LOCALIZATION THEOREMS FOR ALGEBRAIC STACKS

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Abstract. In this paper we consider three types of localization theorems for algebraic stacks:

(i) Concentration, or cohomological localization. Given an algebraic group acting on a scheme or stack, we give a sufficient criterion for its localized equivariant Borel–Moore homology to be concentrated in a given closed substack. We deduce this from a new kind of stacky concentration theorem.

(ii) Atiyah–Bott localization, or localization of (virtual) fundamental classes to fixed loci of torus actions. In particular, this gives a conceptual new proof of the Graber–Pandharipande formula without global embedding or global resolution hypotheses.

(iii) Cosection localization, or localization of virtual fundamental classes to degeneracy loci of cosections (of the "obstruction sheaf"). We recast this in terms of a notion of "cohomological reductions" of (-1)-shifted 1-forms on derived stacks.

These results also apply to oriented Borel–Moore homology theories, such as (higher) Chow homology and algebraic bordism, and hold over arbitrary fields and even in mixed characteristic. In an appendix, we study various types of fixed loci for algebraic group actions on algebraic stacks.

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Introduction

Localization techniques are indispensable in moduli theory. For many modern applications, it is important to consider moduli problems as algebraic stacks. In this paper we extend three types of localization theorems to the setting of algebraic stacks. This builds on the recent developments of new formalisms for cohomology and intersection theory on stacks, based on advances in homotopical methods and derived algebraic geometry.

0.1. **Concentration, a.k.a. cohomological localization.** The first version of localization we consider is localization in the sense of Borel [Bor1]. Following Thomason [Tho3], we call this "concentration". To formulate it, let T be a split torus over a field k and consider the T-equivariant cohomology ring

$$\mathrm{H}^*(BT) = \mathrm{H}^*_T(\operatorname{Spec} k) \coloneqq \bigoplus_{m,n\in\mathbf{Z}} \mathrm{H}^n_T(\operatorname{Spec} k)(m).$$

For an Artin stack X locally of finite type over k we have the equivariant Borel–Moore homology

$$\mathrm{H}^{\mathrm{BM},T}_*(X) \coloneqq \bigoplus_{m,n\in\mathbf{Z}} \mathrm{H}^{\mathrm{BM},T}_n(X)(m) \tag{0.1}$$

where (m) denotes the Tate twist. Let $\mathrm{H}^*(BT)_{\mathrm{loc}}$ denote the localization at the set of first Chern classes of all nontrivial characters of T, and for the $\mathrm{H}^*(BT)$-module $\mathrm{H}^{\mathrm{BM},T}(X)$ write

$$\mathrm{H}^{\mathrm{BM},T}(X)_{\mathrm{loc}} \coloneqq \mathrm{H}^{\mathrm{BM},T}(X) \otimes_{H^*(BT)} H^*(BT)_{\mathrm{loc}}.$$

Theorem A (Torus concentration)**.** *Let X be an Artin stack of finite type over k with T-action. Let $Z \subseteq X$ be a T-invariant closed substack away from which every point has T-stabilizer properly contained in T. Then*

$$i_* : \mathrm{H}^{\mathrm{BM},T}_*(Z)_{\mathrm{loc}} \to \mathrm{H}^{\mathrm{BM},T}_*(X)_{\mathrm{loc}} \tag{0.2}$$

is an isomorphism.

See Theorem 3.1. Informally speaking, the condition on Z means that it contains every T-fixed point. The notion of G-stabilizers, for an action of group scheme G, is developed in Appendix A (see also Subsect. 0.4 of the introduction below).

Remark 0.3. Borel–Moore homology, i.e., cohomology of the dualizing complex, can be formed with respect to any reasonable six functor formalism, such as Betti or étale sheaves; see [LZ1] for extensions to stacks. We will also work with motivic six functor formalisms, such as Voevodsky's triangulated categories of motives; these also extend to stacks by [Kha2, Kha5], and for quotient stacks the corresponding Borel–Moore homology theories recover equivariant Chow groups (see [Kha6]). For example, for the T-equivariant higher Chow groups of Edidin–Graham [EG1] we have the isomorphism

$$i_* : \mathrm{CH}^T_*([Z/G], *)_{\mathrm{loc}} \to \mathrm{CH}^T_*([X/G], *)_{\mathrm{loc}}$$

where G is a linear algebraic group and X is a k-scheme with $G \times T$-action such that the T-stabilizers of points $x \in X \smallsetminus Z$ are properly contained in T. Similarly, using MGL-modules as the sheaf theory we get the same result for T-equivariant higher algebraic cobordism of [Kha6].

We may reformulate torus-equivariant concentration (Theorem A) in terms of the induced morphism of quotient stacks $[Z/T] \to [X/T]$. The following more general form of concentration applies to arbitrary stacks that need not arise from a torus action, or from any group action at all.

Theorem B (Stacky concentration)**.** *Let $\mathcal{X}$ be an Artin stack of finite type over a field k with affine stabilizers and $\mathcal{Z} \subseteq \mathcal{X}$ a closed substack.*

(i) *Let $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$ be a subset such that for every geometric point x of $\mathcal{X} \setminus \mathcal{Z}$, there is a line bundle $\mathscr{L}(x) \in \Sigma$ whose restriction to $B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$ is trivial. Then i_* induces an isomorphism*
$$i_* : \mathrm{H}^{\mathrm{BM}}_*(\mathcal{Z})[\Sigma^{-1}] \to \mathrm{H}^{\mathrm{BM}}_*(\mathcal{X})[\Sigma^{-1}]$$
where Σ acts via the first Chern class map $c_1 : \mathrm{Pic}(X) \to \mathrm{H}^*(\mathcal{X})$.

(ii) *Let $\Sigma \subseteq \mathrm{K}_0(\mathcal{X})$ be a subset such that for every geometric point x of $\mathcal{X} \setminus \mathcal{Z}$, there is a K-theory class $\alpha(x) \in \Sigma$ whose restriction to $B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$ is trivial. Then i_* induces an isomorphism*
$$i_* : \widehat{\mathrm{H}}^{\mathrm{BM}}_*(\mathcal{Z})_{\mathbf{Q}}[\Sigma^{-1}] \to \widehat{\mathrm{H}}^{\mathrm{BM}}_*(\mathcal{X})_{\mathbf{Q}}[\Sigma^{-1}]$$
where Σ acts via the Chern character isomorphism $\mathrm{ch} : \mathrm{K}_0(X)_{\mathbf{Q}} \to \widehat{\mathrm{H}}^*(\mathcal{X})_{\mathbf{Q}}$.

