Lecture 1 ∞-Categories

1.1. Simplicial sets

Def (simplicial sets)

$\Delta = \{[n] \mid n \in \mathbb{N}\}$ $[n] = \{0, 1, \dots, n\}$

$\mathrm{Hom}_{\Delta}([m], [n])$ = order-preserving maps

a simplicial set is a functor $X : \Delta^{op} \to \mathrm{Sets}$

$\Leftrightarrow$ $(X_n)_{n \in \mathbb{N}}$ $\forall \alpha : [m] \to [n]$ in Δ

$\alpha^* : X_n \longrightarrow X_m$ (functorially ...)

Notation: $n \in \mathbb{N}$, $0 \le i \le n$

$\delta_n^i : [n-1] \longrightarrow [n]$ injective map which "skips" i

$\sigma_n^i : [n+1] \to [n]$ surjective map which "doubles" i

$X \in \mathrm{SSet}$ $d_n^i : X_n \longrightarrow X_{n-1}$ induced by δ_n^i (face maps)

$s_n^i : X_n \to X_{n+1}$ σ_n^i (degeneracy maps)

Ex: X set $\Rightarrow$ $c(X) \in \mathrm{SSet}$ constant

$c(X)_n = X \quad \forall n$

$\alpha : [m] \to [n] \quad \Rightarrow \quad \alpha^* = \mathrm{id}$

Claim: $\mathrm{Set} \overset{c}{\hookrightarrow} \mathrm{SSets}$ is fully faithful.

Ex: $\Delta^n \in \mathrm{SSet}$ $\forall n \in \mathbb{N}$ $\Delta^n_i = \mathrm{Hom}_{\Delta}([i], [n])$

$\in$

$(a_0, \dots, a_i)$ $0 \le a_i \le a_j \le n \quad \forall i \le j$

Ex: $\partial\Delta^n$ $\quad \Delta^{n-1} \longrightarrow \Delta^n \quad \delta^k_n : [n-1] \to [n]$

$$([i] \to [n-1]) \longmapsto ([i] \to [n-1] \xrightarrow{\delta^k_n} [n])$$

$\partial^k\Delta^n \subseteq \Delta^n$ image of $\Delta^{n-1} \longrightarrow \Delta^n$ <u>faces</u> of Δ^n

$\partial\Delta^n := \bigcup_k \partial^k\Delta^n$ <u>boundary</u> of Δ^n

Ex: $\Lambda^n_k \subseteq \Delta^n$ union of $\partial^j\Delta^n$ $j \neq k$

($\partial\Delta^n$ minus k^{th} face) <u>horns</u> of Δ^n

1.2 Categories as simplicial sets

<u>Constr</u>: C category $\rightsquigarrow N(C) \in SSet$ <u>nerve of C</u>

$$N(C)_n = \underset{Cat}{Fun}([n], C) = \{ c_0 \to c_1 \to \cdots \to c_n \text{ strings of morphisms in } C \}$$

$\alpha : [m] \to [n]$ in Δ $\qquad \alpha^* : N(C)_n \to N(C)_m$

$$([n] \to C) \longmapsto ([m] \to [n] \to C)$$

- $N(C)_0$ = objects of C
- $N(C)_1$ = morphisms of C
- $N(C)_2$ = diagrams $c_0 \to c_1 \to c_2$ in C
- ...

$N(C)$ has all info. about C

<u>Exercise</u>: $C \mapsto N(C)$ defines a <u>fully faithful</u> functor $N : Cat \to SSets$

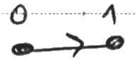
- Moreover: There is a left adjoint $\tau : SSets \to Cat$.

