VIRTUAL FUNDAMENTAL CLASSES OF DERIVED STACKS I

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Abstract. We construct the étale motivic Borel–Moore homology of derived Artin stacks. Using a derived version of the intrinsic normal cone, we construct fundamental classes of quasi-smooth derived Artin stacks and demonstrate functoriality, base change, excess intersection, and Grothendieck–Riemann–Roch formulas. These classes also satisfy a general cohomological Bézout theorem which holds without any transversity hypotheses. The construction is new even for classical stacks and as one application we extend Gabber's proof of the absolute purity conjecture to Artin stacks.

Introduction

In this paper we revisit the foundations of the theory of virtual fundamental classes using the language of derived algebraic geometry.

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Quasi-smoothness. Let X be a smooth algebraic variety of dimension m over a field k. Any collection of regular functions $f_1, \ldots, f_n \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ determines a quasi-smooth derived subscheme $\mathrm{Z} = \mathrm{Z}(f_1, \ldots, f_n)$ of X. Its underlying classical scheme Z_{cl} is the usual zero locus, but Z admits a perfect 2-term cotangent complex of the form

$$\mathcal{L}_{\mathrm{Z}} = \left(\mathcal{O}_{\mathrm{Z}}^{\oplus n} \to \Omega_{\mathrm{X}}|_{\mathrm{Z}}\right)$$

whose virtual rank encodes the virtual dimension $d = m - n$. Every quasi-smooth derived Artin stack $\mathcal{Z}$ is given by this construction, locally on some smooth atlas.

To any such $\mathcal{Z}$, the main construction of this paper assigns a *virtual fundamental class* $[\mathcal{Z}]^{\mathrm{vir}}$. More generally, for any quasi-smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived Artin stacks, we define a relative virtual fundamental class $[\mathcal{X}/\mathcal{Y}]^{\mathrm{vir}}$.

The normal bundle stack. We begin in Sect. 1 by introducing a derived version of the intrinsic normal cone of Behrend–Fantechi [BF]. For any quasi-smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived Artin stacks, this is a vector bundle stack $\mathrm{N}_{\mathcal{X}/\mathcal{Y}}$ over $\mathcal{X}$. When $\mathcal{X}$ and $\mathcal{Y}$ are classical 1-Artin stacks and f is a local complete intersection morphism that is representable by Deligne–Mumford stacks, then $\mathrm{N}_{\mathcal{X}/\mathcal{Y}}$ is the relative intrinsic normal cone defined in [BF, Sect. 7]. If f is not representable by Deligne–Mumford stacks, then $\mathrm{N}_{\mathcal{X}/\mathcal{Y}}$ is only a 2-Artin stack. The key geometric construction, which is joint with D. Rydh, is called "deformation to the normal bundle stack". For any quasi-smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ it provides a family of quasi-smooth morphisms parametrized by $\mathbf{A}^1$, with generic fibre $f : \mathcal{X} \to \mathcal{Y}$ and special fibre the zero section $0 : \mathcal{X} \to \mathrm{N}_{\mathcal{X}/\mathcal{Y}}$.

Motivic Borel–Moore homology theories. In Sect. 2 we construct étale motivic Borel–Moore homology theories on derived Artin stacks. If SH(S) denotes Voevodsky's stable motivic homotopy category over a scheme S, any object $\mathcal{F} \in \mathrm{SH}(\mathrm{S})$ gives rise to relative Borel–Moore homology groups

$$\mathrm{H}_s^{\mathrm{BM}}(\mathrm{X}/\mathrm{S}, \mathcal{F}(r)) := \mathrm{Hom}_{\mathrm{SH}(\mathrm{S})}(\mathbf{1}_{\mathrm{S}}(r)[s], f_* f^!(\mathcal{F})),$$

bigraded by integers $r, s \in \mathbf{Z}$ (where (r) denotes the Tate twist), where X is a locally of finite type S-scheme with structural morphism $f : \mathrm{X} \to \mathrm{S}$. It was observed in [De2] that as X and S vary, these groups behave just like a bivariant theory in the sense of [FM] except that they are bigraded. Appropriate choices of the coefficient $\mathcal{F}$ give rise to bivariant versions of such theories as motivic cohomology, algebraic cobordism, étale cohomology with finite or adic coefficients, and singular cohomology. Using the extension of SH to derived schemes constructed in [Kh1], we also obtain derived extensions of all these bivariant theories. Moreover, for coefficients $\mathcal{F}$ satisfying étale descent, these bivariant theories extend further to derived Artin stacks (this is done by extending the étale-local motivic homotopy category $\mathrm{SH}_{\text{ét}}$ and its six operations to derived Artin stacks, see Appendix A). In Subsect. 2.4 we demonstrate the expected properties: long exact localization sequences, homotopy invariance for vector bundle stacks, and Poincaré duality for smooth stacks.

Fundamental classes. Sect. 3 contains our construction of the virtual class $[\mathcal{X}/\mathcal{Y}]^{\mathrm{vir}}$ of a quasi-smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ of relative virtual dimension d. Assume $\mathcal{F}$ is oriented for simplicity. The idea is that there are canonical isomorphisms

$$\mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}/\mathcal{Y}, \mathcal{F}(d)) \simeq \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}_{\mathrm{cl}}, \mathcal{F}(d))$$

through which the virtual class corresponds to a more intrinsic fundamental class $[\mathcal{X}/\mathcal{Y}] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}/\mathcal{Y}, \mathcal{F}(d))$. The latter is constructed, much as in Fulton's intersection theory, by using deformation to the normal bundle stack to define a specialization map

$$\mathrm{sp}_{\mathcal{X}/\mathcal{Y}} : \mathrm{H}^{\mathrm{BM}}_{s}(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_{s}(\mathrm{N}_{\mathcal{X}/\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)),$$

see Subsect. 3.1. By homotopy invariance for vector bundle stacks, the target is identified with $\mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathcal{X}/\mathcal{Y}, \mathcal{F}(r+d))$, so we get a Gysin map

$$f^! : \mathrm{H}^{\mathrm{BM}}_{s}(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r+d)). \tag{0.1}$$

The fundamental class $[\mathcal{X}/\mathcal{Y}]$ is the image of the unit $1 \in \mathrm{H}^{\mathrm{BM}}_{0}(\mathcal{Y}/\mathcal{Y}, \mathcal{F})$, where we take $\mathcal{S} = \mathcal{Y}$.

The two key properties of the fundamental class are functoriality and stability under arbitrary derived base change, see Theorems 3.12 and 3.13. We also have excess intersection, self-intersection, and blow-up formulas (Subsect. 3.2). In the sequel we intend to prove analogues of the virtual Atiyah–Bott localization and cosection formulas in this framework.

Non-transverse Bézout theorem. The fundamental classes satisfy a cohomological Bézout theorem that holds without any transversity hypotheses (Subsect. 3.4). For schemes, it can be stated in the Chow group as follows. Let X be a smooth quasi-projective scheme over a field k. Let $f : \mathrm{Y} \to \mathrm{X}$ and $g : \mathrm{Z} \to \mathrm{X}$ be quasi-smooth projective morphisms of derived schemes of relative virtual dimensions $-d$ and $-e$, respectively. Then the intersection product of the fundamental classes $[\mathrm{Y}] \in \mathrm{A}^d(\mathrm{X})$ and $[\mathrm{Z}] \in \mathrm{A}^e(\mathrm{X})$ is given by the fundamental class of the derived fibred product:

$$[\mathrm{Y}] \cdot [\mathrm{Z}] = [\mathrm{Y} \overset{\mathbf{R}}{\underset{\mathrm{X}}{\times}} \mathrm{Z}] \tag{0.2}$$

in $\mathrm{A}^{d+e}(\mathrm{X})$.

If k is of characteristic zero, this formula completely characterizes the intersection product in $\mathrm{A}^*(\mathrm{X})$, since by resolution of singularities the Chow group is generated by fundamental classes $[\mathrm{Z}]$ where $f : \mathrm{Z} \to \mathrm{X}$ is a projective morphism with Z smooth (so that f is automatically quasi-smooth).

Grothendieck–Riemann–Roch. In Subsect. 3.5 we prove a generalization of the Grothendieck–Riemann–Roch theorem to derived Artin stacks. For a locally noetherian derived Artin stack $\mathcal{X}$, denote by $\mathrm{G}(\mathcal{X})$ the G-theory of $\mathcal{X}$, i.e., the Grothendieck group of coherent sheaves on $\mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived Artin stacks, locally of finite type over some regular noetherian base scheme. Then there is a commutative diagram

$$\begin{tikzcd} \mathrm{G}(\mathcal{Y}) \arrow[r, "f^*"] \arrow[d, "\tau_{\mathcal{Y}}"] & \mathrm{G}(\mathcal{X}) \arrow[d, "\tau_{\mathcal{X}}"] \\ \mathrm{A}_*(\mathcal{Y})_{\mathbf{Q}} \arrow[r, "\mathrm{Td}_{\mathcal{X}/\mathcal{Y}} \cap f^!"] & \mathrm{A}_*(\mathcal{X})_{\mathbf{Q}}, \end{tikzcd} \tag{0.3}$$

where $\mathrm{Td}_{\mathcal{X}/\mathcal{Y}}$ is the Todd class of the relative cotangent complex $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}$.

In particular, if $\mathcal{X}$ is a quasi-smooth derived Artin stack over a field, this gives the following formula for the fundamental class of $\mathcal{X}$ in $\mathrm{A}_*(\mathcal{X})_{\mathbf{Q}}$:

$$[\mathcal{X}] = \mathrm{Td}_{\mathcal{X}}^{-1} \cap \tau_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}). \tag{0.4}$$

Through the canonical isomorphisms $\mathrm{A}_*(\mathcal{X})_{\mathbf{Q}} \simeq \mathrm{A}_*(\mathcal{X}_{\mathrm{cl}})_{\mathbf{Q}}$ and $\mathrm{G}(\mathcal{X}) \simeq \mathrm{G}(\mathcal{X}_{\mathrm{cl}})$, this becomes the formula

$$[\mathcal{X}]^{\mathrm{vir}} = (\mathrm{Td}_{\mathcal{X}}^{\mathrm{vir}})^{-1} \cap \left(\sum_{i \in \mathbf{Z}} (-1)^i \cdot \tau_{\mathcal{X}_{\mathrm{cl}}}(\pi_i(\mathcal{O}_{\mathcal{X}})) \right)$$

in $\mathrm{A}_*(\mathcal{X}_{\mathrm{cl}})_{\mathbf{Q}}$, relating the virtual class $[\mathcal{X}]^{\mathrm{vir}} \in \mathrm{A}_*(\mathcal{X}_{\mathrm{cl}})_{\mathbf{Q}}$ with the K-theoretic fundamental class in $\mathrm{G}(\mathcal{X}_{\mathrm{cl}})$. The virtual Todd class $\mathrm{Td}_{\mathcal{X}}^{\mathrm{vir}}$ is the Todd class of the perfect complex $\mathcal{L}_{\mathcal{X}}|_{\mathcal{X}_{\mathrm{cl}}}$ on $\mathcal{X}_{\mathrm{cl}}$. This extends the formula predicted by Kontsevich in the case of schemes [Ko, 1.4.2].

Absolute purity. Our construction of fundamental classes is interesting even when we restrict to classical algebraic geometry; in this case quasi-smoothness translates to being a local complete intersection morphism (which need not admit a *global* factorization through a regular immersion and smooth morphism). For example, we get Gysin maps for proper lci morphisms between Artin stacks in étale cohomology and mixed Weil cohomology theories such as Betti and de Rham cohomology. In terms of the six operations, if $f : \mathcal{X} \to \mathcal{Y}$ is an lci morphism of virtual dimension d between Artin stacks, then the fundamental class can be viewed as a canonical morphism

$$f^*\mathcal{F}(d)[2d] \to f^!(\mathcal{F}) \tag{0.5}$$

for any coefficient $\mathcal{F}$. In the context of étale cohomology, such morphisms were constructed previously by Gabber [Fu], [ILO, Exp. XVI] in the case of schemes and assuming the existence of a global factorization of f. Thus taking $\mathcal{F}$ to be the étale motivic cohomology spectrum $\Lambda^{\text{ét}}$ (with coefficients in $\Lambda = \mathbf{Z}/n\mathbf{Z}$, n invertible on $\mathcal{Y}$) gives a generalization of Gabber's construction. In Subsect. 3.6 we prove that the morphism (0.5) is invertible when $\mathcal{F} = \Lambda^{\text{ét}}$ and $\mathcal{X}$ and $\mathcal{Y}$ are regular Artin stacks. This extends Gabber's proof of the absolute purity conjecture to Artin stacks (and drops the global factorization hypothesis in the case of schemes).

Related work. The yoga of fundamental classes in motivic bivariant theories was developed in [De2] and [DJK]. This paper extends these constructions on one hand from classical to derived algebraic geometry, and on the other hand from schemes to algebraic stacks (at least for étale coefficients).

The notion of perfect obstruction theory introduced by K. Behrend and B. Fantechi [BF] is a useful approximation to a quasi-smooth derived structure on a scheme or Deligne–Mumford stack, and actually suffices for the construction of virtual fundamental classes on Deligne–Mumford stacks. This construction was done in [BF] in Chow groups, and has been refined to algebraic cobordism and other Borel–Moore homology theories recently by M. Levine [Le2] and Y.-H. Kiem and H. Park [KP]. Our construction agrees with the P.O.T. approach when both are defined (see Subsect. 3.3), but it is worth noting that a quasi-smooth derived Artin stack typically has a 3-term cotangent complex, so that the P.O.T. formalism does not apply (in fact, there is no associated intrinsic normal cone in the world of classical 1-stacks).

Virtual fundamental classes have been studied using the language of derived algebraic geometry previously in the setting of algebraic cobordism by P. Lowrey and T. Schürg [LS]. They were also studied using the older language of dg-schemes by I. Ciocan-Fontanine and M. Kapranov [CK] in rational Chow groups and G-theory. These approaches only work for derived *schemes* and also require other unpleasant hypotheses such as existence of a characteristic zero base field and embeddings into smooth ambient schemes. The Bézout formula (0.2) mentioned above was inspired by a similar formula announced by J. Lurie [Lu] in Betti cohomology.

Classical Borel–Moore homology was recently extended to Artin stacks by M. Kapranov and E. Vasserot [KV], for the purpose of defining a cohomological Hall algebra whose underlying vector space is the Borel–Moore homology of the moduli stack of coherent sheaves on a surface. Our formalism gives a streamlined approach to the construction of this algebra, whose multiplicative structure arises from the quasi-smooth structure on the moduli stack. Moreover it shows that the same structure exists on the Borel–Moore homology with coefficients in any étale motivic spectrum.

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1. The intrinsic normal bundle

1.1. **Stacks.** In this paper, we define a stack to be a "higher stack" in the sense of [HS]. That is, it is a functor

$$\mathrm{R} \mapsto \mathcal{X}(\mathrm{R})$$

assigning to any commutative ring R an ∞-groupoid $\mathcal{X}(\mathrm{R})$ of R-valued points, and satisfying hyperdescent with respect to the étale topology (in the sense of ∞-category theory, see e.g. [To2, p. 183]).