See Corollary 2.6 and Theorem 2.10. Here $\widehat{\mathrm{H}}^*$ and $\widehat{\mathrm{H}}^{\mathrm{BM}}_*$ are defined like their unhatted versions (0.1), except that the direct sum over $n \in \mathbf{Z}$ is replaced by the product.

We can now specialize Theorem B to quotient stacks, to get the following generalization of Theorem A to actions of general algebraic groups (see Corollary 2.15).

Corollary C (Equivariant concentration)**.** *Let G be an fppf group scheme acting on an Artin stack X of finite type over a field k with affine stabilizers. Let $\Sigma \subseteq \mathrm{K}_0(BG)$ be a subset of nonzero elements. Let $Z \subseteq X$ be a G-invariant closed substack containing every point x of X such that no element of Σ is sent to zero by $\mathrm{K}_0(BG) \to \mathrm{K}_0(B\mathrm{St}^G_X(x))$. Then*
$$i_* : \widehat{\mathrm{H}}^{\mathrm{BM},G}_*(Z)_{\mathbf{Q}}[\Sigma^{-1}] \to \widehat{\mathrm{H}}^{\mathrm{BM},G}_*(X)_{\mathbf{Q}}[\Sigma^{-1}]$$
is an isomorphism, where Σ acts via the Chern character.

Remark 0.4. Our methods also give analogues of all the above results in G-theory (= algebraic K-theory of coherent sheaves). This will be explained elsewhere.

In topology, torus concentration goes back to Borel [Bor1] and was further generalized by Atiyah–Segal [AS] and Quillen [Qui]. The statement of Corollary C is very closely analogous to [Hsi, §3.2, Thm. III.1]. In algebraic geometry, these statements were generalized (over the complex numbers) to sheaf cohomology in [EM] and [GKM]. For Chow groups, an analogue was proven by Edidin–Graham [EG2] for torus actions on schemes, and generalized to Deligne–Mumford stacks by Kresch [Kre] (for rank one torii and assuming the base field is algebraically closed). The analogous statement for Levine–Morel algebraic bordism was proven for smooth and projective varieties over a field of characteristic zero by A. Krishna [Kri], and generalized recently in [KP] to (possibly singular) quasi-projective schemes[1]. For global quotient stacks, a concentration theorem was proven by A. Minets [Min] under a fairly restrictive technical condition.

Our stacky concentration theorem (Theorem B) appears to be completely new. We also briefly summarize some key new aspects of our torus concentration theorem (Theorem A):

(i) In the case of schemes, it can be regarded as unifying the sheaf cohomology versions with more recent versions in Chow groups and algebraic bordism. Moreover, we also lift concentration to a statement about Voevodsky motives (or MGL-modules).

(ii) For Deligne–Mumford stacks, our Chow statement generalizes [Kre] to higher rank tori (and arbitrary bases). For singular Borel–Moore and étale homology, the statement appears to be new in this case (and indeed definitions of equivariant Borel–Moore homology of stacks have only become possible relatively recently).

(iii) For Artin stacks, our result is new, except for the special case considered in [Min]. Note also that our torus concentration theorem even works without affine stabilizers assumptions.

(iv) We also prove a variant of concentration for non-quasi-compact schemes and stacks; this requires some care as the naïve formulation of the result does not hold. See Theorem 7.13 for an application to moduli stacks of Higgs sheaves.

0.2. **Atiyah–Bott localization.** Let T be a split torus acting on a scheme X. In the situation of torus concentration (Theorem A), suppose moreover that X is smooth. Then the fixed locus X^T is also smooth and the Atiyah-Bott localization formula computes the inverse of the isomorphism i_* (0.2) in terms of the Gysin map $i^!$: we have

$$(i_*)^{-1} = i^!(-) \cap e(N)^{-1} \tag{0.5}$$

where $e(N)$, the Euler class (= top Chern class) of the normal bundle, is shown to be invertible in localized equivariant cohomology.

Now let X be a Deligne–Mumford stack with T-action. There is still a closed substack Z, smooth when X is, which may be substituted for the fixed locus (this is the reparametrized homotopy fixed point stack introduced in Subsect. 0.4 below), and we show that the Atiyah–Bott localization formula (0.5) still holds in this setting.

The Atiyah–Bott localization formula is one of the main computational tools in enumerative geometry. For most applications, one wants to apply

[1]Krishna assumed the right-exact localization sequence for equivariant algebraic bordism, which is still not known. The argument of [KP] bypassed this issue.

it to a moduli stack which is singular but still *quasi-smooth* when regarded with its derived structure. (For example, this is the case for moduli stacks of stable maps and stable sheaves which are used in Gromov–Witten and Donaldson–Thomas theory.) To that end, we prove a generalization of the Atiyah–Bott formula to the case of quasi-smooth Deligne–Mumford stacks.

In fact, we show that the formula (0.5) applies word-for-word in this situation, up to the following difficulties:

(a) The first problem is that, even though the fixed locus Z is still quasi-smooth when X is, the inclusion of the fixed locus $i : Z \to X$ is not quasi-smooth (its relative cotangent complex has an extra H^{-2} when X is not smooth). Therefore, there is no (virtual) Gysin map $i^!$. Nevertheless, we construct an $i^!_T$ in *localized* T-equivariant Borel–Moore homology.

(b) The normal bundle now has to be replaced by the "virtual normal bundle", i.e., the 1-shifted relative tangent bundle of $i : Z \to X$. Since this is only a perfect complex, which we do not assume to have a global resolution, we have to define an Euler class $e^T(N^{\mathrm{vir}})$ in localized equivariant cohomology (and again show it is invertible).

With these ingredients we have the following result (see Corollary 5.30):

Theorem D (Virtual localization formula)**.** *Let X be a derived Deligne–Mumford stack of finite type over k with T-action. Let $Z \subseteq X$ be the reparametrized homotopy fixed point stack. If X is quasi-smooth, then so is Z and we have an equality*

$$(i_*)^{-1} = i^!_T(-) \cap e^T(N^{\mathrm{vir}})^{-1}$$

of maps $\mathrm{H}^{\mathrm{BM},T}_*(X)_{\mathrm{loc}} \to \mathrm{H}^{\mathrm{BM},T}_*(Z)_{\mathrm{loc}}$, where $i : Z \to X$ is the inclusion. Moreover, we have

$$[X]^{\mathrm{vir}} = i_*([Z]^{\mathrm{vir}} \cap e^T(N^{\mathrm{vir}})^{-1}) \tag{0.6}$$

in $\mathrm{H}^{\mathrm{BM},T}_*(X)_{\mathrm{loc}}$.