Def: $X \in SSet$

- objects of X: 0-simplices $x \in X_0 \Leftrightarrow \Delta^0 \to X$
- morphisms of X: 1-simplices $f \in X_1 \Leftrightarrow \Delta^1 \to X$
- source/target of $f \in X_1$

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_1^0"] \arrow[r, shift right, "d_1^1"'] & X_0 \end{tikzcd} \qquad s(f) := d_1^1(f) \qquad t(f) := d_1^0(f)$$

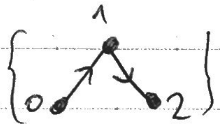
$$\begin{tikzcd} \Delta^0 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \Delta^1 \arrow[r, "f"] & X \end{tikzcd}$$



- identity of $x \in X_0$ $\qquad s_0^0 : X_0 \longrightarrow X_1$

$$x \mapsto id_x \qquad s(id_x) = x$$
$$t(id_x) = x$$

Def $\Lambda^2_1 = \{$



$\}$

$$\begin{tikzcd} \Delta^0 \arrow[r, hook] \arrow[d, hook] & \Delta^1 \arrow[d, hook] \\ \Delta^1 \arrow[r, hook] & \Lambda^2_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$X \in SSet$ a composable pair in X is a map $\Lambda^2_1 \to X$

a composition of a composable pair is a lift

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "\sigma"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "\tilde{\sigma}"'] & \end{tikzcd}$$

Q: When do compositions exist (uniquely)?

Prop (Grothendieck-Segal): $X \in SSet$ belongs to the ess. image of $N : Cat \hookrightarrow SSet$ iff:

$$Hom(\Delta^n, X) \longrightarrow Hom(\Lambda^n_k, X)$$

~~$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}($~~

res: $\mathrm{Hom}(\Delta^n, X) \longrightarrow \mathrm{Hom}(\Lambda^n_k, X)$ is bijective $\forall n \geq 2$, $0 < k < n$

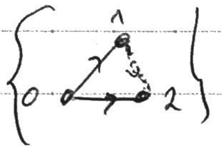
($\Longrightarrow$ compositions exist + unique)

1.3 Groupoids and Kan complexes

Remark: C groupoid $\overset{\text{def}}{\Longleftrightarrow}$ all morphisms in $\mathcal{C}$ are invertible

$\Longleftrightarrow$ $N(C) \in \mathrm{sSet}$ satisfies the following:

$$\mathrm{Hom}(\Delta^n, N(C)) \xrightarrow{\sim} \mathrm{Hom}(\Lambda^n_k, N(C)) \quad \text{bijective}$$
$$\forall n \geq 2, \quad 0 \leq k \leq n$$

$\Lambda^2_0 =$ 

Def (Kan complex): $X \in \mathrm{sSet}$ is a Kan complex

$\Longleftrightarrow$ $\mathrm{Hom}(\Delta^n, X) \xrightarrow{\text{res}} \mathrm{Hom}(\Lambda^n_k, X)$ surjective $\forall\, 0 \leq k \leq n$

Ex: C category $\Rightarrow$ $N(C)$ Kan complex iff C groupoid

"Kan complex = generalized groupoids where composition + inverses exist but not uniquely"

Theorem (Milnor): There is an equivalence between

(homotopy category of CW complexes)(up to w.h.e.) $\xrightarrow{\sim}$ {homotopy cat. of Kan comp}(up to w.h.e.)

~~(up to weak htpy)~~

which is given by $X \longmapsto \mathrm{Sing}(X)_\bullet \in \mathrm{SSet}$

$\mathrm{Sing}(X)_n = \{ \underbrace{\Delta^n_{top}}_{\text{topological n-simplex}} \longrightarrow X \text{ continuous maps} \}$ (Kan complex)

composition in a Kan complex $\leftrightsquigarrow$ composition of paths in a space

1.4. ∞-Categories as weak Kan complexes

groupoid :: category
Kan cpx :: ???

(a.k.a. quasi-category)

Def (Boardman-Vogt) $X \in \mathrm{SSet}$ is a weak Kan complex

$\Leftrightarrow$ $\mathrm{Hom}(\Delta^n, X) \xrightarrow{res} \mathrm{Hom}(\Lambda^n_k, X)$ surjective $\forall n \geq 2$, $0 < k < n$.