We say $\mathcal{X}$ is 0-*Artin* if it is (representable by) an algebraic space. We define k-Artin stacks inductively, following [To1, §3.1].

For an integer $k \geqslant 0$, a morphism $f : \mathcal{X} \to \mathcal{Y}$ is k-*representable* if for every k-Artin $\mathcal{Y}'$ and every morphism $\mathcal{Y}' \to \mathcal{Y}$, the fibred product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is k-Artin. An k-representable morphism f is *smooth* if for every scheme Y, every morphism $\mathrm{Y} \to \mathcal{Y}$, and every smooth atlas $\mathrm{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathrm{Y}$, the composite $\mathrm{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathrm{Y} \to \mathrm{Y}$ is a smooth morphism of schemes. A stack $\mathcal{X}$ is $(k+1)$-*Artin* if its diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by k-Artin stacks, and there exists a scheme X and a morphism $\mathrm{X} \to \mathcal{X}$ (automatically k-representable) which is smooth and surjective. The morphism $\mathrm{X} \to \mathcal{X}$ is called a *smooth atlas* for $\mathcal{X}$.

An k-Artin stack $\mathcal{X}$ always takes values in k-groupoids: for every commutative ring R, the ∞-groupoid $\mathcal{X}(\mathrm{R})$ is k-truncated. We say a stack is *Artin* if it is k-Artin for some $k \geqslant 0$. Artin stacks in this sense form an ∞-category, whose full subcategory spanned by 1-Artin stacks is equivalent to the $(2,1)$-category of Artin stacks in the usual sense.

Now replace the category of commutative rings by its nonabelian derived ∞-category, i.e., the ∞-category of simplicial commutative rings. This is the natural target for derived functors on the nonabelian category of commutative rings, such as the derived tensor product. A simplicial commutative ring R has an underlying ordinary commutative ring $\pi_0(\mathrm{R})$ as well as $\pi_0(\mathrm{R})$-modules $\pi_i(\mathrm{R})$. We say R is *discrete* if $\pi_i(\mathrm{R}) = 0$ for all $i > 0$ (i.e., $\mathrm{R} \simeq \pi_0(\mathrm{R})$); the discrete simplicial commutative rings span a full subcategory equivalent to the ordinary category of commutative rings. The notions of étale and smooth homomorphism admit natural extensions to simplicial commutative rings. See [SAG, Chap. 25] or [To1, §4].

A *derived stack* $\mathcal{X}$ is a functor $\mathrm{R} \mapsto \mathcal{X}(\mathrm{R})$, assigning an ∞-groupoid of R-points to every simplicial commutative ring R, that satisfies étale hyperdescent. Derived k-Artin and Artin stacks are defined following the pattern outlined above, see e.g. [To1, §5.2] for details.

1.2. **Vector bundle stacks.** Let $\mathcal{X}$ be a derived Artin stack and $\mathcal{E}$ a perfect complex on $\mathcal{X}$ of Tor-amplitude $[-k, 1]$, for some integer $k \geqslant -1$. The associated *vector bundle stack*

$$\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]) \to \mathcal{X}$$

is the moduli stack of co-sections of $\mathcal{E}[-1]$. That is, for any affine derived scheme S over $\mathcal{X}$, the ∞-groupoid of $\mathcal{X}$-morphisms $\mathrm{S} \to \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1])$ is naturally equivalent to the ∞-groupoid of $\mathcal{O}_{\mathrm{S}}$-linear morphisms of perfect complexes $\mathcal{E}[-1]|_{\mathrm{S}} \to \mathcal{O}_{\mathrm{S}}$.

Since $\mathcal{E}[-1]$ is perfect of Tor-amplitude $[-k-1, 0]$, $\mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1])$ is a smooth $(k+1)$-Artin derived stack over $\mathcal{X}$ of relative dimension $-d$, where d is the virtual rank of $\mathcal{E}$. See [To2, Subsect. 3.3, p. 201].

1.3. **Normal bundle stacks.** The normal bundle stack is a derived version of the relative intrinsic normal cone of [BF].

A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived Artin stacks is *quasi-smooth* if it is locally of finite presentation and the relative cotangent complex $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}$ is of Tor-amplitude $(-\infty, 1]$. Note that we use homological grading: this means that, for every discrete quasi-coherent sheaf $\mathcal{E}$ on $\mathcal{X}$, we have

$$\pi_i(\mathcal{L}_{\mathcal{X}/\mathcal{Y}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathrm{X}}} \mathcal{E}) = 0$$

for $i > 1$. If $\mathcal{X} = \mathrm{X}$ and $\mathcal{Y} = \mathrm{Y}$ are derived schemes, this is equivalent to the following condition: Zariski-locally on X, f factors through a smooth morphism $\mathrm{M} \to \mathrm{Y}$ and a morphism $\mathrm{X} \to \mathrm{M}$ which exhibits X as the derived zero-locus of some functions on M [KR, 2.3.14]. If $f : \mathcal{X} \to \mathcal{Y}$ is k-representable, then it is quasi-smooth if and only if for every derived scheme Y, every morphism $\mathrm{Y} \to \mathcal{Y}$, and every smooth atlas $\mathrm{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathrm{Y}$, the composite $\mathrm{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathrm{Y} \to \mathrm{Y}$ is a quasi-smooth morphism of derived schemes. The *relative virtual dimension* of a quasi-smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ is

$$\mathrm{vd}(\mathcal{X}/\mathcal{Y}) \coloneqq \mathrm{rk}(\mathcal{L}_{\mathcal{X}/\mathcal{Y}}),$$

the virtual rank (Euler characteristic) of the relative cotangent complex.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a k-representable quasi-smooth morphism. The cotangent complex $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}$ is perfect of Tor-amplitude $[-k, 1]$, so the associated vector bundle stack $\mathbf{V}_{\mathcal{X}}(\mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1])$ is a smooth $(k+1)$-Artin stack of relative virtual dimension $-\mathrm{vd}(\mathcal{X}/\mathcal{Y})$.

Definition 1.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived Artin stacks. The* normal bundle stack *is the vector bundle stack*

$$\mathrm{N}_{\mathcal{X}/\mathcal{Y}} = \mathbf{V}_{\mathcal{X}}(\mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1]) \to \mathcal{X}.$$

If f is a closed immersion, then $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1]$ is of Tor-amplitude $[0, 0]$, and the normal bundle stack is just the normal bundle. If f is smooth, then $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1]$ is of Tor-amplitude $(-\infty, -1]$, and the normal bundle stack is the classifying stack of the tangent bundle $\mathrm{T}_{\mathcal{X}/\mathcal{Y}}$. If f factors through a closed immersion $i : \mathcal{X} \to \mathcal{Y}'$ and a smooth morphism $p : \mathcal{Y}' \to \mathcal{Y}$, then the normal bundle stack is the quotient

$$\mathrm{N}_{\mathcal{X}/\mathcal{Y}} = [\mathrm{N}_{\mathcal{X}/\mathcal{Y}'}/i^*\mathrm{T}_{\mathcal{Y}'/\mathcal{Y}}].$$

Proposition 1.2.

(i) *The construction $\mathrm{N}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is stable under derived arbitrary base change in $\mathcal{X}$. That is, for any homotopy cartesian square of derived Artin stacks*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

with f quasi-smooth, there is a canonical isomorphism

$$\mathrm{N}_{\mathcal{X}/\mathcal{Y}} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathcal{X}' \to \mathrm{N}_{\mathcal{X}'/\mathcal{Y}'}$$

of derived Artin stacks over $\mathcal{X}'$.

(ii) *Suppose given a commutative square*

$$\begin{tikzcd} \mathrm{X} \arrow[r, "i"] \arrow[d, "p"] & \mathrm{Y} \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

with f quasi-smooth, p and q smooth surjections with X *and* Y *schematic, and i a quasi-smooth closed immersion. Then* $\mathrm{N}_{\mathcal{X}/\mathcal{Y}}$ *is the quotient of the groupoid*

$$\mathrm{N}_{\check{\mathrm{C}}(\mathrm{X}/\mathcal{X})_\bullet/\check{\mathrm{C}}(\mathrm{Y}/\mathcal{Y})_\bullet} \coloneqq \Big[\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathrm{N}_{\mathrm{X}\times^{\mathbf{R}}_{\mathcal{X}}\mathrm{X}/\mathrm{Y}\times^{\mathbf{R}}_{\mathcal{Y}}\mathrm{Y}} \rightrightarrows \mathrm{N}_{\mathrm{X}/\mathrm{Y}}\Big],$$

i.e., the geometric realization of this simplicial diagram.

Proof. The first claim follows from the fact that the cotangent complex is stable under derived base change [Lu, Prop. 3.2.10]. The second follows from the fact that the cotangent complex satifies descent for smooth surjections [Bh, Cor. 2.7]. □

1.4. **Deformation to the normal bundle stack.** For any quasi-smooth morphism $f : \mathcal{X} \to \mathcal{Y}$, there is a canonical $\mathbf{A}^1$-deformation to the zero section $0 : \mathcal{X} \to \mathrm{N}_{\mathcal{X}/\mathcal{Y}}$, generalizing the classical construction of Verdier. This construction is joint with D. Rydh.

Theorem 1.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived Artin stacks.*

(i) *There exists a quasi-smooth derived Artin stack $\mathrm{D}_{\mathcal{X}/\mathcal{Y}}$ over $\mathcal{Y} \times \mathbf{A}^1$, and a quasi-smooth morphism*

$$\mathcal{X} \times \mathbf{A}^1 \to \mathrm{D}_{\mathcal{X}/\mathcal{Y}}$$

over $\mathcal{Y} \times \mathbf{A}^1$. The fibre over $\mathbf{G}_m = \mathbf{A}^1 \smallsetminus \{0\}$ is the quasi-smooth morphism $\mathcal{X} \times \mathbf{G}_m \to \mathcal{Y} \times \mathbf{G}_m$ and the fibre over $\{0\}$ is the quasi-smooth morphism $0 : \mathcal{X} \to \mathrm{N}_{\mathcal{X}/\mathcal{Y}}$.

(ii) *The construction $\mathrm{D}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{Y}$ is stable under arbitrary derived base change in $\mathcal{Y}$.*

In the case where f is a closed immersion, $\mathrm{D}_{\mathcal{X}/\mathcal{Y}}$ was already constructed in [KR, Thm. 4.1.13]. For a general quasi-smooth morphism with a presentation as in Proposition 1.2(ii), it can be described as the quotient of the groupoid

$$\mathrm{D}_{\check{\mathrm{C}}(\mathrm{X}/\mathcal{X})_\bullet/\check{\mathrm{C}}(\mathrm{Y}/\mathcal{Y})_\bullet} \coloneqq \Big[\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathrm{D}_{\mathrm{X}\times^{\mathbf{R}}_{\mathcal{X}}\mathrm{X}/\mathrm{Y}\times^{\mathbf{R}}_{\mathcal{Y}}\mathrm{Y}} \rightrightarrows \mathrm{D}_{\mathrm{X}/\mathrm{Y}}\Big].$$

Without choosing a presentation, it can be described simply as the Weil restriction

$$\mathrm{D}_{\mathcal{X}/\mathcal{Y}} = \mathrm{Res}_{\mathcal{Y}/\mathcal{Y}\times\mathbf{A}^1}(\mathcal{X})$$

of $\mathcal{X}$ along $\mathcal{Y} = \mathcal{Y} \times \{0\} \to \mathcal{Y} \times \mathbf{A}^1$. Details will be provided elsewhere.

2. Motivic Borel–Moore homology of derived stacks

In this section we construct, given a "coefficient" $\mathcal{F}$ over a derived Artin stack $\mathcal{S}$, a (relative) Borel–Moore homology theory with coefficients in $\mathcal{F}$. The main example is $\mathcal{F} = \mathbf{Q}_{\mathcal{S}}$, the rational motivic cohomology spectrum. The construction requires a formalism of six operations on derived Artin stacks such as that developed in Appendix A.

2.1. Definition and examples.

Let $\mathcal{S}$ be a derived Artin stack and let $\mathcal{F} \in \mathrm{SH}_{\text{ét}}(\mathcal{S})$ be an étale motivic spectrum (see Appendix A).

Definition 2.1. *For a derived Artin stack $\mathcal{X}$ locally of finite type over $\mathcal{S}$ with structural morphism $f : \mathcal{X} \to \mathcal{S}$, we define Borel–Moore homology with coefficients in $\mathcal{F}$ by the formula*

$$\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) = \mathrm{Hom}_{\mathrm{SH}_{\text{ét}}(\mathcal{S})}(\mathbf{1}_{\mathcal{S}}(r)[s], f_* f^! \mathcal{F}), \qquad r, s \in \mathbf{Z} \tag{2.2}$$

where $\mathbf{1}_{\mathcal{S}} \in \mathrm{SH}_{\text{ét}}(\mathcal{S})$ is the monoidal unit. Similarly we define cohomology with coefficients in $\mathcal{F}$ by

$$\mathrm{H}^s(\mathcal{X}, \mathcal{F}(r)) = \mathrm{Hom}_{\mathrm{SH}_{\text{ét}}(\mathcal{S})}(\mathbf{1}_{\mathcal{S}}, f_* f^* \mathcal{F}(r)[s]) \tag{2.3}$$

for any derived Artin stack $\mathcal{X}$ over $\mathcal{S}$.