In particular, when X is proper we deduce a virtual integration formula by push-forward to the point.

For smooth manifolds, the localization and integration formulas were first proven by Atiyah–Bott [AB] and Berline–Vergne [BV]. In algebraic geometry, the analogous formulas in Chow homology were proven for smooth schemes by Edidin–Graham [EG2]. Kresch [Kre] generalized them to smooth Deligne–Mumford stacks, for T rank one and k algebraically closed.

The virtual localization formula was proven in Chow homology by [GP] using the language of perfect obstruction theories of Behrend–Fantechi [BF]. They worked over the field of complex numbers and with rank one T, and assumed the existence of a global T-equivariant embedding into an ambient smooth stack, as well as a global resolution of the perfect obstruction theory. Later, Chang–Kiem–Li [CKL] gave a different proof avoiding the global embedding and only requiring a global resolution of the virtual normal bundle N^{vir}. Their argument is close in spirit to ours, but their use of the global resolution (which is essential in their construction) makes our approach more direct. A general virtual localization formula with no global resolution hypotheses was desired by D. Joyce, see [Joy, Rem. 2.20]. Note that we also prove a variant of Theorem D using the language of perfect obstruction theories; this is convenient for applications, even though in practice we always expect perfect obstruction theories of interest to come from derived structures.

In Subsect. 5.6 we prove a simple wall-crossing formula as in [KL2, §2.1, App. A], [CKL, §4], and [Joy, Cor. 2.21], with no global resolution assumption and over arbitrary base fields. This proves a non-symmetric analogue of [KL2, Conj. 1.2]. We expect that this could be useful in the context of wall-crossing for non-projective moduli spaces, e.g. moduli spaces of Bridgeland-stable perfect complexes.

It could be interesting to use torus localization to compute enumerative invariants in positive characteristic, such as Gromov–Witten, Donaldson–Thomas, or Mochizuki's Donaldson-type invariants, where the moduli stacks are Artin with finite stabilizers but typically not Deligne–Mumford. For a smooth *quasi*-projective variety X, the moduli space of stable objects on X may not be proper. In that case, it may happen that there is a torus action on X such that the fixed locus of the induced action on the moduli space is proper. Then we may use the virtual localization formula to *define* the enumerative invariants of X. For example, see the theories of *local* Gromov–Witten and Donaldson–Thomas invariants [BP, MNOP, OP], and the theory of Vafa–Witten invariants [TT1, TT2].

Let us briefly sketch the idea behind the construction of the Gysin pull-back $i^!_T$. First, recall from [Kha2] that when $f : X \to Y$ is a quasi-smooth morphism (or admits a relative perfect obstruction theory), one can define a specialization map

$$\mathrm{sp}_{X/Y} : \mathrm{H}^{\mathrm{BM}}_*(Y) \to \mathrm{H}^{\mathrm{BM}}_*(T_{X/Y}[1]),$$

where $T_{X/Y}[1]$ is the 1-shifted tangent bundle (resp. the vector bundle stack associated to the perfect obstruction theory). By homotopy invariance for vector bundle stacks there is a canonical isomorphism $\mathrm{H}^{\mathrm{BM}}_*(T_{X/Y}[1]) \simeq \mathrm{H}^{\mathrm{BM}}_{*+2d}(X)(-d)$, where d is the relative virtual dimension of $X \to Y$. The composite

$$f^! : \mathrm{H}^{\mathrm{BM}}_*(Y) \to \mathrm{H}^{\mathrm{BM}}_{*+2d}(X)(-d)$$

is the *virtual Gysin pull-back* (see also [Man] in the Deligne–Mumford case).

As mentioned above, in the situation of Theorem D the inclusion of the fixed locus i is usually not quasi-smooth. Nevertheless, the specialization map above still exists without the quasi-smoothness hypothesis. Moreover, we prove that in localized equivariant Borel–Moore homology there is an analogue of the homotopy invariance property above which yields an isomorphism $\mathrm{H}^{\mathrm{BM},T}_*(T_{Z/X}[1])_{\mathrm{loc}} \simeq \mathrm{H}^{\mathrm{BM},T}_*(Z)_{\mathrm{loc}}$. Combining these two ingredients then gives the virtual Gysin pull-back $i_T^!$

$$i_T^! : \mathrm{H}_*^{\mathrm{BM},T}(X)_{\mathrm{loc}} \to \mathrm{H}_*^{\mathrm{BM},T}(Z)_{\mathrm{loc}}.$$

The remaining input into Theorem D is then a functoriality property, which in particular yields

$$i_T^![X]^{\mathrm{vir}} = [Z]^{\mathrm{vir}}.$$

0.3. **Cosection localization.** In this part we give a new point of view on the theory of cosection localization of Y.-H. Kiem and J. Li [KL].

Let X be a derived stack with perfect cotangent complex. A cosection in the sense of *op. cit.* is a morphism[2]

$$h^0(L_X^\vee[1]) \to h^0(\mathscr{O}_X)$$

in the category $\mathrm{Qcoh}(X_{\mathrm{cl}})$ of quasi-coherent sheaves on the classical truncation X_{cl}. For us, a cosection will instead be a morphism in the derived ∞-category $\mathbf{D}_{\mathrm{qc}}(X)$ of the form $L_X^\vee[1] \to \mathscr{O}_X$. In fact, we will prefer to look at the dual morphism

$$\mathscr{O}_X \to L_X[-1]$$

which we can regard as a (-1)-shifted 1-form on X.

We introduce a notion of *cohomological reduction* of a (-1)-shifted 1-form on X, and show:

Theorem E. *Let X be a quasi-smooth derived Artin stack over k. For any (-1)-shifted 1-form $\alpha : \mathscr{O}_X \to L_X[-1]$ which admits a cohomological reduction ρ, there exists a localized virtual fundamental class*

$$[X]_\rho^{\mathrm{vir}} \in \mathrm{H}_*^{\mathrm{BM}}(X(\alpha))$$

where $X(\alpha)$ is the derived zero locus of α, such that

$$i_*[X]_\rho^{\mathrm{vir}} = [X]^{\mathrm{vir}}$$

in $\mathrm{H}_*^{\mathrm{BM}}(X)$ where $i : X(\alpha) \to X$ is the inclusion.

We also show an analogous "relative" statement for the (virtual) Gysin map of a quasi-smooth morphism. At first approximation, a cohomological reduction is a reduction of the specialization map in Borel–Moore homology. However, the precise definition (and the proof of Theorem E) requires this to be at the level of Borel–Moore chain complexes (as objects of the derived ∞-category), rather than homology. See Definition 6.2.