Constr: X WKC $f, g : x \to y$

~~Def~~ a homotopy $f \simeq g$ is a 2-simplex $\Delta^2 \xrightarrow{\sigma} X$

$$\begin{tikzcd} & y \arrow[dr, "id"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & \sigma & y \end{tikzcd}$$

h(X)

~~h(X)~~ := homotopy category : objects: X_0
morphisms: $\mathrm{Hom}(x,y)$
$= \{\text{equiv. classes of morphisms in } X\}$

~~$X \xrightarrow{unit} N_\tau(X)$~~

~~Def~~:

Def: X wKc. $f : x \to y$ is an isomorphism

$\iff f$ invertible, i.e. $\exists\, g : y \to x$, $f \circ g \simeq id$, ~~$g \circ f \simeq id$~~ $g \circ f \simeq id$

$\iff f$ isomorphism in $h(X)$.

X is an ∞-groupoid $\iff$ every morphism in X is an iso.

$(X \xrightarrow{unit} N_\tau(X))$

Theorem (Joyal): X w. Kan cpx. TFAE:

- X is a Kan complex
- X is an ∞-groupoid.

Ex: (category theory of weak Kan complexes):

- $Fun(X, Y) = \underline{Hom}(X, Y)$ internal hom of SSet

  $\underline{Hom}(X, Y)_n = Hom(\Delta^n \times X, Y)$

- $\underset{X}{Maps}(x, y)$ X w. Kan cpx, ~~x~~ $x, y \in X_0$

$$\begin{tikzcd} \underset{X}{Maps}(x,y) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{Hom}(\Delta^1, X) = Fun(\Delta^1, X) \arrow[d, "{(s,t)}"] \\ \Delta^0 \arrow[r, "{(x,y)}"'] & X \times X \end{tikzcd}$$

- co/limits
- adjoint functors

Def: An ∞-category is a weak Kan complex.

$\uparrow$ platonic concept $\qquad\qquad \uparrow$ shadow

1.5. ∞-Category of Kan complexes

Constr: Kan (large) simplicial set ($\neq N$(cat. of Kan cpx's))

- $\mathrm{Kan}_0 = \{$(small) Kan complexes$\}$
- $\mathrm{Kan}_1 = \{$maps of Kan complexes$\} = \{(X, Y, f),\ f : X \to Y\}$ X, Y Kan cpx's
- $\mathrm{Kan}_2 = \{(X, Y, Z, f, g, h, \sigma) \mid f : X \to Y,\ g : Y \to Z,\ h : X \to Z$, $\sigma \in \underline{\mathrm{Hom}}(X, Z)_1$, $d^0_1(\sigma) = g \circ f,\ d^1_1(\sigma) = h\}$ $(g \circ f \sim h)$

  $\vdots$
- $\mathrm{Kan}_n = \{$tuples of Kan cpx's $X_0, \ldots, X_n$ $X_i \to X_j$ compatible up to coherent homotopy$\}$

  $\vdots$

$\mathrm{Kan}_0 \to \mathrm{Kan}_1 \qquad X \mapsto \mathrm{id}_X$

$(s, t) : \mathrm{Kan}_1 \rightrightarrows \mathrm{Kan}_0$

Rmk: This is the homotopy coherent nerve of the simplicially enriched category of Kan complexes

Def: There is a fully faithful functor (of ∞-categories)

$$\mathrm{Set} \hookrightarrow \mathrm{Kan}$$

An object $X \in \mathrm{Kan}$ is in the ess. image
$\iff X$ homotopy equivalent to a constant simplicial set

$\iff \pi_i(X) = 0 \quad \forall i > 0$.

References

- Lurie, Higher Topos Theory [HTT]
- Cisinski, Higher Cat's and Homotopical Algebra [HCHA].