The observation that $\mathrm{H}^s(\mathcal{X}, \mathcal{F}(r)) = \mathrm{H}^{\mathrm{BM}}_{-s}(\mathcal{X}/\mathcal{X}, \mathcal{F}(-r))$ (by adjunction) allows us to pass freely from Borel–Moore homology statements to their cohomological counterparts, which is why we generally stick with the former perspective. For an immersion $i : \mathcal{Y} \to \mathcal{X}$, we have also cohomology with support:

$$\mathrm{H}^s_{\mathcal{Y}}(\mathcal{X}, \mathcal{F}(r)) = \mathrm{H}^{\mathrm{BM}}_{-s}(\mathcal{X}/\mathcal{Y}, \mathcal{F}(-r)). \tag{2.4}$$

Remark 2.5. The Borel–Moore homology groups $\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r))$ only depend on the *homotopy category* (underlying triangulated category) of the stable ∞-category $\mathrm{SH}_{\text{ét}}(\mathcal{S})$. A more refined object is the *spectrum* (in the sense of homotopical algebra)

$$\mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \coloneqq \mathrm{Maps}_{\mathrm{SH}_{\text{ét}}(\mathcal{S})}(\mathbf{1}_{\mathrm{S}}(r), f_* f^! \mathcal{F}),$$

defined using the spectral enrichment of $\mathrm{SH}_{\text{ét}}(\mathcal{S})$. The groups $\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r))$ are the homotopy groups $\pi_s \mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r))$. Similarly, there is a cohomology spectrum

$$\mathbf{R}\Gamma(\mathcal{X}, \mathcal{F}(r)) \coloneqq \mathrm{Maps}_{\mathrm{SH}_{\text{ét}}(\mathcal{S})}(\mathbf{1}_{\mathrm{S}}(r), f_* f^* \mathcal{F}).$$

Remark 2.6. Let $\mathcal{S} = \mathrm{S}$ be a derived algebraic space and $\mathcal{X}$ a locally of finite type derived Artin stack over S. The formula (A.4) implies that the Borel–Moore spectra $\mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}/\mathrm{S}, \mathcal{F}(r))$ can be computed by the homotopy limit

$$\mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}/\mathrm{S}, \mathcal{F}(r)) = \varprojlim_u \mathbf{R}\Gamma^{\mathrm{BM}}(\mathrm{X}/\mathrm{S}, \mathcal{F}(r + d_u))[-2d_u] \tag{2.7}$$

over the ∞-category of smooth morphisms $u : \mathrm{X} \to \mathcal{X}$ with X a scheme, where d_u is the relative dimension of u. Similarly, the cohomology spectrum is computed as the homotopy limit

$$\mathbf{R}\Gamma(\mathcal{X}, \mathcal{F}(r)) = \varprojlim_{u} \mathbf{R}\Gamma(\mathrm{X}, \mathcal{F}(r)). \tag{2.8}$$

Alternatively, we can fix a smooth atlas $\mathrm{X} \to \mathcal{X}$ and use (A.3) to write $\mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}/\mathrm{S}, \mathcal{F}(r))$ as the homotopy limit or totalization of the cosimplicial diagram

$$\mathbf{R}\Gamma^{\mathrm{BM}}(\mathrm{X}/\mathrm{S}, \mathcal{F}(r+d))[-2d] \rightrightarrows \mathbf{R}\Gamma^{\mathrm{BM}}(\mathrm{X} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathrm{X}/\mathrm{S}, \mathcal{F}(r+2d))[-4d] \tag{2.9}$$
$$\rightrightarrows\!\!\!\!\to \mathbf{R}\Gamma^{\mathrm{BM}}(\mathrm{X} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathrm{X} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathrm{X}/\mathrm{S}, \mathcal{F}(r+3d)[-6d]) \rightrightarrows\!\!\!\!\rightrightarrows \cdots$$

where $d = \mathrm{vd}(\mathrm{X}/\mathcal{X})$, and again similarly for $\mathbf{R}\Gamma(\mathcal{X}, \mathcal{F}(r))$.

Example 2.10. Let $\mathbf{Q}$ denote the rational motivic cohomology spectrum over $\mathrm{Spec}(\mathbf{Z})$ (see [CD1, Chap. 14], [Sp]). It satisfies étale (hyper)descent [CD2, Prop. 2.2.10], so for any derived Artin stack $\mathcal{S}$ we may define $\mathbf{Q}_{\mathcal{S}}$ as its inverse image along $\mathcal{S} \to \mathrm{Spec}(\mathbf{Z})$. The groups

$$\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathbf{Q}(r)), \quad \mathrm{H}^s(\mathcal{X}, \mathbf{Q}(r))$$

are simply called the (rational) *motivic Borel–Moore homology* and *motivic cohomology* groups. If $\mathcal{S}$ is the spectrum of a field k and $\mathcal{X} = \mathrm{X}$ is a quasi-projective classical scheme, then motivic Borel–Moore homology is computed as the cohomology of Bloch's cycle complex; in particular

$$\mathrm{H}^{\mathrm{BM}}_{2n}(\mathrm{X}/\mathrm{Spec}(k), \mathbf{Q}(n)) = \mathrm{A}_n(\mathrm{X})_{\mathbf{Q}}.$$

For X a classical scheme locally of finite type over a field k, it is computed by the Zariski hypercohomology of the same complex. More precisely, $\mathbf{R}\Gamma^{\mathrm{BM}}(\mathrm{X}/\mathrm{Spec}(k), \mathbf{Q}(r))$ is the Zariski localization of Bloch's cycle complex. Thus for $\mathcal{X}$ an Artin stack locally of finite type over k, $\mathrm{H}^{\mathrm{BM}}_*(\mathcal{X}/\mathrm{Spec}(k), \mathbf{Q}(r))$ is computed according to the formula (2.9) by the étale hypercohomology of the complex

$$\varprojlim_{[m]\in\mathbf{\Delta}} z_{r+md}(\mathrm{X} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \cdots \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathrm{X}, *)_{\mathbf{Q}}[-2md]$$

where there are m terms in the fibred product. These are thus the same as the rational higher Chow groups defined by Joshua [Jo], and they agree with the rationalization of Kresch's Chow groups [Kr].

Example 2.11. Integrally, we can take the étale motivic cohomology spectrum $\mathbf{Z}^{\text{ét}}$. More generally for every commutative ring Λ, let $\Lambda^{\text{ét}}$ denote the étale hyperlocalization of the Λ-linear motivic cohomology spectrum and write $\Lambda^{\text{ét}}_{\mathcal{S}}$ for its inverse image to any derived Artin stack $\mathcal{S}$ (along the structural morphism $\mathcal{S} \to \mathrm{Spec}(\mathbf{Z})$). The resulting groups are called *étale motivic Borel–Moore homology* and *étale motivic cohomology* (or "Lichtenbaum motivic cohomology"), respectively:

$$\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \Lambda^{\text{ét}}(r)), \quad \mathrm{H}^s(\mathcal{X}, \Lambda^{\text{ét}}(r)).$$

Rationally these give back the groups just defined above since $\mathbf{Q}_{\mathcal{S}}$ already satisfies étale hyperdescent. With finite coefficients $\Lambda = \mathbf{Z}/n\mathbf{Z}$ it follows from Remark 2.6 and [CD2, Thm. 4.5.2] that these agree with étale Borel–Moore homology and étale cohomology [La, Ol, LO1], over classical Artin stacks with n invertible. Taking $\mathbf{Z}^{\wedge}_{\ell,\mathcal{S}}$ to be the ℓ-adic completion of $\mathbf{Z}_{\mathcal{S}}$ as in [CD2, Subsect. 7.2], for a prime ℓ, we also recover ℓ-adic Borel–Moore homology and cohomology, respectively.

Example 2.12. Let MGL denote Voevodsky's algebraic cobordism spectrum. If X is a smooth algebraic space over a perfect field k, then the cohomology groups $\mathrm{H}^{2n}(\mathrm{X}, \mathrm{MGL}(n))$ are computed for $n \geqslant 0$ by the Nisnevich hypercohomology of a certain presheaf of spectra built out of finite quasi-smooth derived schemes over X [EHKSY]. If k is of characteristic zero, then they are identified with Levine–Morel's algebraic cobordism $\Omega^n(\mathrm{X})$, and moreover the Borel–Moore homology groups $\mathrm{H}^{\mathrm{BM}}_{2n}(\mathrm{X}, \mathrm{MGL}(n))$ are identified with $\Omega_n(\mathrm{X})$ for all $n \in \mathbf{Z}$, also for X singular [Le1].

Let $\mathrm{MGL}^{\text{ét}}$ denote the étale hyperlocalization of MGL. For a derived Artin stack $\mathcal{S}$, let $\mathrm{MGL}^{\text{ét}}_{\mathcal{S}}$ denote the inverse image along the structural morphism $\mathcal{S} \to \mathrm{Spec}(\mathbf{Z})$. This gives étale algebraic cobordism and bordism groups for derived Artin stacks

$$\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathrm{MGL}^{\text{ét}}(r)), \quad \mathrm{H}^s(\mathcal{X}, \mathrm{MGL}^{\text{ét}}(r)).$$

If $\mathcal{X}$ is smooth over a perfect field k, then $\mathrm{H}^{2n}(\mathcal{X}, \mathrm{MGL}^{\text{ét}}(n))$ is computed using the construction of Remark 2.6 by the same presheaf of spectra mentioned above (for $n \geqslant 0$).

The rationalization $\mathrm{MGL}_{\mathbf{Q}}$ already satisfies étale hyperdescent ($\mathrm{MGL}_{\mathbf{Q}} \simeq \mathrm{MGL}^{\text{ét}}_{\mathbf{Q}}$) and is identified with

$$\mathrm{MGL}_{\mathbf{Q}} \simeq \mathbf{Q}[c_1, c_2, \ldots],$$

where c_i is a generator of bidegree $(2i, i)$, by [NSØ]. We thus define $\mathrm{MGL}_{\mathbf{Q},\mathcal{S}}$ as the inverse image of $\mathrm{MGL}_{\mathbf{Q}}$ for any derived Artin stack $\mathcal{S}$. There are canonical maps

$$\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathrm{MGL}^{\text{ét}}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathrm{MGL}_{\mathbf{Q}}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathbf{Q}(r))$$

for all $\mathcal{X}$ locally of finite type over $\mathcal{S}$.

Example 2.13. Let $\mathrm{KGL}^{\text{ét}}_{\mathcal{S}}$ denote the étale hyperlocalization of the algebraic K-theory spectrum. Assuming that $\mathcal{S}$ is a regular (classical) stack, such as the spectrum of $\mathbf{Z}$ or a field, the Borel–Moore homology represented by $\mathrm{KGL}^{\text{ét}}_{\mathcal{S}}$ coincides with étale hypercohomology with coefficients in G-theory, and the proper covariance and smooth Gysin maps are compatible with the respective intrinsic operations in G-theory [Ji, Cor. 3.3.7]. Note that in this case the formula of Remark 2.6 simplifies since there are Bott periodicity isomorphisms

$$\mathrm{KGL}(n)[2n] \simeq \mathrm{KGL}$$

for all $n \in \mathbf{Z}$.

Remark 2.14. If we restrict to derived schemes or algebraic spaces, then we are allowed to take coefficients that do not satisfy étale descent, such as the *integral* motivic cohomology spectrum $\mathbf{Z}$ or the algebraic cobordism spectrum MGL. Indeed, for derived algebraic spaces the formalism of six operations is already available before imposing étale descent (see Subsect. A.1). The basic operations discussed in the next section will also carry over to that setting. Moreover, the fundamental class can still be defined at least for *smoothable* quasi-smooth morphisms (Variant 3.11).

2.2. **Basic operations.** The formalism of six operations (see Appendix A) immediately yields the following structure on Borel–Moore homology groups. Here $\mathcal{F}$ is any coefficient defined over $\mathcal{S}$, though for simplicity we assume that $\mathcal{F}$ is *multiplicative* (a motivic ring spectrum) and *oriented*[1]. In particular there is a unit element

$$1 \in \mathrm{H}_0^{\mathrm{BM}}(\mathcal{X}/\mathcal{X}, \mathcal{F}) = \mathrm{H}^0(\mathcal{X}, \mathcal{F})$$

induced by the unit $\eta_{\mathcal{F}} : \mathbf{1}_{\mathcal{S}} \to \mathcal{F}$.

2.2.1. *Proper direct image.* If $f : \mathcal{X} \to \mathcal{Y}$ is a representable proper morphism of derived Artin stacks locally of finite type over $\mathcal{S}$, then there are functorial direct image homomorphisms

$$f_* : \mathrm{H}_s^{\mathrm{BM}}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}_s^{\mathrm{BM}}(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)).$$

These are induced by the co-unit $f_* f^! = f_! f^! \to \mathrm{id}$. If $\mathcal{F}$ satisfies h-descent, e.g., $\mathcal{F}$ is $\mathbf{Q}$ or $\mathrm{MGL}_{\mathbf{Q}}$, then by Theorem A.7 this extends to arbitrary proper morphisms $f : \mathcal{X} \to \mathcal{Y}$ as long as $\mathcal{X}$ and $\mathcal{Y}$ are Deligne–Mumford (see Example A.8 for some milder assumptions that work).

2.2.2. *Smooth contravariance.* If $f : \mathcal{X} \to \mathcal{Y}$ is a smooth morphism of relative dimension d between derived Artin stacks locally of finite type over $\mathcal{S}$, then there are functorial Gysin homomorphisms

$$f^! : \mathrm{H}_s^{\mathrm{BM}}(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}_{s+2d}^{\mathrm{BM}}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r+d)).$$

These are compatible with proper direct images by a base change formula. They are induced by the co-trace transformation $\mathrm{id} \to f_* \Sigma^{-\mathcal{L}_{\mathcal{X}/\mathcal{Y}}} f^!$, right transpose of the purity equivalence $\Sigma^{\mathcal{L}_{\mathcal{X}/\mathcal{Y}}} f^* = f^!$ (Theorem A.13).

2.2.3. *Change of base.* If $f : \mathcal{T} \to \mathcal{S}$ is a morphism of derived Artin stacks and $\mathcal{X}$ is a derived Artin stack locally of finite type over S, then there are change of base homomorphisms

$$f^* : \mathrm{H}_s^{\mathrm{BM}}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}_s^{\mathrm{BM}}(\mathcal{X}_{\mathcal{T}}/\mathcal{T}, \mathcal{F}(r)),$$

where $\mathcal{X}_{\mathcal{T}} = \mathcal{X} \times_{\mathcal{S}}^{\mathbf{R}} \mathcal{T}$ is the derived fibred product. More generally, for any commutative square

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta"] & \mathcal{T} \arrow[d, "f"] \\
\mathcal{X} \arrow[r] & \mathcal{S}
\end{tikzcd}$$

which is cartesian on underlying classical stacks, there are homomorphisms

$$f_\Delta^* : \mathrm{H}_s^{\mathrm{BM}}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}_s^{\mathrm{BM}}(\mathcal{Y}/\mathcal{T}, \mathcal{F}(r)).$$

[1] This essentially amounts to admitting a theory of Chern classes. The constructions also work for non-oriented spectra [DJK], but are more notationally complex due to the necessity of grading by K-theory classes instead of just pairs of integers. We are only interested in oriented examples here.

These are induced by the unit map $\mathrm{id} \to f_* f^*$ and the base change formula (Theorem A.5).

Remark 2.15. Note that f^* always denotes contravariant functoriality in the *base* (change of base homomorphisms, 2.2.3), while $f^!$ denotes contravariant functoriality in the *source* (Gysin homomorphisms, 2.2.2). Potentially the notation also clashes with that of the six operations (Appendix A), but there should be no risk of confusion.

2.2.4. *Top Chern class.* Let $\mathcal{E}$ be a finite locally free sheaf of rank r on a derived Artin stack $\mathcal{X}$ over $\mathcal{S}$. Then there is a top Chern class (Euler class)

$$c_r(\mathcal{E}) \in \mathrm{H}^{2r}(\mathcal{X}, \mathcal{F}(r)).$$

This is induced by the Euler transformation $\mathrm{id} \to \Sigma^{\mathcal{E}}$ (Construction A.16). There is a general theory of Chern classes $c_i(\mathcal{E})$ (when $\mathcal{F}$ is oriented), as in [De1, Sect. 2.1], but we will not need it here.

2.2.5. *Composition product.* Given a derived Artin stack $\mathcal{T}$ locally of finite type over $\mathcal{S}$ and a derived Artin stack $\mathcal{X}$ locally of finite type over $\mathcal{T}$, there is a pairing

$$\circ : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{T}, \mathcal{F}(r)) \otimes \mathrm{H}^{\mathrm{BM}}_{s'}(\mathcal{T}/\mathcal{S}, \mathcal{F}(r')) \to \mathrm{H}^{\mathrm{BM}}_{s+s'}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r+r')).$$

This comes from the multiplication map $m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$, see [DJK, 2.2.7(4)] for details.