In [KL, KL3, KP] the idea is to use *cone reductions*, i.e., they show that for X nice enough, the reduced intrinsic normal cone is contained in the "kernel cone stack" of the cosection (compare Definition 6.4). We show that cone reductions give rise to cohomological reductions (Proposition 6.5), but the latter exist much more generally and have the advantage of also working in positive characteristic and in the relative case (see Examples 6.8 and 6.9).

[2]Actually, in [KL] the authors look at "meromorphic cosections", which are only required to be defined on an open of X. For simplicity we will only consider globally defined cosections in this paper.

Our new approach using cohomological reduction relies crucially on the "homotopical enhancements" of homology and intersection theory (specifically, the specialization map) considered in [Kha2]. The proof of Theorem E is then essentially immediate from the definitions. For example, in [KL3] studied cosection localization in the setting of singular Borel–Moore homology, but they had to pass to certain intersection homology groups [KL3, Rem. 3.2]. This is because, unlike our construction, they work at the level of homology groups rather than complexes (as objects of the derived ∞-category).

We give some applications of this technique in enumerative geometry. For example, let S be a smooth projective surface with $h^1(S) = 0$ over an arbitrary base field k. We define a (-1)-shifted 1-form on a derived enhancement of the Hilbert scheme of divisors on S, which on the classical truncation is dual to the cosection used in [CK] (for $k = \mathbf{C}$). We also define a (-1)-shifted 1-form on the derived moduli stack of stable maps to S. This gives rise to a derived reduction of the latter, which agrees with the construction of [STV] when S is a K3 surface (and $k = \mathbf{C}$). In particular in that case we recover the reduced virtual cycle in Chow homology constructed by [MPT].

If X is a smooth projective threefold with holomorphic 2-form θ, we define a (-1)-shifted 1-form on the moduli stack of Pandharipande–Thomas stable pairs on X. We conjecture that this 1-form is closed (in the sense of [PTVV]). Assuming this we obtain, if k is algebraically closed of characteristic zero, either a reduced or localized virtual class for the moduli of stable pairs with fixed curve class β and Euler characteristic n (depending on whether our (-1)-shifted 1-form is nowhere zero on this component).

As for torus localization, the cosection localization technique can also be used to define enumerative invariants when the ambient moduli stack is not proper. For example, see the theory of stable maps with p-fields [CL, CJW] and the theory of Vafa-Witten invariants for K3 surfaces [TT1, JT, MT].

0.4. **Fixed loci on stacks.** If T is a split torus acting on a scheme X, one may think of the fixed locus X^T as the locus where the action is trivial. In the case of stacks, T can act nontrivially on the stabilizers of X, so there are several non-equivalent ways we can make this precise. This leads to several versions of the notion of "fixed locus" which we study in the appendix. Even in the Deligne–Mumford case, careful treatments of such questions (that are sufficient for applications in enumerative geometry, say) have only appeared in the literature relatively recently in work of Alper–Hall–Rydh [AHR].

Let us begin by describing the various fixed loci we will consider at the level of points. For any (field-valued) point x of X, consider the exact sequence

$$1 \to \underline{\mathrm{Aut}}_X(x) \to \underline{\mathrm{Aut}}_{\mathcal{X}}(x) \xrightarrow{\alpha} T_{k(x)}$$

where $\mathcal{X} = [X/T]$ is the quotient stack and $T_{k(x)} = T \times_k k(x)$ is the base change to the residue field. The image of the homomorphism α may be thought of as the T-*stabilizer* $\mathrm{St}^T(x)$ at x. Then we may consider the following possible variants of the condition "$\mathrm{St}^T(x) = T_x$":

(i) α is surjective;

(ii) α restricts to a surjection on a maximal subtorus $T' \subseteq \underline{\mathrm{Aut}}_{\mathfrak{X}}(x)$;

(iii) α admits a group-theoretic section.

We will see that these all define closed subsets of $|X|$ (for nice enough X), such that $|X^{hT'}| = |X^{sT}|$ and $|X^{hT'}| \to |X^T|$ is surjective, up to some reparametrization $T' \twoheadrightarrow T$. Here $|X|$ denotes the underlying topological space of X.

To enhance these subsets to stacks, it is appropriate to replace these conditions with *properties.* We will prove:

Theorem F. *Let X be an Artin stack of finite type over k with affine stabilizers. Then we have:*

(i) *Let X^{hT} be the stack of group-theoretic sections of α, i.e. the mapping stack of T-equivariant morphisms*

$$\underline{\mathrm{Map}}_S^T(S, X),$$

where S is regarded with trivial T-action, or equivalently the Weil restriction of $\mathfrak{X} = [X/T] \to BT$ along $BT \to \mathrm{Spec}(k)$. Then $X^{hT} \to X$ is a closed immersion if X is Deligne–Mumford.

(ii) *Let X^{sT} be the stack $\mathfrak{X}_r \times_{\mathfrak{X}} X$, where $\mathfrak{X}_r \subseteq \mathfrak{X}$ is the locus of r-dimensional stabilizers where $r = \mathrm{rk}(T)$ (compare* [ER, App. C]*, and see Definition A.51 for the precise definition). Then $X^{sT} \to X$ is a closed immersion if X is tame Artin.*

(iii) *There exists a reparametrization $T' \twoheadrightarrow T$ such that (a) $X^{hT'} \to X^T$ is surjective on field-valued points and (b) $X^{hT'} = X^{sT}$ as closed substacks of X.*

The stack X^{hT} has been studied extensively by Romagny [Rom2, Rom3, Rom4]. See also [CGP, Prop. A.8.10] for the case where X is a scheme and [Dri] for the case of X an algebraic space and T of rank one. Informally speaking, group-theoretic sections of α correspond to fixed points x *together* with isomorphisms $t \cdot x \simeq x$ for all $t \in T$. This is the homotopy theorists' notion of *homotopy fixed points* (see e.g. [Tho1]).

The reparametrized version $X^{hT'}$ has been studied, for T of rank one, by Alper–Hall–Rydh [AHR, §5.4] (see also [Kre, 5.3.4] for k algebraically closed).

Our definition of X^{sT} is inspired by that of Edidin–Rydh's locus of maximal-dimensional stabilizers [ER, App. C].

0.5. **Conventions, notation and terminology.**

We work over a fixed base ring k.

0.5.1. *Stacks.* Except in the introduction, we work with *higher* stacks throughout the paper.

A *prestack* is a presheaf of ∞-groupoids on the site of k-schemes. A *stack* is a prestack that satisfies hyperdescent with respect to the étale topology.

