mathrm{Mat}_{n\times m}(\mathbb{Z})$
- composition $\leftrightarrow$ matrix multiplication

$Ab \simeq \mathrm{Fun}_{\Pi}(F_{Ab}^{op}, Set)$

Object of RHS $\Leftrightarrow$ $F_0, F_1, \dots, F_n, \dots \in Set$

$F_0 \simeq pt, \quad F_n \simeq (F_1)^{\times n}$

$\phi \in \mathrm{Mat}_{n\times m}(\mathbb{Z}) \Rightarrow F_\phi : F_m \to F_n$

$\rightsquigarrow$ underlying set: $G := F_1 \in Set$

- operations: maps $G^{\times n} \to G \Leftrightarrow (n\times 1)$-matrices

$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \rightsquigarrow$ operation of forming a linear combination w/ coeffs. a_i

$(x_1, \dots, x_n) \longmapsto \sum_i a_i x_i$

- addition: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow (x_1, x_2) \longmapsto x_1 + x_2$

$G \times G \to G$

- zero elt. $[\,]$ empty (0×1)-matrix $\Rightarrow pt \xrightarrow{0} G$

- additive inverse: $[-1] \Leftrightarrow G \to G$

Rmk: There is a canonical choice of $F_{\mathcal{C}}$, namely the full subcategory of compact projective objects

- $X \in \mathcal{C}$ compact $\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathrm{Set}$ preserves filtered colimits
- $X \in \mathcal{C}$ projective $\Leftrightarrow$ preserves reflexive coequalizers

Ex: In Set, $X \in \mathrm{Set}$ is cpt. proj. $\Leftrightarrow X$ finite.
In Ab, $X \in \mathrm{Ab}$ is cpt proj $\Leftrightarrow X$ f.g. free

Claim: Moreover, if $\mathcal{C}$ algebraic then $\mathrm{Fun}_{\pi}(F_{\mathcal{C}}^{op}, \mathrm{Set}) \simeq \mathcal{C}$ is the free completion of $F_{\mathcal{C}}$ by filtered colimits and reflexive coequalizers:

For every category $\mathcal{D}$ (with filt colimits + reflexive coeq), there is an equivalence

$$\mathrm{Fun}'(\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(F_{\mathcal{C}},\mathcal{D})$$
$$\parallel$$
$$\{\text{functors } \mathcal{C} \to \mathcal{D} \text{ that preserve filt colims. and refl. coeqs.}\}$$

Reminder: a reflexive pair in $\mathcal{C}$ is a diagram

$$x \underset{g}{\overset{f}{\rightrightarrows}} y, \quad s : y \to x \qquad \text{s.th.} \quad f \circ s = g \circ s$$

Reflexive coeqs. are colimits indexed by reflexive pairs. These generalize quotients by equivalence relations.

$$\left(\underset{\text{eq. relatn}}{R \subseteq X \times X} \quad \rightsquigarrow \quad R \rightrightarrows X \ \text{reflexive pair} \right)$$

2.2. Animation of categories

Def: $\mathcal{C}$ an ∞-category, $X_\bullet : \Delta^{op} \to \mathcal{C}$ simplicial diagram in $\mathcal{C}$.
The colimit of $X_\bullet$ is denoted

$$|X_\bullet| := \varinjlim_{[n]\in\Delta^{op}} X_n \qquad \cdots X_2 \rightrightarrows X_1 \rightleftarrows X_0 \to |X_\bullet|$$

and is called the geometric ~~rela~~ realization.

Def: $\mathcal{C}$ algebraic category. An animation of $\mathcal{C}$ is an ∞-category $\mathrm{Anim}(\mathcal{C})$ equipped with a fully faithful functor $F_\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ such that:

$\forall\ \mathcal{D}$ ∞-category (admitting filt. colimits + geo. realizations),

$$\underset{\{F:\ \mathrm{Anim}(\mathcal{C}) \to \mathcal{D} \text{ preserve filt. colims + geo. realizations}\}}{\overset{\shortparallel}{\mathrm{Fun}'(\mathrm{Anim}(\mathcal{C}), \mathcal{D})}} \xrightarrow{\sim} \mathrm{Fun}(F_\mathcal{C}, \mathcal{D}) \quad \text{is an equivalence.}$$

- The ∞-category of anima is an animation of the category of sets. We denote it by: Anim.

Theorem: (Quillen, Lurie)

1) The fully faithful functor $\mathrm{Fin} \hookrightarrow \mathrm{Kan}$ exhibits the ∞-category Kan as an ∞-category of anima.