Special cases of the composition product are cap and cup products:

2.2.6. *Cap product.* Given a derived Artin stack $\mathcal{X}$ locally of finite type over $\mathcal{S}$, there is a pairing

$$\cap : \mathrm{H}^s(\mathcal{X}, \mathcal{F}(r)) \otimes \mathrm{H}^{\mathrm{BM}}_{s'}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r')) \to \mathrm{H}^{\mathrm{BM}}_{s'-s}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r'-r)). \tag{2.16}$$

2.2.7. *Cup product.* Given a derived Artin stack $\mathcal{X}$ over $\mathcal{S}$, there is a pairing

$$\cup : \mathrm{H}^s(\mathcal{X}, \mathcal{F}(r)) \otimes \mathrm{H}^{s'}(\mathcal{X}, \mathcal{F}(r')) \to \mathrm{H}^{s+s'}(\mathcal{X}, \mathcal{F}(r+r')). \tag{2.17}$$

From now on, whenever we consider a Borel–Moore homology group $\mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r))$, we will implicitly assume that $\mathcal{X}$ is locally of finite type over $\mathcal{S}$ (so that the exceptional inverse image functor $f^!$ exists, see Subsect. A.2).

2.3. **Basic compatibilities.** The operations on Borel–Moore homology are subject to the following compatibilities, direct analogues of the axioms of a bivariant theory in the sense of Fulton–MacPherson [FM, Sect. 2.2].

2.3.1. *Change of base and composition product.* Suppose given a commutative diagram

$$\begin{tikzcd} \mathcal{X}_{\mathcal{T}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}_{\mathcal{T}} \arrow[r, "g"] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathcal{T} \arrow[r, "f"] & \mathcal{S} \end{tikzcd}$$

where the squares are cartesian. Then for classes $\alpha \in \mathrm{H}^{\mathrm{BM}}_r(\mathcal{X}/\mathcal{Y}, \mathcal{F}(s))$, $\beta \in \mathrm{H}^{\mathrm{BM}}_{r'}(\mathcal{Y}/\mathcal{S}, \mathcal{F}(s'))$, we have

$$f^*(\alpha \circ \beta) = g^*(\alpha) \circ f^*(\beta)$$

in $\mathrm{H}^{\mathrm{BM}}_{s+s'}(\mathcal{X}_{\mathcal{T}}/\mathcal{T}, \mathcal{F}(r + r'))$.

2.3.2. *Change of base and direct image.* Suppose given a commutative diagram

$$\begin{tikzcd} \mathcal{X}_{\mathcal{T}} \arrow[r] \arrow[d, "h'"] & \mathcal{X} \arrow[d, "h"] \\ \mathcal{Y}_{\mathcal{T}} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathcal{T} \arrow[r, "f"] & \mathcal{S} \end{tikzcd}$$

where the squares are cartesian. Then for any class $\alpha \in \mathrm{H}^{\mathrm{BM}}_r(\mathcal{X}/\mathcal{S}, \mathcal{F}(s))$, we have

$$f^* h_*(\alpha) = h'_* f^*(\alpha)$$

in $\mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}_{\mathcal{T}}/\mathcal{T}, \mathcal{F}(r))$.

2.3.3. *Direct image and composition product (on the right).* Suppose given a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{S} \arrow[r] & \mathcal{T} \end{tikzcd}$$

with f representable and proper. Then for classes $\alpha \in \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r))$, $\beta \in \mathrm{H}^{\mathrm{BM}}_{s'}(\mathcal{S}/\mathcal{T}, \mathcal{F}(r'))$, we have

$$f_*(\alpha) \circ \beta = f_*(\alpha \circ \beta)$$

in $\mathrm{H}^{\mathrm{BM}}_{s+s'}(\mathcal{Y}/\mathcal{T}, \mathcal{F}(r + r'))$.

2.3.4. *Direct image and composition product (on the left).* Suppose given a commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "g"] \arrow[d] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r, "f"] \arrow[dr] & \mathcal{Y} \arrow[d] \\
& \mathcal{S}
\end{tikzcd}$$

where the square is cartesian. Then for classes $\alpha \in \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r))$ and $\beta \in \mathrm{H}^{\mathrm{BM}}_{s'}(\mathcal{Y}'/\mathcal{Y}, \mathcal{F}(r'))$, we have

$$\beta \circ f_*(\alpha) = g_*(f^*(\beta) \circ \alpha)$$

in $\mathrm{H}^{\mathrm{BM}}_{s+s'}(\mathcal{Y}'/\mathcal{S}, \mathcal{F}(r+r'))$.

2.4. **Properties.** The following two statements follow immediately from Theorem A.9.

Theorem 2.18 (Localization)**.** *Let $i : \mathcal{Z} \to \mathcal{X}$ be a closed immersion of derived Artin stacks over $\mathcal{S}$, with open complement $j : \mathcal{U} \to \mathcal{X}$. Then for every integer r there is a long exact sequence*

$$\cdots \xrightarrow{\partial} \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathcal{Z}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{i_*} \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{j^!} \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathcal{U}/\mathcal{S}, \mathcal{F}(r))$$
$$\xrightarrow{\partial} \mathrm{H}^{\mathrm{BM}}_{s}(\mathcal{Z}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{i_*} \mathrm{H}^{\mathrm{BM}}_{s}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{j^!} \cdots$$

Theorem 2.19 (Derived invariance)**.** *Let $\mathcal{X}$ be a derived Artin stack over $\mathcal{S}$.*

(i) *Let $i_{\mathcal{S}} : \mathcal{S}_{\mathrm{cl}} \to \mathcal{S}$ denote the inclusion of the underlying classical stack. Then the change of base homomorphisms*

$$i_{\mathcal{S}}^* : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X} \overset{\mathbf{R}}{\underset{\mathcal{S}}{\times}} \mathcal{S}_{\mathrm{cl}}/\mathcal{S}_{\mathrm{cl}}, \mathcal{F}(r))$$

are bijective for all $r, s \in \mathbf{Z}$.

(ii) *Let $i_{\mathcal{X}} : \mathcal{X}_{\mathrm{cl}} \to \mathcal{X}$ denote the inclusion of the underlying classical stack. Then the direct image homomorphisms*

$$(i_{\mathcal{X}})_* : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}_{\mathrm{cl}}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r))$$

are bijective for all $r, s \in \mathbf{Z}$.

Proposition 2.20 (Homotopy invariance)**.** *Let $\mathcal{X}$ be a derived Artin stack over $\mathcal{S}$. For a perfect complex $\mathcal{E}$ on $\mathcal{X}$ of Tor-amplitude $[-k, 1]$, where $k \geqslant -1$, denote by $\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]) \to \mathcal{X}$ the associated vector bundle stack. Then for every $r, s \in \mathbf{Z}$ there is a canonical isomorphism*

$$\pi^! : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{X}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_{s-2d}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1])/\mathcal{S}, \mathcal{F}(r-d)),$$

where d is the virtual rank of $\mathcal{E}$.

Proof. The map is induced by the natural transformation

$$f_* f^!(\mathcal{F}) \xrightarrow{\text{unit}} f_*\pi_*\pi^* f^!(\mathcal{F}) \xrightarrow{\text{pur}_\pi} f_*\pi_*\Sigma^{-\mathcal{L}_\pi}\pi^! f^!(\mathcal{F}) = f_*\pi_*\Sigma^{\pi^*(\mathcal{E})}\pi^! f^!(\mathcal{F}),$$

which is invertible by Props. A.10 and A.13. ☐

Definition 2.21. *Let $\mathcal{X}$ be a derived Artin stack over $\mathcal{S}$. If $\mathcal{X}$ is smooth of relative dimension d, then there is a relative* fundamental class

$$[\mathcal{X}/\mathcal{S}] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}/\mathcal{S}, \mathcal{F}(d))$$

defined as the image of the unit by the Gysin map $f^!$ (2.2.2). More explicitly, this class is induced by the morphism

$$\mathbf{1}_{\mathcal{S}} \to f_*\Sigma^{-\mathcal{L}_{\mathcal{X}/\mathcal{S}}} f^!(\mathcal{F}) = f_* f^!(\mathcal{F})(-d)[-2d]$$

coming by adjunction from the purity isomorphism $f^! = \Sigma^{\mathcal{L}_{\mathcal{X}/\mathcal{S}}} f^*$ (Theorem A.13), where $f : \mathcal{X} \to \mathcal{S}$ is the structural morphism.

Remark 2.22. For $\mathcal{X}$ smooth over $\mathcal{S}$ as above, the fundamental class $[\mathcal{X}/\mathcal{S}]$ is "classical", in the sense that it is insensitive to the derived structure. That is, under the canonical isomorphism (Theorem 2.19)

$$\mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}/\mathcal{S}, \mathcal{F}(d)) \simeq \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}_{\mathrm{cl}}/\mathcal{S}_{\mathrm{cl}}, \mathcal{F}(d)),$$

the class $[\mathcal{X}/\mathcal{S}]$ corresponds to $[\mathcal{X}_{\mathrm{cl}}/\mathcal{S}_{\mathrm{cl}}]$, the fundamental class of the morphism $\mathcal{X}_{\mathrm{cl}} \to \mathcal{S}_{\mathrm{cl}}$. Note that this makes sense because the latter is again a smooth morphism of relative dimension d. This is in contrast to the more general case of quasi-smooth morphisms (Sect. 3).

Theorem 2.23 (Poincaré duality)**.** *Let $\mathcal{X}$ be a smooth derived Artin stack over $\mathcal{S}$. Then cap product (2.2.6) with the fundamental class $[\mathcal{X}/\mathcal{S}]$ induces a canonical isomorphism*

$$\mathrm{H}^s(\mathcal{X}, \mathcal{F}(r)) \xrightarrow{\cap[\mathcal{X}/\mathcal{S}]} \mathrm{H}^{\mathrm{BM}}_{2d-s}(\mathcal{X}/\mathcal{S}, \mathcal{F}(d-r))$$

for all $r, s, \in \mathbf{Z}$.

Proof. Unraveling definitions, this follows from the fact that the morphism $\mathbf{1}_{\mathcal{S}} \to f_* f^!(\mathcal{F})(-d)[-2d]$ defining $[\mathcal{X}/\mathcal{S}]$ is the "right transpose" of an isomorphism. See the discussion after [DJK, Def. 2.3.11]. ☐

3. Fundamental classes

We develop some basic tools of intersection theory, namely the specialization and Gysin maps, for Borel–Moore homology of derived Artin stacks. We follow the constructions of Déglise–Jin–Khan [DJK] closely, the main difference being the introduction of the normal bundle stack to handle the cases of quasi-smooth closed immersions and smooth morphisms simultaneously.

3.1. **Construction.** Let $\mathcal{S}$ be a derived Artin stack and fix a coefficient $\mathcal{F}$ as in Sect. 2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived Artin stacks over $\mathcal{S}$, say of relative virtual dimension d. Denote by $\mathrm{N}_{\mathcal{X}/\mathcal{Y}}$ the normal bundle stack (Definition 1.1).

Construction 3.1 (Specialization map)**.** We define a specialization map

$$\mathrm{sp}_{\mathcal{X}/\mathcal{Y}} : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathrm{N}_{\mathcal{X}/\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)) \tag{3.2}$$

for all $r, s \in \mathbf{Z}$. First, the localization long exact sequence associated to the closed immersion $\mathcal{Y} = \mathcal{Y} \times \{0\} \to \mathcal{Y} \times \mathbf{A}^1$ splits into short exact sequences

$$0 \to \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathbf{A}^1_{\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathbf{G}_{m,\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{\partial} \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to 0.$$

The homomorphism ∂ admits a canonical section

$$\gamma_t : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathbf{G}_{m,\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)), \tag{3.3}$$

see [DJK, 3.2.2] for details.

Let $\mathrm{D}_{\mathcal{X}/\mathcal{Y}}$ be the deformation space (Subsect. 1.4). Let $i : \mathrm{N}_{\mathcal{X}/\mathcal{Y}} \to \mathrm{D}_{\mathcal{X}/\mathcal{Y}}$ denote the inclusion of the exceptional fibre and $j : \mathcal{Y} \times \mathbf{G}_m \to \mathrm{D}_{\mathcal{X}/\mathcal{Y}}$ its complement. The associated localization long exact sequence has boundary map

$$\partial : \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathbf{G}_{m,\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathrm{N}_{\mathcal{X}/\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)),$$

and we define (3.2) as the composite

$$\mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{\gamma_t} \mathrm{H}^{\mathrm{BM}}_{s+1}(\mathbf{G}_{m,\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{\partial} \mathrm{H}^{\mathrm{BM}}_s(\mathrm{N}_{\mathcal{X}/\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)).$$

Construction 3.4 (Gysin map)**.** We now construct the Gysin map

$$f^! : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r+d)), \tag{3.5}$$

where $f : \mathcal{X} \to \mathcal{Y}$ is as above. Let $\pi : \mathrm{N}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ denote the projection. The Gysin map (3.5) is the composite

$$\mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{\mathrm{sp}_{\mathcal{X}/\mathcal{Y}}} \mathrm{H}^{\mathrm{BM}}_s(\mathrm{N}_{\mathcal{X}/\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)) \xrightarrow{(\pi^!)^{-1}} \mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r+d)).$$

where $\pi^!$ is the isomorphism of Proposition 2.20.

Construction 3.6 (Fundamental class)**.** The (relative) *fundamental class* of $f : \mathcal{X} \to \mathcal{Y}$ is the class

$$[\mathcal{X}/\mathcal{Y}] \coloneqq f^!(1) \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}/\mathcal{Y}, \mathcal{F}(d))$$

which is the image of $1 \in \mathrm{H}^{\mathrm{BM}}_0(\mathcal{Y}/\mathcal{Y}, \mathcal{F})$. When f is smooth, this is the fundamental class already defined (see before Theorem 2.23).

The (relative) *virtual* fundamental class is defined to be the unique class

$$[\mathcal{X}/\mathcal{Y}]^{\mathrm{vir}} \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}_{\mathrm{cl}}, \mathcal{F}(d))$$

corresponding to $[\mathcal{X}/\mathcal{Y}]$ under the canonical isomorphisms of Theorem 2.19.

Remark 3.7. The Gysin map and fundamental class are essentially interchangeable data, as we can recover the former via the composition product (2.2.5) with $[\mathcal{X}/\mathcal{Y}]$:

$$f^!(x) = [\mathcal{X}/\mathcal{Y}] \circ x \in \mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r+d))$$

for all $x \in \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r))$.