A stack is 0-*Artin* if it is an algebraic space, i.e., if it has schematic and monomorphic ($=(-1)$-truncated) diagonal[3] and admits a surjective étale morphism from a k-scheme. A stack is n-*Artin*, for $n > 0$, if it has $(n-1)$-representable diagonal and admits a surjective smooth morphism from a scheme. A stack is *Artin* if it is n-Artin for some n. A stack is *Deligne–Mumford* if it has representable ($=$ 0-representable) diagonal and admits a surjective étale morphism from a scheme, or equivalently if it is 1-Artin with unramified diagonal.

In the above definitions, a morphism of prestacks $f : X \to Y$ is called *schematic*, resp. $(n-1)$-*representable*, if for every scheme S and every morphism $S \to Y$, the base change $X \times_Y S$ is a scheme, resp. $(n-1)$-Artin. An $(n-1)$-representable morphism $f : X \to Y$ is *étale* (resp. *smooth*, *flat*, *surjective*), if for every scheme S and every morphism $S \to Y$, there exists a scheme U such that the morphism of $(n-1)$-representable stacks $X \times_Y S \to S$ is étale (resp. smooth, flat, surjective).

See [Gai, §4.2] or [Toë2, §3.1] for more details (our particular conventions agree with the former).

0.5.2. *Points of stacks.* A *point* of a prestack X is a field-valued point, i.e., a morphism $x : \mathrm{Spec}(k(x)) \to X$ where $k(x)$ is a field (which we call the *residue field* at x). A *geometric point* of X is a field-valued point x whose residue field $k(x)$ is an algebraic closure of a residue field of k at a prime ideal $\mathfrak{p}$. A morphism of stacks is *surjective* if it is surjective on geometric points.

The *set of points* of X, denoted $|X|$, is the colimit

$$\varinjlim_{\kappa} \pi_0 X(\kappa),$$

taken over fields κ, in the category of sets. Here $\pi_0 X(\kappa)$ is the set of connected components of the ∞-groupoid $X(\kappa)$, and given a field extension $\kappa' \to \kappa$, the corresponding transition arrow is the map induced by $X(\kappa') \to X(\kappa)$ on sets of connected components. When X is 1-Artin, $|X|$ admits a canonical structure of topological space (see e.g. [SP, Tag 04XL]).

0.5.3. *Stabilizers of stacks.* Let X be a 1-Artin stack and x a point. The *stabilizer* at x is the group algebraic space $\underline{\mathrm{Aut}}_X(x)$ of automorphisms of x. This can be defined equivalently as the fibred product $\mathrm{Spec}(k(x)) \times_X \mathrm{Spec}(k(x))$, the fibre of the diagonal $X \to X \times X$ over (x, x), or the fibre of the projection of the inertia stack $I_X \to X$ over x.

We say a 1-Artin stack X has *affine stabilizers* if for every point x of X, the stabilizer $\underline{\mathrm{Aut}}_X(x)$ is affine. If X has affine inertia or affine diagonal, then it has affine stabilizers.

[3] hence in particular takes values in sets ($=$ 0-truncated ∞-groupoids)

We say that X has *finite stabilizers* if for every point x the stabilizer $\underline{\mathrm{Aut}}_X(x)$ is finite (over $\mathrm{Spec}(k(x))$), or equivalently if X has quasi-finite diagonal. If X is Deligne–Mumford or has quasi-finite inertia, then it has finite stabilizers.

0.5.4. *Derived stacks.* Replacing "scheme" by "derived scheme" everywhere in (0.5.1), we get the notions of *derived* (n-Artin) (pre)stacks. See e.g. [Gai, §4.2] or [Toë2, §5.2].

Any derived (pre)stack X has an underlying classical truncation X_{cl}, which is a (pre)stack equipped with a canonical morphism $X_{\mathrm{cl}} \to X$ which induces an isomorphism on S-valued points $X_{\mathrm{cl}}(S) \simeq X(S)$ whenever S is a classical scheme. If X is derived n-Artin, then X_{cl} is n-Artin. Recall that for a morphism $f : X \to Y$ the properties of affineness, representability, separatedness, and properness are all detected on classical truncations.

If X is a derived 1-Artin stack, it has the same (field-valued) points as its classical truncation X_{cl}, and we define the *stabilizer* at a point x to be the stabilizer at x of X_{cl}, i.e., $\underline{\mathrm{Aut}}_X(x) \coloneqq \underline{\mathrm{Aut}}_{X_{\mathrm{cl}}}(x)$ (0.5.3).

0.5.5. *Finite type and finite presentation hypotheses.* A morphism of derived Artin stacks is *(locally) of finite type* or *(locally) of finite presentation* if it the induced morphism on classical truncations has the respective property. It is *homotopically of finite presentation* if it is locally of finite presentation and the relative cotangent complex is perfect.

We denote by Stk_S (resp. dStk_S) the ∞-category of (resp. derived) Artin stacks that are locally of finite type over S and have quasi-compact and separated diagonal.

0.5.6. *Perfect and coherent complexes.* Let X be a derived Artin stack. We have the stable ∞-category $\mathbf{D}_{\mathrm{qc}}(X)$ of quasi-coherent complexes on X and the full subcategories $\mathbf{D}_{\mathrm{coh}}(X)$ and $\mathbf{D}_{\mathrm{perf}}(X)$ of coherent and perfect complexes, respectively. We write $\mathbf{D}^{\geqslant n}_{\mathrm{perf}}(X)$ for the full subcategory of perfect complexes of Tor-amplitude $\geqslant n$ (or $[n, \infty)$), using cohomological grading. See e.g. [Kha4, §1].

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1. Intersection theory on stacks

In this section we briefly summarize the definitions and main properties of Borel–Moore homology of Artin stacks, and the derived specialization map, from [Kha2].

1.1. **Borel–Moore homology.** We let $\mathbf{D}$ be a six functor formalism, incarnated as a constructible ∞-category in the sense of [Kha3] (a.k.a. a motivic ∞-category in the sense of [Kha1, CD1]), on locally of finite type k-schemes. We will assume that $\mathbf{D}$ is *oriented*, meaning that it admits a theory of Thom isomorphisms (and hence that the corresponding cohomology admits Chern classes). For example:

(i) *Betti:* Let k be a $\mathbf{C}$-algebra, Λ a commutative ring, and $\mathbf{D}(X)$ the derived ∞-category of sheaves of Λ-modules on the topological space $X(\mathbf{C})$.

(ii) *Étale:* Let Λ be a commutative ring of characteristic $n > 0$, $n \in k^\times$, and $\mathbf{D}(X)$ the derived ∞-category of sheaves of Λ-modules on the small étale site of X. (One may also take ℓ-adic coefficients, see e.g. [LZ2]).