Set

2) $\mathcal{C}$ algebraic category. Then the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(F_\mathcal{C}^{op}, \mathrm{Anim})$ factors through

$$\mathcal{C} \hookrightarrow \mathrm{Fun}_\Pi(F_\mathcal{C}^{op}, \mathrm{Anim}) = \{\text{finite product-preserving functors}\}$$

and exhibits the target as an animation of $\mathcal{C}$.

Def: Recall: $\mathrm{Set} \hookrightarrow \mathrm{Kan}$ ~~[illegible]~~
$$\cong$$
$$\mathrm{Anim}$$

An anima $X \in \mathrm{Anim}$ is <u>discrete</u> if it is iso. to an object in the essential image.

$\mathcal{C}$ algebraic category. An object $X \in \mathrm{Anim}(\mathcal{C})$ is <u>discrete</u> if $X : (F_\mathcal{C})^{op} \to \mathrm{Anim}$ factors through $\mathrm{Set} \hookrightarrow \mathrm{Anim}$.
$\iff$ the <u>underlying anima</u> of X is discrete.

$$\mathrm{Anim}(\mathcal{C})^{\heartsuit} = \{\text{discrete objects}\} \subseteq \mathrm{Anim}(\mathcal{C})$$

<u>Claim</u>: 1) The assignment

$$\left((F_\mathcal{C})^{op} \to \underset{\mathrm{Set}}{\cancel{\mathrm{Anim}}}\right) \longmapsto \left((F_\mathcal{C})^{op} \to \mathrm{Set} \hookrightarrow \mathrm{Anim}\right)$$

defines a fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ with ess. image $\mathrm{Anim}(\mathcal{C})^{\heartsuit}$ $(\mathcal{C} \simeq \mathrm{Anim}(\mathcal{C})^{\heartsuit})$.

2) The functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ admits a left adjoint $\pi_0 : \mathrm{Anim}(\mathcal{C}) \to \mathcal{C}$, given by composition with $\pi_0 : \mathrm{Anim} \to \mathrm{Set}$ $(\mathrm{Kan} \xrightarrow{\pi_0} \mathrm{Set}$ connected component functor$)$.

<u>Constr</u> (Underlying anima): $X \in \mathrm{Anim}(\mathcal{C})$

$$X^\circ := X(1) \in \mathrm{Anim} \qquad X(1) = \mathrm{Maps}_{\mathrm{Anim}(\mathcal{C})}(1, X)$$

$1 \in F_\mathcal{C}$
(Assume $F_\mathcal{C}$ is generated under finite coproducts by one obj.)
$$F_{Ab} = \{\mathbb{Z}, \mathbb{Z}^{\oplus 2}, \ldots\}$$

2.3 Animated modules

Def: A comm. ring $\qquad FA := \{A^{\oplus n} \mid n \in \mathbb{N}\} \subseteq \mathrm{Mod}_A$
f.g. free A-modules

An *animated A-module* is a (finite) product-preserving functor

$$M : (FA)^{op} \to \mathrm{Anim}$$

(cohomologically: $D(A)^{\leq 0}$)

Notation: $D(A)_{\geq 0} := \mathrm{Anim}(\mathrm{Mod}_A)$.

Later: $D(A)_{\geq 0}$ equiv. to usual ∞-category of connective chain complexes.

Rmk: $M \in D(A)_{\geq 0}$ consists of:

- $M_n \in \mathrm{Anim} \quad \forall n \in \mathbb{N}$
- $M_m \to M_n \quad \forall \varphi \in \mathrm{Mat}_{m \times n}(A)$
- $\forall \varphi \in \mathrm{Mat}_{m \times n}(A)$, $\forall \psi \in \mathrm{Mat}_{\ell \times m}(A)$
  a homotopy between $(M_{\psi\varphi} : M_\ell \to M_n)$ and $(M_\ell \xrightarrow{M_\psi} M_m \xrightarrow{M_\varphi} M_n)$.
- + a homotopy coherent system of compatibilities between these homotopies.

subject to the condition: $M_n \xrightarrow{\sim} (M_1)^{\times n} \quad n \geq 0.\ \forall n$
$M_0 \simeq pt.$

The data of relevance:

- Underlying anima $M^\circ := M_1 \in \mathrm{Anim}$
- Operations $(M^\circ)^{\times n} \to M^\circ$
  $\Longleftrightarrow \varphi \in \mathrm{Mat}_{n \times 1}(A)$
- action of A on M°: $\quad A \to \mathrm{End}(M^\circ)$

$$A \subseteq M_{1\times 1}(A) = \mathrm{Hom}_{FA}(1,1) \xrightarrow{M} \mathrm{Maps}_{\mathrm{Anim}}(M_1, M_1) = \mathrm{End}(M^\circ)$$