Remark 3.8 (Purity transformation)**.** In terms of the six operations, the fundamental class can be interpreted as a canonical natural transformation

$$\mathrm{pur}_f : \Sigma^{\mathcal{L}_{\mathcal{X}/\mathcal{Y}}} f^* \to f^! \tag{3.9}$$

of functors $\mathrm{SH}_{\text{ét}}(\mathcal{Y}) \to \mathrm{SH}_{\text{ét}}(\mathcal{X})$, where $\Sigma^{\mathcal{L}_{\mathcal{X}/\mathcal{Y}}}$ is the operation defined in (A.12). Through the orientation of $\mathcal{F}$, this induces a canonical isomorphism

$$f^*(\mathcal{F})(d)[2d] \simeq \Sigma^{\mathcal{L}_{\mathcal{X}/\mathcal{Y}}} f^*(\mathcal{F}) \to f^!(\mathcal{F}). \tag{3.10}$$

See [DJK, Subsects. 2.5, 4.3] for details on this perspective.

Variant 3.11. Let's restrict our attention to derived schemes or algebraic spaces. As explained in Remark 2.14, there is a well-behaved theory of Borel–Moore homology with coefficients in any $\mathcal{F}$, not necessarily satisfying étale descent (such as the integral motivic cohomology or algebraic cobordism spectrum). Following the constructions of [DJK, §3], we can still define the fundamental class $[\mathrm{X}/\mathrm{Y}] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{X}/\mathrm{Y}, \mathcal{F}(d))$ for *smoothable* quasi-smooth morphisms $f : \mathrm{X} \to \mathrm{Y}$ (where $d = \mathrm{vd}(\mathrm{X}/\mathrm{Y})$).

First let $i : \mathrm{Z} \to \mathrm{X}$ be a quasi-smooth closed immersion (or quasi-smooth unramified morphism [KR, §5.2]). In this case the normal bundle stack $\mathcal{N}_{\mathrm{Z}/\mathrm{X}}$ is a vector bundle (as opposed to a vector bundle stack), and Constructions 3.1 and 3.4 only involve derived algebraic spaces. Thus we get the fundamental class $[\mathrm{Z}/\mathrm{X}] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Z}/\mathrm{X}, \mathcal{F}(d))$, where $d = \mathrm{vd}(\mathrm{Z}/\mathrm{X})$. These fundamental classes also satisfy the properties asserted in the next section.

Now let $f : \mathrm{X} \to \mathrm{Y}$ be a smoothable quasi-smooth morphism of derived algebraic spaces, i.e., one that admits a global factorization

$$\mathrm{X} \xrightarrow{i} \mathrm{M} \xrightarrow{p} \mathrm{Y}$$

with p smooth and i a (quasi-smooth) closed immersion. Define the fundamental class $[\mathrm{X}/\mathrm{Y}] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{X}/\mathrm{Y}, \mathcal{F}(d))$, where $d = \mathrm{vd}(\mathrm{X}/\mathrm{Y})$, by

$$[\mathrm{X}/\mathrm{Y}] = [\mathrm{X}/\mathrm{M}] \circ [\mathrm{M}/\mathrm{Y}].$$

Exactly as in [DJK, §3.3], one verifies that this is independent of the factorization and that the resulting system of fundamental classes still satisfies the properties stated in the next subsection.

3.2. **Properties.**

We record the basic properties of the fundamental class. These could equivalently be stated for the Gysin maps.

Theorem 3.12 (Functoriality)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be quasi-smooth morphisms of derived Artin stacks, of relative virtual dimensions d and e, respectively. Then we have*

$$[\mathcal{X}/\mathcal{Y}] \circ [\mathcal{Y}/\mathcal{Z}] = [\mathcal{X}/\mathcal{Z}]$$

in $\mathrm{H}^{\mathrm{BM}}_{2d+2e}(\mathcal{X}/\mathcal{Z}, \mathcal{F}(d+e))$.

Use the double deformation space as in [DJK, Prop. 3.2.19].

Theorem 3.13 (Base change)**.** *Suppose given a cartesian square of derived Artin stacks*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g"] \arrow[d, "p"] & \mathcal{Y}' \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \tag{3.14}$$

over $\mathcal{S}$, where f is quasi-smooth. Then there is an equality

$$p^*[\mathcal{X}/\mathcal{Y}] = [\mathcal{X}'/\mathcal{Y}'] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}'/\mathcal{Y}', \mathcal{F}(d)),$$

where d is the relative virtual dimension of f (and hence of g).

This follows easily from the stability of the deformation space $\mathrm{D}_{\mathcal{X}/\mathcal{Y}}$ under base change (Theorem 1.3(ii)). More generally:

Proposition 3.15 (Excess intersection formula)**.** *Suppose given a commutative square of derived Artin stacks*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g"] \arrow[d, "p"] \arrow[dr, phantom, "\Delta"] & \mathcal{Y}' \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd} \tag{3.16}$$

over $\mathcal{S}$, where f and g are quasi-smooth. Assume that Δ is an excess intersection square*, i.e., that it is cartesian on underlying classical stacks and that the fibre $\mathcal{E}$ of the canonical map*

$$p^*\mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1] \to \mathcal{L}_{\mathcal{X}'/\mathcal{Y}'}[-1]$$

is a locally free $\mathcal{O}_{\mathrm{X}}$-module of finite rank. Then there is an equality

$$q^*_\Delta[\mathcal{X}/\mathcal{Y}] = c_r(\mathcal{E}) \cap [\mathcal{X}'/\mathcal{Y}'] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}'/\mathcal{Y}', \mathcal{F}(d)),$$

where q^*_Δ denotes the change of base homomorphism (2.2.3), $d = \mathrm{vd}(\mathcal{X}/\mathcal{Y})$, and $r = \mathrm{rk}(\mathcal{E})$.

Same as the proof of [DJK, Prop. 3.2.8]. We call $\mathcal{E}$ the *excess sheaf* associated to Δ. Note that its rank is $r = \mathrm{vd}(\mathcal{X}'/\mathcal{Y}') - \mathrm{vd}(\mathcal{X}/\mathcal{Y})$.

Corollary 3.17 (Self-intersection formula)**.** *Let $i : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth closed immersion of relative virtual codimension n. Consider the self-intersection square*

$$\begin{tikzcd} \mathcal{X} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Delta"] & \mathcal{X} \arrow[d, "i"] \\ \mathcal{X} \arrow[r, "i"'] & \mathcal{Y}. \end{tikzcd}$$

We have

$$i^*_\Delta[\mathcal{X}/\mathcal{Y}] = c_n(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) \in \mathrm{H}^{\mathrm{BM}}_{-2n}(\mathcal{X}/\mathcal{X}, \mathcal{F}(-n)) = \mathrm{H}^{2n}(\mathcal{X}, \mathcal{F}(n)),$$

where $\mathcal{N}_{\mathcal{X}/\mathcal{Y}} = \mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1]$ is the conormal sheaf.

Corollary 3.18 (Key formula)**.** *Let $i : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth closed immersion of relative virtual codimension n. Form the blow-up square* [KR, Thm. 4.1.5]*:*

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\Delta"] & \mathrm{Bl}_{\mathcal{X}/\mathcal{Y}} \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "i"'] & \mathcal{Y}, \end{tikzcd}$$

where $\mathcal{D} = \mathbf{P}_{\mathcal{X}}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}})$ is the virtual exceptional divisor. We have

$$q^*_\Delta[\mathcal{X}/\mathcal{Y}] = c_{n-1}(\mathcal{E}) \cap [\mathcal{D}/\mathrm{Bl}_{\mathcal{X}/\mathcal{Y}}] \in \mathrm{H}^{\mathrm{BM}}_{-2n}(\mathcal{D}/\mathrm{Bl}_{\mathcal{X}/\mathcal{Y}}, \mathcal{F}(-n)),$$

where $\mathcal{E}$ is the excess sheaf.

3.3. **Comparison with Behrend–Fantechi.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived 1-Artin stacks. Assume that f is representable by derived Deligne–Mumford stacks and $\mathcal{Y}$ is classical. In this case the virtual fundamental class

$$[\mathcal{X}/\mathcal{Y}]^{\mathrm{vir}} \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}, \mathcal{F}(d))$$

can also be defined using the approach of Behrend–Fantechi [BF]. Below, we give a variant of the construction of the Gysin map $f^!$ (3.5) which will visibly agree with the "virtual pullback" of Manolache [Ma]. By Corollary 3.12 of *op. cit.*, this will therefore identify our virtual fundamental class $[\mathcal{X}/\mathcal{Y}]^{\mathrm{vir}}$ with the construction of Behrend–Fantechi.

Let $\mathrm{C}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}}$ denote the relative intrinsic normal cone [BF, Sect. 7] of the morphism $\mathcal{X}_{\mathrm{cl}} \to \mathcal{X} \to \mathcal{Y}$, and let $\mathrm{D}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}}$ denote Kresch's deformation to the intrinsic normal cone [Ma, Thm. 2.31]. There is a commutative diagram

$$\begin{tikzcd} \mathrm{C}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}} \arrow[r] \arrow[d, "a"] & \mathrm{D}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}} \arrow[d] & \mathcal{Y} \times \mathbf{G}_m \arrow[l] \arrow[d, equal] \\ \mathrm{N}_{\mathcal{X}/\mathcal{Y}} \arrow[r] & \mathrm{D}_{\mathcal{X}/\mathcal{Y}} & \mathcal{Y} \times \mathbf{G}_m \arrow[l] \end{tikzcd}$$

where the vertical arrows are closed immersions. Using the upper row, one constructs just as in (3.2) a specialization map

$$\mathrm{sp}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}} : \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathrm{C}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r)).$$

By naturality of the localization triangle with respect to proper covariance (e.g. [DJK, Prop. 2.2.10]), we have an equality

$$a_* \circ \mathrm{sp}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}} = \mathrm{sp}_{\mathcal{X}/\mathcal{Y}}$$

of morphisms $\mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathrm{N}_{\mathcal{X}/\mathcal{Y}}/\mathcal{S}, \mathcal{F}(r))$. In particular we get

$$f^! = (\pi^!)^{-1} \circ \mathrm{sp}_{\mathcal{X}/\mathcal{Y}} = (\pi^!)^{-1} \circ a_* \circ \mathrm{sp}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}},$$

where $\pi : \mathrm{N}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is the projection. Now the right-hand side is precisely the virtual pullback $f^!_{\mathrm{N}_{\mathcal{X}/\mathcal{Y}}}$ [Ma, Constr. 3.6] constructed with respect to the vector bundle stack $\mathrm{N}_{\mathcal{X}/\mathcal{Y}}$.

3.4. **Non-transverse Bézout theorem.** Let $f : \mathcal{Z} \to \mathcal{X}$ be a morphism of derived Artin stacks over $\mathcal{S}$. Suppose that f is quasi-smooth of relative virtual dimension d. The fundamental class $[\mathcal{Z}/\mathcal{X}]$ induces a cohomological Gysin map

$$f_! : \mathrm{H}^r(\mathcal{Z}, \mathcal{F}(s)) \to \mathrm{H}^{r-2d}_{\mathcal{Z}}(\mathcal{X}, \mathcal{F}(s-d)) \tag{3.19}$$

where the target is the cohomology of $\mathcal{X}$ with support in $\mathcal{Z}$ (2.4). This map is the composite

$$\mathrm{H}^{\mathrm{BM}}_{-r}(\mathcal{Z}/\mathcal{Z}, \mathcal{F}(-s)) \xrightarrow{\circ[\mathcal{Z}/\mathcal{X}]} \mathrm{H}^{\mathrm{BM}}_{-r+2d}(\mathcal{Z}/\mathcal{X}, \mathcal{F}(-s+d)).$$

Composing further with the Borel–Moore direct image (2.2.1)

$$f_* : \mathrm{H}^{\mathrm{BM}}_{-r+2d}(\mathcal{Z}/\mathcal{X}, \mathcal{F}(-s+d)) \to \mathrm{H}^{\mathrm{BM}}_{-r+2d}(\mathcal{X}/\mathcal{X}, \mathcal{F}(-s+d)),$$

when it exists, gives rise to the Gysin map

$$f_! : \mathrm{H}^r(\mathcal{Z}, \mathcal{F}(s)) \to \mathrm{H}^{r-2d}(\mathcal{X}, \mathcal{F}(s-d)) \tag{3.20}$$

valued in the cohomology of $\mathcal{X}$. For example, this exists when f is proper and representable, or just proper if $\mathcal{X}$ is Deligne–Mumford and $\mathcal{F} = \mathbf{Q}$ or $\mathrm{MGL}_{\mathbf{Q}}$.

In particular we have a cohomological fundamental class

$$[\mathcal{Z}] = f_!(1) \in \mathrm{H}^{-2d}(\mathcal{X}, \mathcal{F}(-d)) \tag{3.21}$$

under these assumptions. For simplicity we'll state Theorem 3.22 below only for the representable case, but the proof only requires the existence of proper direct images.

The following is a generalized cohomological Bézout theorem, where no transversity assumptions are imposed.

Theorem 3.22. *Let $\mathcal{X}$ be derived Artin stack over $\mathcal{S}$, and let $f : \mathcal{Y} \to \mathcal{X}$ and $g : \mathcal{Z} \to \mathcal{X}$ be representable proper quasi-smooth morphisms of relative virtual dimension $-d$ and $-e$, respectively. Then we have*

$$[\mathcal{Y}] \cdot [\mathcal{Z}] = [\mathcal{Y} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathcal{Z}] \in \mathrm{H}^{2d+2e}(\mathcal{X}, \mathcal{F}(d+e)).$$

Proof. Consider the homotopy cartesian square

$$\begin{tikzcd} \mathcal{W} \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, "h"] & \mathcal{Z} \arrow[d, "g"] \\ \mathcal{Y} \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

where $\mathcal{W} = \mathcal{Y} \times^{\mathbf{R}}_{\mathcal{X}} \mathcal{Z}$. Under the identification

$$\mathrm{H}^{2d+2e}(\mathcal{X}, \mathcal{F}(d+e)) = \mathrm{H}^{\mathrm{BM}}_{-2d-2e}(\mathcal{X}/\mathcal{X}, \mathcal{F}(-d-e)),$$

the desired equality is

$$\begin{aligned} h_*[\mathcal{W}/\mathcal{X}] &= g_*p_*([\mathcal{W}/\mathcal{Z}]\circ[\mathcal{Z}/\mathcal{X}]) \\ &= g_*p_*(g^*[\mathcal{Y}/\mathcal{X}]\circ[\mathcal{Z}/\mathcal{X}]) \\ &= g_*(p_*g^*[\mathcal{Y}/\mathcal{X}]\circ[\mathcal{Z}/\mathcal{X}]) \\ &= g_*(g^*f_*[\mathcal{Y}/\mathcal{X}]\circ[\mathcal{Z}/\mathcal{X}]) \\ &= f_*[\mathcal{Y}/\mathcal{X}]\circ g_*[\mathcal{Z}/\mathcal{X}]. \end{aligned}$$

The first and second equalities follow from the functoriality and base change properties of the fundamental class (Theorems 3.12 and 3.13). For the rest we use the formulas (2.3.3), (2.3.2), and (2.3.4), in that order. □

The variant for schemes stated in the introduction (0.2) is obtained by applying this to the integral motivic cohomology spectrum $\mathcal{F} = \mathbf{Z}$ and using the fundamental classes of Variant 3.11.