(iii) *Rational motives:* Let k be a noetherian commutative ring, assume Λ contains $\mathbf{Q}$, and let $\mathbf{D}(X)$ be the ∞-category of Beilinson motives as in [CD1, §14].

(iv) *Integral motives:* Let Λ be a commutative ring and $\mathbf{D}(X)$ the ∞-category $\mathbf{D}_{\mathrm{H}\Lambda}(X)$ of modules over the motivic Eilenberg–MacLane spectrum $H\Lambda_X$ as in [Spi]. When k is a field whose characteristic is zero or invertible in Λ, this is equivalent to the ∞-category of integral motives defined in [CD2] (by Theorem 5.1 in *op. cit.*). (This is a relative version of Voevodsky motives: for $X = \mathrm{Spec}(k)$ it recovers the construction of [Voe].)

(v) *Cobordism motives:* Let Λ be a commutative ring and $\mathbf{D}(X)$ the ∞-category $\mathbf{D}_{\mathrm{MGL}}(X)$ of modules over Voevodsky's algebraic cobordism spectrum MGL_X (see e.g. [EHKSY]). (This may be regarded as the universal example: see Remark 1.7 below.)

We extend this to locally of finite type Artin stacks over k as in [Kha2, App. A] (see also [Kha5]). (When $\mathbf{D}$ satisfies étale descent, as in the first three examples, this coincides with the construction of [LZ1].) For every $X \in \mathrm{dStk}_k$ we let $\mathbf{1}_X \in \mathbf{D}(X)$ denote the monoidal unit, and we introduce the notation $\langle n \rangle$ for the endofunctor $(n)[2n] : \mathbf{D}(X) \to \mathbf{D}(X)$ where $n \in \mathbf{Z}$ and (n) denotes Tate twist.

Definition 1.1. Let $X \in \mathrm{dStk}_k$ with projection $a_X : X \to \mathrm{Spec}(k)$. Given an object $\Lambda \in \mathbf{D}(\mathrm{Spec}(k))$ we define the following objects of $\mathbf{D}(\mathrm{Spec}(k))$:

(i) *Cochains*: $\mathrm{C}^\bullet(X;\Lambda) \coloneqq a_{X,*}a_X^*(\Lambda)$.

(ii) *Borel–Moore chains*: $\mathrm{C}^{\mathrm{BM}}_\bullet(X;\Lambda) \coloneqq a_{X,*}a_X^!(\Lambda)$.

If Λ has a commutative ring structure, then it induces one on $\mathrm{C}^\bullet(X;\Lambda)$ (cup product), and $\mathrm{C}^{\mathrm{BM}}_\bullet(X;\Lambda)$ is a module over it (cap product). In the following, we will keep Λ fixed and leave it implicit in the notation.

Remark 1.2. It follows from the definition of $\mathbf{D}(X)$ for $X \in \mathrm{dStk}_k$ that $\mathrm{C}^{\mathrm{BM}}_\bullet(X;\Lambda)$ can be described as a homotopy limit

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X;\Lambda) \simeq \varprojlim_{(U,u)} \mathrm{C}^{\mathrm{BM}}_\bullet(U)\langle -d_u\rangle$$

over the category of pairs (U,u) where $U \in \mathrm{dStk}_k$ is schematic and $u : U \to X$ is a smooth morphism of relative dimension d_u, and smooth morphisms between them; the transition arrows are given by smooth Gysin pull-backs.

Variant 1.3 (Relative)**.** If $f : X \to S$ is locally of finite type where $S \in \mathrm{dStk}_k$, we have *relative Borel–Moore chains*:

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}) \coloneqq f_*f^!(\Lambda_S) \in \mathbf{D}(S),$$

where $\Lambda_S = a_S^*(\Lambda)$ for $a_S : S \to \mathrm{Spec}(k)$ the projection. (As X and S vary these form a "homotopically enhanced" version of a bivariant theory in the sense of [FM]; see [DJK] for this perspective). Note that $\mathrm{C}^{\mathrm{BM}}_\bullet(X) = \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/k})$ and $\mathrm{C}^\bullet(X) = \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/X})$.

Variant 1.4 (Equivariant)**.** Let $S \in \mathrm{dStk}_k$ and $X \in \mathrm{dStk}_S$. For G an fppf group algebraic space over k we write

$$\mathrm{C}^\bullet_G(X) \coloneqq \mathrm{C}^\bullet([X/G]),$$
$$\mathrm{C}^{\mathrm{BM},G}_\bullet(X) \coloneqq \mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/BG}),$$

for the G-equivariant cochains and G-equivariant Borel–Moore chains of X.

Notation 1.5. We consider the "periodizations":

$$\mathrm{C}^\bullet(X)\langle *\rangle \coloneqq \bigoplus_{n\in\mathbf{Z}} \mathrm{C}^\bullet(X)\langle n\rangle,$$
$$\mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle *\rangle \coloneqq \bigoplus_{n\in\mathbf{Z}} \mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -n\rangle,$$

and similarly for relative Borel–Moore chains. Thus for example

$$\pi_*\big(\mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle *\rangle\big) = \mathrm{H}^{\mathrm{BM}}_*(X)$$

where the right-hand side is as in the introduction. We also define $\widehat{\mathrm{C}}^\bullet(X)\langle *\rangle$ and $\widehat{\mathrm{C}}^{\mathrm{BM}}_\bullet(X)\langle *\rangle$ similarly, but with direct sum replaced by direct product.

Example 1.6. Let $\mathbf{D} = \mathbf{D}_{\mathrm{H}\Lambda}$ be the constructible ∞-category of motives with coefficients in a commutative ring Λ, and suppose k is a field whose characteristic is zero or invertible in Λ. Then for every finite type k-scheme X, the derived global sections of the (twisted) Borel–Moore motive of X is

$$R\Gamma(\mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -n\rangle) \simeq \mathbf{z}_n(X)_\Lambda$$

where the right-hand side is the Λ-linear Bloch cycle complex. Taking homotopy groups, we have

$$\pi_s R\Gamma(\mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -n\rangle) \simeq \mathrm{CH}_n(X,s)_\Lambda$$

where the right-hand side is Bloch's higher Chow group (with coefficients in Λ). In particular, for $s=0$ we recover the usual Chow group of n-cycles on X with coefficients in Λ, and $\mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle *\rangle$ is a spectrum whose π_0 is the total Chow group $\mathrm{CH}_*(X)_\Lambda$.