- Assoc. + commutativity up to coherent homotopy:

$\forall x,y,z \in M \quad (\Leftrightarrow \; x,y,z : pt \to M^\circ)$

$a(a(x,y),z) \simeq a(x, a(y,z)) \quad$ where $\quad a : M^\circ \times M^\circ \to M^\circ$

$$\begin{tikzcd} M^\circ \times M^\circ \times M^\circ \arrow[r, "a \times id"] \arrow[d, "id \times a"'] & M^\circ \times M^\circ \arrow[d, "a"] \\ M^\circ \times M^\circ \arrow[r] & M^\circ \end{tikzcd}$$

2.4. <u>Derived functors</u>

<u>Constr</u> (left-derived functor): $\quad F : \mathcal{C} \to \mathcal{D}$ functor of algebraic categories

If F preserves filt colims + ~~fc~~ refl. coeq's, then

$\Rightarrow \; \mathbb{L}F : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{D})$ unique functor s.th.

- $\mathbb{L}F$ preserves filt. colims + geom. realizations
- $$\begin{tikzcd} F_{\mathcal{C}} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, hook] & \mathrm{Anim}(\mathcal{D}) \\ \mathrm{Anim}(\mathcal{C}) \arrow[urrr, dashed, "\exists! \, \mathbb{L}F"'] & & & \end{tikzcd}$$ commutes
- $\forall X \in \mathrm{Anim}(\mathcal{C}) \qquad \pi_0 \mathbb{L}F(X) \simeq F(\pi_0(X)) \in \mathcal{D}$
- If F preserves fin. coproducts, then $\mathbb{L}F$ does also.

<u>Def</u>: $\mathbb{L}F$ = left-derived functor of F.

Prop: $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ functors of alg. cats. preserving filt colims + reflexive coeqs.

Assume one of the following holds:

a) F sends $F_{\mathcal{C}} \to F_{\mathcal{D}} \subseteq \mathcal{D}$

$$\begin{tikzcd} F_{\mathcal{C}} \arrow[r, "F"] \arrow[d, hook] & F_{\mathcal{D}} \arrow[d, hook] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

[More generally: sends $F_{\mathcal{C}}$ to filtered colimits of objs. in $F_{\mathcal{D}}$.]

b) $\mathbb{L}G : \mathrm{Anim}(\mathcal{D}) \to \mathrm{Anim}(\mathcal{E})$ preserves discrete objs.

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\mathbb{L}G"] \arrow[d, hook] & \mathcal{E} \arrow[d, hook] \\ \mathrm{Anim}(\mathcal{D}) \arrow[r, "\mathbb{L}G"] & \mathrm{Anim}(\mathcal{E}) \end{tikzcd}$$

[$\forall X \in F_{\mathcal{C}}$ $\quad \mathbb{L}G(F(X))$ is discrete in $\mathrm{Anim}(\mathcal{E})$.]

Then: $\cancel{\mathbb{L}F \circ \mathbb{L}G} \simeq \mathbb{L}($

$$\boxed{\mathbb{L}G \circ \mathbb{L}F \simeq \mathbb{L}(G \circ F)} : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{E}).$$

Ex: $\varphi : A \to B$ ring homo.

$\varphi^* : \mathrm{Mod}_A \to \mathrm{Mod}_B \qquad M \mapsto M \otimes_A B$

$\Rightarrow \mathbb{L}\varphi^* : D(A)_{\geq 0} \to D(B)_{\geq 0}$ preserves colimits

$\pi_0 \mathbb{L}\varphi^*(M) \simeq \varphi^*(\pi_0 M)$

$$\begin{tikzcd} F_A \arrow[r, "\varphi^* = \otimes_A B"] \arrow[d, hook] & F_B \arrow[d, hook] \\ D(A)_{\geq 0} \arrow[r, "\mathbb{L}\varphi^*"'] & D(B)_{\geq 0} \end{tikzcd}$$

$$\left(- \overset{\mathbb{L}}{\underset{A}{\otimes}} B = \mathbb{L}\varphi^* \right)$$

~~$\mathrm{CAlg}_A \to \mathrm{CAlg}_B$~~

~~$A' \mapsto A' \otimes_A B$~~

~~$(A \to B \to B') \leftarrow\!\shortmid (B \to B')$~~

[HTT] §5.5.8

[CS] Cesnavicius-Scholze, §5 (Purity for flat cohomology)

[Adamek-Rosicky-Vitale] ~~Algebraic theories~~ Algebraic theories