3.5. **Grothendieck–Riemann–Roch.** Let $\mathcal{F}$ and $\mathcal{G}$ be two (multiplicative) coefficients over $\mathcal{S}$ and $\phi : \mathcal{F} \to \mathcal{G}$ a ring morphism. The morphism ϕ induces a homomorphism

$$\phi_* : \mathrm{H}^s(\mathcal{X}, \mathcal{F}(r)) \to \mathrm{H}^s(\mathcal{X}, \mathcal{G}(r))$$

for every $\mathcal{X}$ over $\mathcal{S}$. Given a quasi-smooth morphism of derived Artin stacks $f : \mathcal{X} \to \mathcal{Y}$, let $[\mathcal{X}/\mathcal{Y}]^{\mathcal{F}}$ and $[\mathcal{X}/\mathcal{Y}]^{\mathcal{G}}$ denote the fundamental classes formed with respect to $\mathcal{F}$ and $\mathcal{G}$, respectively. The following Grothendieck–Riemann–Roch formula compares these two classes in terms of a certain class $\mathrm{Td}^{\phi}_{\mathcal{X}/\mathcal{Y}} \in \mathrm{H}^0(\mathcal{X}, \mathcal{G})$.

Theorem 3.23. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived Artin stacks over $\mathcal{S}$. Then we have*

$$\phi_*([\mathcal{X}/\mathcal{Y}]^{\mathcal{F}}) = \mathrm{Td}^{\phi}_{\mathcal{X}/\mathcal{Y}} \cap [\mathcal{X}/\mathcal{Y}]^{\mathcal{G}} \tag{3.24}$$

in $\mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}/\mathcal{Y}, \mathcal{G}(d))$, where $d = \mathrm{vd}(\mathcal{X}/\mathcal{Y})$ and where $\cap$ denotes the cap product (2.2.6).

These immediately gives the usual formulas in Borel–Moore homology and cohomology:

Corollary 3.25. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived Artin stacks over $\mathcal{S}$. Then the square*

$$\begin{tikzcd} \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{F}(r)) \arrow[r, "f^!"] \arrow[d, "\phi_*"] & \mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathcal{X}/\mathcal{S}, \mathcal{F}(r+d)) \arrow[d, "\phi_*"] \\ \mathrm{H}^{\mathrm{BM}}_s(\mathcal{Y}/\mathcal{S}, \mathcal{G}(r)) \arrow[r, "\mathrm{Td}^{\phi}_{\mathcal{X}/\mathcal{Y}} \cap f^!"] & \mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathcal{X}/\mathcal{S}, \mathcal{G}(r+d)) \end{tikzcd}$$

commutes. If f is moreover proper, then the square

$$\begin{tikzcd} \mathrm{H}^s(\mathcal{X}, \mathcal{F}(r)) \arrow[r, "f_!"] \arrow[d, "\phi_*"] & \mathrm{H}^{s-2d}(\mathcal{Y}, \mathcal{F}(r-d)) \arrow[d, "\phi_*"] \\ \mathrm{H}^s(\mathcal{X}, \mathcal{G}(r)) \arrow[r, "f_!(\mathrm{Td}^{\phi}_{\mathcal{X}/\mathcal{Y}} \cup -)"] & \mathrm{H}^{s-2d}(\mathcal{Y}, \mathcal{G}(r-d)) \end{tikzcd}$$

also commutes.

Let $\mathcal{E}$ be a perfect complex of Tor-amplitude $[-k,1]$, $k \geqslant -1$, on $\mathcal{X}$ of virtual rank d. We define the Todd class $\mathrm{td}_\phi(\mathcal{E})$. Since capping with the Thom class $\mathrm{th}^{\mathcal{G}}_{\mathcal{X}}(-\mathcal{E})$ defines an isomorphism $\mathrm{H}^0(\mathcal{X},\mathcal{G}) \to \mathrm{H}^{2d}_{\mathcal{X}}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]),\mathcal{G}(d))$, there exists a unique class

$$\mathrm{td}_\phi(\mathcal{E}) \in \mathrm{H}^0(\mathcal{X},\mathcal{G})^\times$$

such that the relation

$$\phi_*(\mathrm{th}^{\mathcal{F}}_{\mathcal{X}}(-\mathcal{E})) = \mathrm{td}_\phi(\mathcal{E}) \cap \mathrm{th}^{\mathcal{G}}_{\mathcal{X}}(-\mathcal{E}) \tag{3.26}$$

holds in $\mathrm{H}^{2d}_{\mathcal{X}}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]),\mathcal{G}(d))$. Exactly as in [De1, Subsect. 5.2], this Todd class can be described explicitly using the formalism of formal group laws. We set $\mathrm{Td}^{\phi}_{\mathcal{X}/\mathcal{Y}} = \mathrm{td}_\phi(\mathcal{L}_{\mathcal{X}/\mathcal{Y}})$ for $f : \mathcal{X} \to \mathcal{Y}$ quasi-smooth.

By deformation to the normal bundle stack (Subsect. 1.4), the formula (3.24) reduces to the case where f is the zero section of a vector bundle stack $\mathcal{Y} = \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1])$, where $\mathcal{E}$ is a perfect complex of Tor-amplitude $[-k,1]$, $k \geqslant -1$. In this case the fundamental class $[\mathcal{X}/\mathcal{Y}]^{\mathcal{F}}$ is nothing else than the Thom class $\mathrm{th}^{\mathcal{F}}_{\mathcal{X}}(\mathcal{E})$, and similarly for $\mathcal{G}$, so the formula reduces to (3.26). Theorem 3.23 is proven.

Let's make this formula slightly more explicit when ϕ is the total Chern character. This is a morphism of motivic ring spectra

$$\mathrm{ch} : \mathrm{KGL} \to \bigoplus_{i\in\mathbf{Z}} \mathbf{Q}(i)[2i]$$

in $\mathrm{SH}(\mathrm{Spec}(\mathbf{Z}))$, which induces an isomorphism $\mathrm{KGL}_{\mathbf{Q}} \simeq \bigoplus_{i\in\mathbf{Z}} \mathbf{Q}(i)[2i]$ upon rationalization [Ri], [De1, 5.3.3]. Since $\mathbf{Q}$ satisfies étale descent, ch factors through the étale localization $\mathrm{KGL}^{\text{ét}}$. For any derived Artin stack $\mathcal{S}$, we obtain by inverse image along the structural morphism a canonical Chern character

$$\mathrm{ch} : \mathrm{KGL}^{\text{ét}}_{\mathcal{S}} \to \bigoplus_{i\in\mathbf{Z}} \mathbf{Q}_{\mathcal{S}}(i)[2i]$$

in $\mathrm{SH}_{\text{ét}}(\mathcal{S})$, which induces an isomorphism $\mathrm{KGL}^{\text{ét}}_{\mathbf{Q},\mathcal{S}} \simeq \bigoplus_{i\in\mathbf{Z}} \mathbf{Q}_{\mathcal{S}}(i)[2i]$. The source and target admit canonical orientations such that the Todd class $\mathrm{Td}_{\mathcal{X}/\mathcal{Y}}$ is the classical Todd class [De1, 5.3.3]. Suppose that $\mathcal{S}$ is the spectrum of a field k, so that the Borel–Moore homology represented by $\mathrm{KGL}^{\text{ét}}_{\mathcal{S}}$ coincides with étale hypercohomology with coefficients in G-theory, and the proper covariance and Gysin maps are compatible with the respective intrinsic operations in G-theory (Example 2.13). The Borel–Moore homology represented by $\mathbf{Q}_{\mathcal{S}}$ coincides with the rational (higher) Chow groups. Under these identifications the Chern character ch induces canonical homomorphisms which we denote

$$\tau_{\mathcal{X}} : \mathrm{H}^0_{\text{ét}}(\mathcal{X},\mathrm{G}) \to \mathrm{A}_*(\mathcal{X})_{\mathbf{Q}}.$$

We also write $\tau_{\mathcal{X}}$ for the composite with the canonical morphism $\mathrm{G}(\mathcal{X}) \to \mathrm{H}^0_{\text{ét}}(\mathcal{X},\mathrm{G})$. Corollary 3.25 now yields the formula

$$\tau_{\mathcal{X}}(\mathcal{O}_{\mathrm{X}}) = \mathrm{Td}_{\mathcal{X}} \cap [\mathcal{X}]$$

in $\mathrm{A}_d(\mathcal{X})_{\mathbf{Q}}$, or equivalently

$$[\mathcal{X}] = \mathrm{Td}_{\mathcal{X}}^{-1} \cap \tau_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}), \tag{3.27}$$

where we write simply $[\mathcal{X}]$ for $[\mathcal{X}/\operatorname{Spec}(k)]$ and similarly for the Todd class. This is an extension of Kontsevich's original conjectural formula for the virtual fundamental class $[\mathcal{X}]^{\mathrm{vir}}$ [Ko, 1.4.2] to Artin stacks.

3.6. **Absolute purity.** In this subsection we extend Gabber's proof of the absolute cohomological purity conjecture [SGA5, Exp. I, 3.1.4] to Artin stacks.

Theorem 3.28 (Absolute purity)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a locally of finite type representable morphism between regular*[2] *Artin stacks over $\mathbf{Z}[\frac{1}{n}]$, for some integer $n \in \mathbf{Z}$. Let $\Lambda = \mathbf{Z}/n\mathbf{Z}$ and denote by $\Lambda^{\text{ét}}$ the Λ-linear étale motivic cohomology spectrum (Example 2.11). Then the purity transformation pur_f (3.10) induces a canonical isomorphism*

$$\Lambda^{\text{ét}}_{\mathcal{X}}(d)[2d] \to f^!(\Lambda^{\text{ét}}_{\mathcal{Y}}) \tag{3.29}$$

of étale motivic spectra over $\mathcal{X}$.

It follows from the rigidity theorem of Cisinski–Déglise [CD2, Thm. 4.5.2] that with finite coefficients, étale motivic cohomology agrees with usual étale cohomology, so this does recover the classical statement when we restrict to schemes. Actually, even in the case of schemes this statement is new, since Gabber's statement [ILO, Exp. XVI, Cor. 3.1.2] requires the schemes to admit ample line bundles.

The new ingredient we use here is the purity transformation (Remark 3.8) which generalizes Gabber's construction of Gysin maps [ILO, Exp. XVI, 2.3]. Neither the statement nor the proof of Theorem 3.28 uses any derived geometry, but it is worth recalling that our construction of pur_f involves the normal bundle stack $\mathrm{N}_{\mathcal{X}/\mathcal{Y}}$, which is a classical 2-Artin stack even when $\mathcal{X}$ and $\mathcal{Y}$ are classical 1-Artin stacks.

Proof of Theorem 3.28. Let $v : \mathrm{Y} \to \mathcal{Y}$ be a smooth surjection with Y schematic, and form the homotopy cartesian square

$$\begin{tikzcd} \mathrm{X} \arrow[r, "f_0"] \arrow[d, "u"] & \mathrm{Y} \arrow[d, "v"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y}. \end{tikzcd}$$

The upper arrow f_0 is a locally of finite type morphism and X (resp. Y) is a regular algebraic space (resp. regular scheme). In terms of the purity transformation, the base change property of the fundamental class (Theorem 3.13)

[2]Recall that an Artin stack $\mathcal{S}$ is regular if and only if, for every smooth morphism $\mathrm{S} \to \mathcal{S}$ with S a scheme, S is regular.

translates to the commutativity of the diagram (cf. [DJK, Prop. 2.5.4(ii)])

$$\begin{tikzcd} u^* f^*(\Lambda^{\text{ét}}_{\mathcal{Y}})(d)[2d] \arrow[r, "\mathrm{pur}_f"] \arrow[d, equal] & u^* f^!(\Lambda^{\text{ét}}_{\mathcal{Y}}) \arrow[d, "\mathrm{Ex}^{*!}"] \\ f_0^* v^*(\Lambda^{\text{ét}}_{\mathcal{Y}})(d)[2d] \arrow[r, "\mathrm{pur}_{f_0}"] & f_0^! v^*(\Lambda^{\text{ét}}_{\mathcal{Y}}). \end{tikzcd}$$

The right-hand vertical arrow is the isomorphism induced by the exchange transformation $\mathrm{Ex}^{*!}$ (Corollary A.15). Therefore, it will suffice to replace f by f_0 and thereby assume that $\mathcal{Y} = \mathrm{Y}$ is a regular scheme and $\mathcal{X} = \mathrm{X}$ is a regular algebraic space.

We can find an étale surjection $p : \mathrm{U} \to \mathrm{X}$ such that U is a (regular) scheme and $f \circ p : \mathrm{U} \to \mathrm{Y}$ is smoothable. The functoriality property of the fundamental class (Theorem 3.12) translates to the commutativity of the diagram (cf. [DJK, Prop. 2.5.4(i)])

$$\begin{tikzcd} p^* f^*(\Lambda^{\text{ét}}_{\mathrm{Y}})(d)[2d] \arrow[r, "\mathrm{pur}_f"] \arrow[d, equal] & p^* f^!(\Lambda^{\text{ét}}_{\mathrm{Y}}) \arrow[r, "\mathrm{pur}_p"] & p^! f^!(\Lambda^{\text{ét}}_{\mathrm{Y}}) \arrow[d, equal] \\ (f \circ p)^*(\Lambda^{\text{ét}}_{\mathrm{Y}})(d)[2d] \arrow[rr, "\mathrm{pur}_{f\circ p}"] & & (f \circ p)^!(\Lambda^{\text{ét}}_{\mathrm{Y}}). \end{tikzcd}$$

Since p is étale, the upper right-hand arrow pur_p is invertible (Theorem A.13). Therefore, replacing X by U and f by $f \circ p : \mathrm{U} \to \mathrm{Y}$, we may assume that $f : \mathrm{X} \to \mathrm{Y}$ is a smoothable morphism between regular schemes.

Choose a factorization of f through a closed immersion $i : \mathrm{X} \to \mathrm{X}'$ and a smooth morphism $g : \mathrm{X}' \to \mathrm{Y}$. Since pur_g is invertible by Theorem A.13, applying the functoriality property again shows that we may replace f by i and thereby assume that $f = i$ is a closed immersion between regular schemes.

The assertion is that pur_i induces an isomorphism

$$\Lambda^{\text{ét}}_{\mathrm{X}}(d)[2d] \to i^!(\Lambda^{\text{ét}}_{\mathrm{Y}}),$$

or equivalently isomorphisms in étale motivic cohomology

$$\mathrm{H}^{2k-2c}(\mathrm{X}, \Lambda^{\text{ét}}(k-c)) \to \mathrm{H}^{2k}_{\mathrm{X}}(\mathrm{Y}, \Lambda^{\text{ét}}(k)) \tag{3.30}$$

for all integers $k \in \mathbf{Z}$, where $c = -d$ is the codimension of i. In this situation the purity transformation pur_i is the same as the one constructed in [DJK, 4.3.1], and by [DJK, 4.4.3] it agrees with the construction of [De2, 2.4.6] when applied to the étale motivic cohomology spectrum. The latter agrees, through the ridigity equivalence [CD2, Thm. 4.5.2] identifying the étale motivic cohomology groups in (3.30) with classical étale cohomology, with Gabber's construction in [ILO, Exp. XVI, 2.3] by design. Thus the claim follows from [ILO, Exp. XVI, Thm. 3.1.1]. □

Remark 3.31. The argument applies more generally to show that for an étale motivic spectrum $\mathcal{F}$, absolute purity holds for locally of finite type representable morphisms of regular Artin stacks (the analogue of Theorem 3.28) if and only if it holds for closed immersions between regular schemes. For example, this also applies to h-motivic cohomology [CD2, Thm. 5.6.2].