Remark 1.7. The constructible ∞-category $\mathbf{D}_{\mathrm{MGL}}$ of modules over Voevodsky's algebraic cobordism spectrum MGL can be regarded as the universal six functor formalism which is oriented in a homotopy coherent sense (cf. [EHKSY]). Oriented constructible ∞-categories in nature are equipped with a canonical morphism $R^* : \mathbf{D}_{\mathrm{MGL}} \to \mathbf{D}$ (which moreover factors through Voevodsky motives $\mathbf{DM}$ when the orientation is via the additive formal group law), i.e. a natural transformation such that $R^* : \mathbf{D}_{\mathrm{MGL}}(X) \to \mathbf{D}(X)$ is a colimit-preserving functor for each $X \in \mathrm{dStk}_k$ and which commutes with $\otimes$, $*$-inverse image, and smooth !-direct image. (Such a universal property can be made precise with rational coefficients, cf. [CD1, Cor. 14.2.16].)

Given a constructible ∞-category $\mathbf{D}$ with a morphism $R : \mathbf{D}_{\mathrm{MGL}} \to \mathbf{D}$ and an object $\Lambda \in \mathbf{D}(k)$, we have by adjunction

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S};\Lambda) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}; R_*\Lambda)$$

where R_* is the right adjoint of the colimit-preserving functor R^*, and the right-hand side is formed with respect to the six functor formalism $\mathbf{D}_{\mathrm{MGL}}$. In particular, it will generally be harmless for our purposes to assume that $\mathbf{D} = \mathbf{D}_{\mathrm{MGL}}$, e.g. that $\mathbf{D}$ is compactly generated or even constructibly generated in the sense of [DFJK, Def. A.7].

1.2. **Localization triangle.**

Theorem 1.8. *Let $S \in \mathrm{dStk}_k$, $X \in \mathrm{dStk}_S$, and $i : Z \to X$, $j : U \to X$ a pair of complementary closed and open immersions. Then there is a canonical exact triangle*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(Z_{/S}) \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}) \xrightarrow{j^!} \mathrm{C}^{\mathrm{BM}}_\bullet(U_{/S})$$

in $\mathbf{D}(S)$.

The localization triangle is compatible with proper push-forward and quasi-smooth Gysin maps:

Lemma 1.9. *Let $S \in \mathrm{dStk}_k$ and suppose given a diagram*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f_Z"] & X' \arrow[d, "f"] & U' \arrow[l, "j'"'] \arrow[d, "f_U"] \\ Z \arrow[r, "i"] & X & U \arrow[l, "j"'] \end{tikzcd}$$

of commutative squares in dStk_S, where (i,j) and (i',j') are pairs of complementary closed and open immersions, where f, f_Z, and f_U are proper.

Then there is a commutative diagram

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(Z'_{/S}) \arrow[r, "i'_*"] \arrow[d, "f_{Z,*}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(X'_{/S}) \arrow[r, "j'^!"] \arrow[d, "f_*"] & \mathrm{C}^{\mathrm{BM}}_\bullet(U'_{/S}) \arrow[d, "f_{U,*}"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(Z_{/S}) \arrow[r, "i_*"] & \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}) \arrow[r, "j^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(U_{/S}). \end{tikzcd}$$

Lemma 1.10. *Let $S \in \mathrm{dStk}_k$ and suppose given a diagram*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f_Z"] & X' \arrow[d, "f"] & U' \arrow[l, "j'"'] \arrow[d, "f_U"] \\ Z \arrow[r, "i"] & X & U \arrow[l, "j"'] \end{tikzcd}$$

of homotopy cartesian squares in dStk_S, where i and j are complementary closed and open immersions and f is quasi-smooth. Then there is a commutative diagram

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(Z_{/S}) \arrow[r, "i_*"] \arrow[d, "f^!_Z"] & \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}) \arrow[r, "j^!"] \arrow[d, "f^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(U_{/S}) \arrow[d, "f^!_U"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(Z'_{/S}) \arrow[r, "i'_*"] & \mathrm{C}^{\mathrm{BM}}_\bullet(X'_{/S}) \arrow[r, "j'^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(U'_{/S}). \end{tikzcd}$$

Proof. The left-hand square commutes by base change for quasi-smooth Gysin maps, and the right-hand square commutes by functoriality of quasi-smooth Gysin maps. □

1.3. **Specialization to the normal bundle.** We recall the derived deformation space from [HKR] (see also [Kha2, Thm. 1.3]).

Theorem 1.11. *Let $f : X \to Y$ be a homotopically finitely presented morphism in dStk_k. Then there exists a commutative diagram of derived Artin stacks*

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d, "0"] & X \times \mathbf{A}^1 \arrow[d, "\widehat{f}"] & X \times \mathbf{G}_m \arrow[l] \arrow[d, "f\times\mathrm{id}"] \\ N_{X/Y} \arrow[r, "i"] \arrow[d] & D_{X/Y} \arrow[d] & Y \times \mathbf{G}_m \arrow[l, "j"'] \arrow[d, equal] \\ Y \arrow[r, "0"] & Y \times \mathbf{A}^1 & Y \times \mathbf{G}_m \arrow[l] \end{tikzcd} \tag{1.12}$$

where each square is homotopy cartesian.

Proof. See [HKR]; we sketch the proof here. One defines $D_{X/Y}$ as the *derived Weil restriction* of $X \to Y$ along the inclusion $0 : Y \hookrightarrow Y \times \mathbf{A}^1$, or equivalently the derived mapping stack $\underline{\mathrm{Map}}_{Y\times\mathbf{A}^1}(Y \times \{0\}, X \times \mathbf{A}^1)$. It is easy to see that this derived stack satisfies the desired properties, and the nontrivial part is the algebraicity (i.e., that it is Artin). If the base k is a derived G-ring which admits a dualizing complex (e.g., k is of finite type over a field or $\mathbf{Z}$), then we can appeal to [HLP, Thm. 5.1.1]. In general, see [HKR]. □

Construction 1.13 (Specialization)**.** Let $S \in \mathrm{dStk}_k$ and $f : X \to Y$ a homotopically of finite presentation morphism in dStk_S. Consider the pair of complementary closed and open immersions

$$N_{X/Y} \xrightarrow{i} D_{X/Y} \xleftarrow{j} Y \times \mathbf{G}_m.$$

In the localization triangle

$$\mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S}) \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM}}_\bullet(D_{X/Y}{}_{/S}) \xrightarrow{j^!} \mathrm{C}^{\mathrm{BM}}_\bullet(Y \times \mathbf{G}_{m/S}),$$

the boundary map

$$\partial : \mathrm{C}^{\mathrm{BM}}_\bullet(Y \times \mathbf{G}_{m/S})[-1] \to \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S})$$

gives rise to the *specialization map*

$$\begin{aligned} \mathrm{sp}_{X/Y} : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) &\xrightarrow{\mathrm{incl}} \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \oplus \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S})(1)[1] \\ &\simeq \mathrm{C}^{\mathrm{BM}}_\bullet(Y \times \mathbf{G}_{m/S})[-1] \xrightarrow{\partial} \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S}), \end{aligned} \tag{1.14}$$

where the splitting comes from the unit section $1 : Y \to Y \times \mathbf{G}_m$.