Appendix A. The six operations for derived Artin stacks

In this appendix we extend the six operations to derived Artin stacks. The category of coefficients we use is $\mathrm{SH}_{\text{ét}}$, the étale-local motivic homotopy category, but the construction works for any motivic ∞-category of coefficients in the sense of [Kh1, Chap. 2] that satisfies étale descent. The notion of "motivic ∞-category of coefficients" is a refinement of that of "motivic triangulated category" studied in [CD1], but every example of the latter that arises in practice can in fact be promoted to a motivic ∞-category. The $(\infty,1)$-categorical refinement is crucial for the construction below. See [To2, Sect. 2] for a quick introduction to the theory of ∞-categories.

The six operations in the (Nisnevich-local) motivic homotopy category SH were already constructed by Ayoub and Voevodsky for schemes. They were extended to derived schemes by Khan [Kh1]. Below we begin by recording the extension from derived schemes to derived algebraic spaces; this is straightforward and will not come as a surprise to certain readers. It is for the further extension to derived Artin stacks that we pass to the étale-local category, so that we can extend the operations essentially "by descent".

A.1. Derived algebraic spaces.

Theorem A.1. *The formalism of six operations on* SH *extends to derived algebraic spaces. In particular:*

(i) *For every derived algebraic space* X*, there is a closed symmetric monoidal structure on* SH(X)*. In particular, there are adjoint bifunctors* $(\otimes, \underline{\mathrm{Hom}})$*.*

(ii) *For any morphism of derived algebraic spaces* $f : \mathrm{X} \to \mathrm{Y}$*, there is an adjunction*

$$f^* : \mathrm{SH}(\mathrm{Y}) \to \mathrm{SH}(\mathrm{X}), \quad f_* : \mathrm{SH}(\mathrm{X}) \to \mathrm{SH}(\mathrm{Y}).$$

The assignments $f \mapsto f^*$*,* $f \mapsto f_*$ *are* 2*-functorial. The functor* f^* *is symmetric monoidal.*

(iii) *For any locally of finite type morphism of derived algebraic spaces* $f : \mathrm{X} \to \mathrm{Y}$*, there is an adjunction*

$$f_! : \mathrm{SH}(\mathrm{X}) \to \mathrm{SH}(\mathrm{Y}), \quad f^! : \mathrm{SH}(\mathrm{Y}) \to \mathrm{SH}(\mathrm{X}).$$

The assignments, $f \mapsto f_!$*,* $f \mapsto f^!$ *are* 2*-functorial.*

(iv) *The operation* $f_!$ *satisfies the base change and projection formulas against* f^**. That is, for any cartesian*[3] *square*

$$\begin{tikzcd} \mathrm{X}' \arrow[r, "f'"] \arrow[d, "p"] & \mathrm{Y}' \arrow[d, "q"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \end{tikzcd}$$

there are identifications

$$q^* f_! = (f')_! p^*$$

and

$$f_!(\mathcal{F}) \otimes \mathcal{G} = f_!(\mathcal{F} \otimes f^*(\mathcal{G}))$$

naturally in $\mathcal{F}$ and $\mathcal{G}$. There is a natural transformation $\alpha_f : f_! \to f_*$, functorial in f, which is invertible if f is proper.

(v) *Let $i : \mathrm{X} \to \mathrm{Y}$ be a closed immersion of derived algebraic spaces, with open complement j. Then the operation $i_* = i_!$ induces a fully faithful functor*
$$i_* : \mathrm{SH}(\mathrm{X}) \to \mathrm{SH}(\mathrm{Y})$$
whose essential image is the kernel of j^*. In particular, if i induces an isomorphism on underlying reduced classical stacks, then i_* is an equivalence.

(vi) *Let X be a derived algebraic space and $\mathcal{E}$ a locally free sheaf on X. If $p : \mathbf{V}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ denotes the associated vector bundle, then the unit transformation*
$$\mathrm{id} \to p_* p^*$$
is invertible.

(vii) *There is a canonical map of presheaves of $\mathcal{E}_\infty$-group spaces on the site of derived algebraic spaces,*
$$\mathrm{K}(-) \to \mathrm{Aut}(\mathrm{SH}(-)), \tag{A.2}$$
from the algebraic K-theory of perfect complexes to the ∞-groupoid of auto-equivalences of SH*. For a perfect complex $\mathcal{E}$ on a derived algebraic space* X*, we let $\Sigma^{\mathcal{E}}$ denote the induced auto-equivalence of* SH(X)*, and $\Sigma^{-\mathcal{E}} = \Sigma^{\mathcal{E}^\vee}$ its inverse. If $\mathcal{E}$ is locally free, then we have*
$$\Sigma^{\mathcal{E}} = s^* p^!, \qquad \Sigma^{-\mathcal{E}} = s^! p^*,$$
where $p : \mathbf{V}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ is the projection of the associated vector bundle and $s : \mathrm{X} \to \mathbf{V}_{\mathrm{X}}(\mathcal{E})$ the zero section.

(viii) *Let $f : \mathrm{X} \to \mathrm{Y}$ be a smooth morphism between derived algebraic spaces. Then there is a purity equivalence*
$$\mathrm{pur}_f : \Sigma^{\mathcal{L}_{\mathrm{X}/\mathrm{Y}}} f^* = f^!$$
which is natural in f.

This was proven in [Kh1] for derived schemes so I only describe the modifications that need to be made for derived algebraic spaces. The idea is that derived algebraic spaces are *Nisnevich*-locally affine (see e.g. the proof of [Kh2, Prop. 2.2.13]), which is good enough since SH satisfies Nisnevich descent. Thus in Chap. 0, one needs to replace "Zariski" by "Nisnevich" in Propositions 5.3.5 and 5.6.2 (the proofs don't change). In the proof of Proposition 6.3.4, one needs to replace the reference to [Con07] by [CLO], where Nagata compactifications are constructed for classical algebraic spaces. The only modification necessary in Chap. 1 is that the proof of Proposition 2.2.9 needs to be replaced by the proof of [Kh2, Prop. 2.2.13]. This extends the proof of the localization theorem [Kh1, Chap. 1, Thm. 7.4.3] to derived algebraic spaces. Chap. 2 then goes through *mutatis mutandis* to give the six operations on derived algebraic spaces.

[3] In fact, item (v) below implies that it suffices to assume that the square is cartesian on underlying classical schemes.

Only item (vii) requires further explanation, as the map (A.2) encodes much more coherence of the assignment $\mathcal{E} \mapsto \Sigma^{\mathcal{E}}$ than was constructed in [Kh1]. On the site of classical schemes, such a map is constructed in [BH, Subsect. 16.2]. It factors through homotopy invariant K-theory KH [BH, Rem. 16.11]. By right Kan extension, the map $\mathrm{KH} \to \mathrm{Aut}(\mathrm{SH})$ extends uniquely to the site of classical algebraic spaces. By derived nil-invariance of KH and SH, see [Kh3, Subsect. 5.4] and [Kh1, Thm. 7.4.3] respectively, we obtain a unique extension of this map to the site of derived algebraic spaces, and we define (A.2) to be the composite $\mathrm{K} \to \mathrm{KH} \to \mathrm{Aut}(\mathrm{SH})$.

Strictly speaking, this only gives the operation $f^!$ for separated morphisms of finite type. Using Zariski descent and the homotopy coherence of the six functor formalism, one extends this to locally of finite type morphisms. Indeed, the coherence of the data in (iii) and (iv) can be encoded using the formalism of Gaitsgory–Rozenblyum [GR, Part III] (as done in [Kh1, Chap. 2, Thm. 5.1.2]) or that of Liu–Zheng [LZ] (as done in [Ro, Sect. 9.4]); the two formalisms are almost equivalent, as explained in [GR, Part III, 1.3]. Then an easy application of the "DESCENT" program [LZ, Thm. 4.1.8] gives the desired extension.

A.2. **Derived algebraic stacks.** We begin with the presheaf of ∞-categories

$$\mathrm{X} \mapsto \mathrm{SH}(\mathrm{X}), \quad f \mapsto f^*$$

on the site of derived algebraic spaces. This is a Nisnevich sheaf, and as such is right Kan-extended from the site of derived schemes or even affine derived schemes.

Now let $\mathrm{SH}_{\text{ét}}$ denote its étale localization. In other words, we force $\mathrm{SH}_{\text{ét}}$ to satisfy descent for Čech covers in the étale topology. We then take its right Kan extension to the site of derived Artin stacks. This is thus the unique extension of $\mathrm{SH}_{\text{ét}}$ to an étale sheaf on derived Artin stacks.

We can be more explicit. If $\mathcal{X}$ is a derived Artin stack and $p : \mathrm{X} \to \mathcal{X}$ is a smooth surjection with X a derived algebraic space, then p is a covering in the étale topology so the ∞-category $\mathrm{SH}_{\text{ét}}(\mathcal{X})$ fits into a homotopy limit diagram of ∞-categories

$$\mathrm{SH}_{\text{ét}}(\mathcal{X}) \xrightarrow{p^*} \mathrm{SH}_{\text{ét}}(\mathrm{X}) \rightrightarrows \mathrm{SH}_{\text{ét}}(\mathrm{X} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathrm{X}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{SH}_{\text{ét}}(\mathrm{X} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathrm{X} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathrm{X}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots. \tag{A.3}$$

More canonically, $\mathrm{SH}_{\text{ét}}(\mathcal{X})$ is identified with the homotopy limit

$$\mathrm{SH}_{\text{ét}}(\mathcal{X}) = \varprojlim \mathrm{SH}_{\text{ét}}(\mathrm{X}) \tag{A.4}$$

taken over the ∞-category $\mathrm{Lis}_{\mathcal{X}}$ of all smooth morphisms $u : \mathrm{X} \to \mathcal{X}$ with X schematic. Roughly speaking, objects $\mathcal{F} \in \mathrm{SH}_{\text{ét}}(\mathcal{X})$ may be viewed as collections $(u^*\mathcal{F})_u$, indexed over $(u : \mathrm{X} \to \mathcal{X}) \in \mathrm{Lis}_{\mathcal{X}}$, compatible up to coherent homotopies. In particular, the family of functors u^* is conservative as u varies in $\mathrm{Lis}_{\mathcal{X}}$.

Theorem A.5.

(i) *For every derived Artin stack $\mathcal{X}$, there is a closed symmetric monoidal structure on $\mathrm{SH}(\mathcal{X})$. In particular, there are adjoint bifunctors $(\otimes, \underline{\mathrm{Hom}})$.*

(ii) *For any morphism of derived Artin stacks $f : \mathcal{X} \to \mathcal{Y}$, there is an adjunction*

$$f^* : \mathrm{SH}_{\text{ét}}(\mathcal{Y}) \to \mathrm{SH}_{\text{ét}}(\mathcal{X}), \quad f_* : \mathrm{SH}_{\text{ét}}(\mathcal{X}) \to \mathrm{SH}_{\text{ét}}(\mathcal{Y}).$$

The assignments $f \mapsto f^*$, $f \mapsto f_*$ are 2-functorial.

(iii) *For any locally of finite type morphism of derived Artin stacks $f : \mathcal{X} \to \mathcal{Y}$, there is an adjunction*

$$f_! : \mathrm{SH}_{\text{ét}}(\mathcal{X}) \to \mathrm{SH}_{\text{ét}}(\mathcal{Y}), \quad f^! : \mathrm{SH}_{\text{ét}}(\mathcal{Y}) \to \mathrm{SH}_{\text{ét}}(\mathcal{X}).$$

The assignments $f \mapsto f_!$, $f \mapsto f^!$ are 2-functorial.

(iv) *The operation $f_!$ satisfies the base change*[4] *and projection formulas against g^*, and $f^!$ satisfies base change against g_*. If f is representable by derived Deligne–Mumford stacks, then there is a natural transformation $\alpha_f : f_! \to f_*$, functorial in $\mathcal{F}$. If f is 0-representable and proper, then α_f is invertible.*

On the site of derived algebraic spaces, we may view $\mathrm{SH}_{\text{ét}}$ as a presheaf valued in the ∞-category of presentably symmetric monoidal ∞-categories and symmetric monoidal left-adjoint functors. Since the forgetful functor to (large) ∞-categories preserves limits [HTT, Prop. 5.5.3.13], the right Kan extension can be performed either way without changing the underlying presheaf of ∞-categories. In particular, we find that $\mathrm{SH}_{\text{ét}}(\mathcal{X})$ is a presentably symmetric monoidal ∞-category for every derived Artin stack $\mathcal{X}$ and that f^* is a symmetric monoidal left-adjoint functor for every morphism f. We let $\otimes$ denote the monoidal product, $\underline{\mathrm{Hom}}$ the internal hom, and f_* the right adjoint of f^*.

Similarly, if we restrict the presheaf $\mathrm{SH}_{\text{ét}}$ to *smooth* morphisms between derived algebraic spaces, then it takes values in presentable ∞-categories and right adjoint functors (as follows from Theorem A.1(viii)). By [HTT, Thm. 5.5.3.18] its right Kan extension to derived Artin stacks will have the same property; that is, f^* admits a left adjoint $f_\sharp$ for every smooth morphism f of derived Artin stacks.

Let $\mathrm{SH}^!_{\text{ét}}$ denote the étale sheaf on the site of derived algebraic spaces, and locally of finite type morphisms, given by

$$\mathrm{X} \mapsto \mathrm{SH}_{\text{ét}}(\mathrm{X}), \qquad f \mapsto f^!$$

and take its right Kan extension to derived Artin stacks. For every $\mathcal{X}$ there is then a canonical equivalence

$$\Theta_{\mathcal{X}} : \mathrm{SH}_{\text{ét}}(\mathcal{X}) \to \mathrm{SH}^!_{\text{ét}}(\mathcal{X})$$

determined by the property that

$$u^!(\Theta_{\mathcal{X}}(\mathcal{F})) = \Sigma^{\mathcal{L}_{\mathrm{X}/\mathcal{X}}} u^*(\mathcal{F})$$

for all $u : \mathrm{X} \to \mathcal{X}$ in $\mathrm{Lis}_{\mathcal{X}}$. For any morphism $f : \mathcal{X} \to \mathcal{Y}$, we define $f^! : \mathrm{SH}_{\text{ét}}(\mathcal{Y}) \to \mathrm{SH}_{\text{ét}}(\mathcal{X})$ by $f^! = \Theta_{\mathcal{X}}^{-1} \circ f^! \circ \Theta_{\mathcal{Y}}$. More concretely, $f^!$ is

[4] From Theorem A.9 below it follows that the base change formula applies also to squares that are only cartesian on underlying classical stacks.

determined by the fact that for any commutative square

$$\begin{tikzcd} \mathrm{X} \arrow[r, "f_0"] \arrow[d, "u"] & \mathrm{Y} \arrow[d, "v"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

with u and v smooth and f_0 a morphism of derived algebraic spaces, we have

$$u^* f^!(\mathcal{F}) = \Sigma^{f_0^*(\mathcal{L}_{\mathrm{Y}/\mathcal{Y}}) - \mathcal{L}_{\mathrm{X}/\mathcal{X}}} f_0^!(v^*\mathcal{F})$$

for all $\mathcal{F} \in \mathrm{SH}_{\text{ét}}(\mathcal{Y})$, or equivalently

$$\Sigma^{\mathcal{L}_{\mathrm{X}/\mathcal{X}}} u^* f^!(\mathcal{F}) = f_0^! \Sigma^{\mathcal{L}_{\mathrm{Y}/\mathcal{Y}}} v^*(\mathcal{F}). \tag{A.6}$$

Moreover, these identifications are subject to a homotopy coherent system of compatibilities as f varies.

On $\mathrm{SH}^!_{\text{ét}}$, the operation $f^!$ automatically admits a left adjoint $f_!$ for every morphism f. Indeed, the right Kan extension can be computed in the ∞-category of presentable ∞-categories and right adjoint functors (as the forgetful functor preserves limits [HTT, Thm. 5.5.3.18]). This induces an operation $f_! : \mathrm{SH}_{\text{ét}}(\mathcal{X}) \to \mathrm{SH}_{\text{ét}}(\mathcal{Y})$ by $f_! = \Theta_{\mathcal{Y}}^{-1} \circ f_! \circ \Theta_{\mathcal{X}}$, so that

$$f_! u_\sharp \Sigma^{-\mathcal{L}_{\mathrm{X}/\mathcal{X}}} = v_\sharp \Sigma^{-\mathcal{L}_{\mathrm{Y}/\mathcal{Y}}} (f_0)_!$$

for all commutative squares as above.

As mentioned in Subsect. A.1, all the data in Theorem A.5 can be encoded using the formalism of either Gaitsgory–Rozenblyum [GR, Part III] or Liu–Zheng [LZ]. In the former case, one may apply [GR, Chap. 8, Thm. 6.1.5] (cf. [GR, Chap. 5, Thm. 3.4.3], [RS, Sect. 2.2]) to glue together the required data from its restriction to algebraic spaces (already constructed in Theorem A.1), via an $(\infty, 2)$-categorical right Kan extension. Alternatively, we apply the "DESCENT" program of [LZ, Thm. 4.1.8], just as in [LZ, Subsect. 5.4].

Under certain assumptions the identification $f_! = f_*$ can be extended to non-representable proper morphisms:

Theorem A.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of derived Artin stacks that is representable by derived Deligne–Mumford stacks. Assume that there exists a finite surjection $g : \mathrm{Z} \to \mathcal{X}$ with Z an algebraic space. For $\mathcal{F} \in \mathrm{SH}_{\text{ét}}(\mathcal{X})$, consider the morphism*

$$\alpha_f : f_!(\mathcal{F}) \to f_*(\mathcal{F})$$

induced by the natural transformation α_f (Theorem A.5(iv)). If f is proper and $\mathcal{F}$ satisfies descent for finite surjections, then this morphism is invertible. In particular, this applies to the rational motivic cohomology spectrum $\mathbf{Q}_{\mathcal{X}}$ (Example 2.10), the rational algebraic cobordism spectrum $\mathrm{MGL}_{\mathbf{Q},\mathcal{X}}$ (Example 2.12), or more generally any $\mathrm{MGL}_{\mathbf{Q},\mathcal{X}}$-module.

Proof. Since $\mathcal{F}$ satisfies descent along the Čech nerve of $g : \mathrm{Z} \to \mathcal{X}$, it will suffice to show that

$$\alpha_f : f_!(h_* h^* \mathcal{F}) \to f_*(h_* h^* \mathcal{F})$$

is invertible for every finite surjection $h : \mathrm{W} \to \mathcal{X}$ with W an algebraic space. Since h and $f \circ h$ are 0-representable and proper, α_h and $\alpha_{f\circ h}$ are invertible by Theorem A.5(iv). Therefore the claim follows from the functoriality of α_f in f. It applies to $\mathbf{Q}_{\mathcal{X}}$ because the latter satisfies descent for the h topology [CD2, Cor. 5.5.5]. ☐

Example A.8. Note that $\mathcal{X}$ admits a finite cover by an algebraic space if and only if the classical stack $\mathcal{X}_{\mathrm{cl}}$ does. This is the case for example if $\mathcal{X}_{\mathrm{cl}}$ has quasi-finite separated diagonal [Ry, Thm. B], or if $\mathcal{X}_{\mathrm{cl}}$ has quasi-finite diagonal and is of finite type over a noetherian scheme [EHKV, Thm. 2.7]. In particular this holds if $\mathcal{X}_{\mathrm{cl}}$ is a Deligne–Mumford stack.

Theorem A.9 (Localization)**.** *Let $i : \mathcal{X} \to \mathcal{Y}$ be a closed immersion of derived Artin stacks, with open complement j. Then the operation $i_* = i_!$ induces a fully faithful functor*

$$i_* : \mathrm{SH}_{\text{ét}}(\mathcal{X}) \to \mathrm{SH}_{\text{ét}}(\mathcal{Y})$$

whose essential image is the kernel of j^*. In particular, if i induces an isomorphism on underlying reduced classical stacks, then i_* is an equivalence.

Proof. For fully faithfulness it suffices to show that the co-unit $i^*i_* \to \mathrm{id}$ is invertible. After base change along a smooth atlas $v : \mathrm{Y} \to \mathcal{Y}$ with Y schematic, we get a closed immersion $i_0 : \mathrm{X} \to \mathrm{Y}$ and an induced atlas $u : \mathrm{X} \to \mathcal{X}$. It suffices to show the co-unit becomes invertible after applying u^* on the left, in which case it is identified with $i_0^*(i_0)_*u^*(\mathcal{F}) \to u^*(\mathcal{F})$, by the base change formula (Theorem A.5(iv)). This is invertible by the localization theorem for derived schemes ([Kh1, Chap. 1, Cor. 7.4.9]).

Since $\mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \smallsetminus \mathcal{X})$ is empty, the base change formula shows that $j^*i_* = 0$. It remains to show that if $\mathcal{F} \in \mathrm{SH}_{\text{ét}}(\mathcal{Y})$ satisfies $j^*(\mathcal{F}) = 0$, then the unit map $\mathcal{F} \to i_*i^*(\mathcal{F})$ is invertible. By descent we reduce again to the schematic case which is [Kh1, Chap. 1, Cor. 7.4.7]. ☐

Thanks to David Rydh for the idea of the inductive argument in the proof below.

Proposition A.10 (Homotopy invariance)**.** *Let $\mathcal{X}$ be a derived Artin stack and $\mathcal{E}$ a perfect complex on $\mathcal{X}$ of Tor-amplitude $[-k, 1]$, for $k \geqslant -1$. If $\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]) \to \mathcal{X}$ denotes the associated vector bundle stack, then the unit transformation*

$$\mathrm{id} \to \pi_*\pi^*$$

is invertible.

Proof. First assume that $\mathcal{E}$ is of Tor-amplitude $[1, 1]$, so that π is a vector bundle. By descent we may assume that $\mathcal{X}$ is schematic, in which case the claim holds almost by construction (see [Kh1, Chap. 2, Subsect. 3.2]).

If $\mathcal{E}$ is of Tor-amplitude $[0, 0]$, then π is the projection of the classifying stack of the vector bundle $\mathbf{V}_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$, and the canonical section $\sigma : \mathcal{X} \to \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1])$ is a smooth surjection. The composite of the two unit maps $\mathrm{id} \to \pi_*\pi^* \to \pi_*\sigma_*\sigma^*\pi^* = \mathrm{id}$ is the identity, so will suffice to show that the unit $\mathrm{id} \to \sigma_*\sigma^*$ is invertible. Since σ is a smooth surjection it suffices moreover to show that $\sigma^! \to \sigma^! \sigma_* \sigma^*$ is invertible. By the base change formula for the square

$$\begin{tikzcd}
\mathbf{V}_{\mathcal{X}}(\mathcal{E}) \arrow[r, "p"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "\sigma"] \\
\mathcal{X} \arrow[r, "\sigma"] & \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]),
\end{tikzcd}$$

we reduce to showing that the unit map $\mathrm{id} \to p_* p^*$ is invertible. This holds by the Tor-amplitude $[1,1]$ case already proven above. Repeating the same argument inductively shows the case of Tor-amplitude $[-k,-k]$ for all $k \geqslant 0$.

For the general case of Tor-amplitude $[-k,1]$, we argue by induction on k to reduce to the $k=-1$ case above. The question being local on $\mathcal{X}$, we may find a surjection $\mathcal{E}_0[-k] \to \mathcal{E}$ with $\mathcal{E}_0$ locally free. If $\mathcal{E}'$ is the fibre of this map, then $\mathcal{E}'[1]$ is then of Tor-amplitude $[-(k-1),1]$, so by indutive assumption we know that the claim holds for $\pi' : \mathbf{V}_{\mathcal{X}}(\mathcal{E}') \to \mathcal{X}$ (i.e., that $\mathrm{id} \to (\pi')_*(\pi')^*$ is invertible). There is a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\sigma'"] \arrow[dr, "\sigma"'] & \mathbf{V}_{\mathcal{X}}(\mathcal{E}') \arrow[r, "\pi'"] \arrow[d, "\tau"] & \mathcal{X} \arrow[d, "\sigma_0"] \\
& \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]) \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \mathbf{V}_{\mathcal{X}}(\mathcal{E}_0[-k-1]) \arrow[d, "\pi_0"] \\
& & \mathcal{X},
\end{tikzcd}$$

where the square is cartesian. As $\mathcal{E}_0[-k-1]$ is of Tor-amplitude $[-k-1,-k-1]$, we already know that the unit $\mathrm{id} \to (\pi_0)_*(\pi_0)^*$ is invertible by above. It remains to show that $\mathrm{id} \to \rho_*\rho^*$ is invertible, which can be done after applying $\sigma_0^!$ on the left. By the base change formula this follows from invertibility of the unit $\mathrm{id} \to (\pi')_*(\pi')^*$. ☐

The canonical map (A.2) of Theorem A.1(vii) also extends to the site of derived Artin stacks:

$$\mathrm{K}(-) \to \mathrm{Aut}(\mathrm{SH}_{\text{ét}}(-)). \tag{A.11}$$

Indeed as the target satisfies étale descent, the map factors through étale K-theory $\mathrm{K}_{\text{ét}}$ and arises via right Kan extension from derived algebraic spaces. We thus also have the (invertible) operations

$$\Sigma^{\mathcal{E}} : \mathrm{SH}(\mathcal{X}) \to \mathrm{SH}(\mathcal{X}) \tag{A.12}$$

for $\mathcal{E} \in \mathrm{Perf}(\mathcal{X})$.

Theorem A.13 (Purity)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth morphism of derived Artin stacks. Then there is a purity equivalence*

$$\mathrm{pur}_f : \Sigma^{\mathcal{L}_{\mathcal{X}/\mathcal{Y}}} f^* \simeq f^!$$

which is natural in f.

Proof. This follows immediately from the characterization of $f^!$ given in the proof of Theorem A.5. ☐

Example A.14. Let $\mathcal{E}$ be a perfect complex of Tor-amplitude $[-k,1]$, $k \geqslant -1$, on a derived Artin stack $\mathcal{X}$. Then $\mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1])$ is a smooth Artin stack over $\mathcal{X}$. Let $\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1]) \to \mathcal{X}$ denote the projection and $\sigma : \mathcal{X} \to \mathbf{V}_{\mathcal{X}}(\mathcal{E}[-1])$ the canonical section. By purity (Theorem A.13) one has the formulas

$$\Sigma^{\mathcal{E}} = \sigma^! \pi^*, \qquad \Sigma^{-\mathcal{E}} = \sigma^* \pi^!.$$

Similarly if $\mathcal{E}$ is of Tor-amplitude $[0,0]$ (= locally free), then

$$\Sigma^{\mathcal{E}} = s^* p^!, \qquad \Sigma^{\mathcal{E}[1]} = s^! p^*,$$

where $p : \mathbf{V}_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ and $s : \mathcal{X} \to \mathbf{V}_{\mathcal{X}}(\mathcal{E})$ denote the projection and zero section, respectively.

Corollary A.15. *Suppose given a commutative square*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "p"] & \mathcal{Y}' \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

of derived Artin stacks which is cartesian on underlying classical stacks. If f is representable and locally of finite type, there is a natural transformation

$$\mathrm{Ex}^{*!} : p^* f^! \to (f')^! q^*.$$

If either f or q is smooth, then $\mathrm{Ex}^{*!}$ is invertible.

Proof. The natural transformation is defined as the composite

$$p^* f^! \xrightarrow{\text{unit}} p^* f^! q_* q^* \simeq p^* p_* (f')^! q^* \xrightarrow{\text{counit}} (f')^! q^*$$

where the isomorphism in the middle is the base change formula, obtained by passage to right adjoints from the base change formula (Theorem A.5(iv)). The second statement follows from Theorem A.13. ☐

Construction A.16 (Euler transformation)**.** Let $\mathcal{E}$ be a locally free sheaf on a derived Artin stack $\mathcal{X}$. There is a natural transformation

$$\mathrm{eul}_{\mathcal{E}} : \mathrm{id} \to \Sigma^{\mathcal{E}} \tag{A.17}$$

of auto-equivalences of $\mathrm{SH}_{\text{ét}}(\mathcal{X})$. More generally for any surjection $\phi : \mathcal{E} \to \mathcal{E}'$ of finite locally free sheaves, there is a natural transformation

$$\Sigma^{\phi} : \Sigma^{\mathcal{E}} \to \Sigma^{\mathcal{E}'}$$

constructed as follows. Consider the commutative triangle

$$\begin{tikzcd} \mathbf{V}_{\mathcal{X}}(\mathcal{E}') \arrow[rr, "i"] \arrow[dr, "q"'] & & \mathbf{V}_{\mathcal{X}}(\mathcal{E}) \arrow[dl, "p"] \\ & \mathcal{X} & \end{tikzcd}$$

and let t and s be the respective zero sections. Then Σ^{ϕ} is the composite

$$t^* q^! = t^* i^! p^! \xrightarrow{\mathrm{Ex}^{*!}} t^* i^* p^! = s^* p^!$$

under the identifications $s^* p^! = \Sigma^{\mathcal{E}}$ and $t^* q^! = \Sigma^{\mathcal{E}'}$ (Example A.14), where $\mathrm{Ex}^{*!} : i^! \to i^*$ is the exchange transformation (Corollary A.15) for the self-intersection square of the closed immersion i.

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