Proposition 1.15 (Specialization and proper push-forward)**.** *Let $S \in \mathrm{dStk}_k$ and suppose given a commutative square Δ*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

in dStk_S. Suppose that q is proper and the square is cartesian on classical truncations. Then there is a canonical homotopy

$$N_{\Delta,*} \circ \mathrm{sp}_{X'/Y'} \simeq \mathrm{sp}_{X/Y} \circ q_*$$

of maps $\mathrm{C}^{\mathrm{BM}}_\bullet(Y'_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S})$.

Proof. As in the proof of Proposition 1.17, the assumption implies that $D_\Delta : D_{X'/Y'} \to D_{X/Y}$ is proper (see [HKR]), so the claim follows from the compatibility of the localization triangle with proper direct image (Lemma 1.9). ☐

Corollary 1.16. *Let $S \in \mathrm{dStk}_k$ and $i : Z \to X$ a closed immersion in dStk_S. Denote by $0 : Z \to N_{Z/X}$ the zero section of the derived normal bundle. Then there is a canonical homotopy*

$$\mathrm{sp}_{Z/X} \circ i_* \simeq 0_*$$

of maps $\mathrm{C}^{\mathrm{BM}}_\bullet(Z_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(N_{Z/X}{}_{/S})$.

Proof. Apply Proposition 1.15 to the self-intersection square

$$\begin{array}{ccc} Z & = & Z \\ \| & & \downarrow{\scriptstyle i} \\ Z & \xrightarrow{i} & X \end{array}$$

and note that $\mathrm{sp}_{Z/Z} = \mathrm{id}$. ☐

Proposition 1.17 (Specialization and quasi-smooth pull-backs)**.** *Let $S \in \mathrm{dStk}_k$ and suppose given a commutative square Δ*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

in dStk_S. Suppose that q and the induced morphism $N_\Delta : N_{X'/Y'} \to N_{X/Y}$ are both quasi-smooth. (For example, suppose q is quasi-smooth and the square Δ is homotopy cartesian.) Then there is a canonical homotopy

$$N_\Delta^! \circ \mathrm{sp}_{X/Y} \simeq \mathrm{sp}_{X'/Y'} \circ q^!$$

of maps $\mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X'/Y'}{}_{/S})\langle -d \rangle$, where d is the relative virtual dimension of q.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} N_{X'/Y'} \arrow[r] \arrow[d, "N_\Delta"] & D_{X'/Y'} \arrow[d, "D_\Delta"] & Y' \times \mathbf{G}_m \arrow[l] \arrow[d, "q\times\mathrm{id}"] \\ N_{X/Y} \arrow[r] \arrow[d] & D_{X/Y} \arrow[d] & Y \times \mathbf{G}_m \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbf{A}^1 & \mathbf{G}_m \arrow[l] \end{tikzcd}$$

Note that both upper squares are homotopy cartesian (since the lower squares and the left-hand and right-hand composite rectangles all are). Therefore, the morphism D_Δ is quasi-smooth (since this can be checked fibrewise). By construction of the specialization maps, it is enough to show the following square commutes:

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(Y \times \mathbf{G}_{m/S})[-1] \arrow[r, "\partial_{N_{X/Y}/D_{X/Y}}"] \arrow[d, "(q\times\mathrm{id})^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S}) \arrow[d, "N_\Delta^!"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(Y' \times \mathbf{G}_{m/S})[-1] \arrow[r, "\partial_{N_{X'/Y'}/D_{X'/Y'}}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X'/Y'}{}_{/S}) \end{tikzcd}$$

where the horizontal arrows are the boundary maps in the respective localization triangles. But this follows from Lemma 1.10 applied to the above diagram. □

Remark 1.18. Let $f : X \to Y$ be a quasi-smooth morphism in dStk_k. Denote by $0 : X \to N_{X/Y}$ the zero section. Applying Proposition 1.17 to the commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

recovers the fact that

$$f^! \simeq 0^! \circ \mathrm{sp}_{X/Y}$$

which was in fact our definition of the left-hand side.

1.4. Projective bundle formula.

Theorem 1.19. *Let $S \in \mathrm{dStk}_k$ and $X \in \mathrm{dStk}_S$. Let $\mathscr{E}$ be a locally free sheaf on X of rank $r+1$ ($r \geqslant 0$), and write $\pi : P = \mathbf{P}(\mathscr{E}) \to X$ for the associated projective bundle. Then the maps*

$$\pi^!(-) \cap e(\mathscr{O}(-1))^{\cup i} : \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(P_{/S})\langle -r+i\rangle,$$

induce a canonical isomorphism of $\mathrm{C}^\bullet(X)$-modules

$$\bigoplus_{0\leqslant i\leqslant r} \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})\langle r-i\rangle \to \mathrm{C}^{\mathrm{BM}}_\bullet(P_{/S}). \tag{1.20}$$

Proof. The map (1.20) is the limit over smooth morphisms $t : T \to X$ with T a scheme of the analogous maps

$$\bigoplus_{0\leqslant i\leqslant r} \mathrm{C}^{\mathrm{BM}}_\bullet(T_{/S})\langle r-i-d_t\rangle \to \mathrm{C}^{\mathrm{BM}}_\bullet(T \underset{S}{\times} P_{/S})\langle -d_t\rangle,$$

where d_t is the relative dimension of $t : T \to X$. We may therefore assume that X is a scheme. By Zariski descent on X, we may further assume that $\mathscr{E}$ admits a trivial direct summand. It is therefore enough to show the following:

($*$) If (1.20) is invertible for $\mathscr{E} = \mathscr{E}_0$ (locally free of rank r), then it is also invertible for $\mathscr{E} = \mathscr{E}_0 \oplus \mathscr{O}$.

Let $P_0 = \mathbf{P}(\mathscr{E}_0)$, $P = \mathbf{P}(\mathscr{E}_0 \oplus \mathscr{O})$, and $E = \mathbf{V}(\mathscr{E}) = P \smallsetminus P_0$. Consider the following diagram:

$$\begin{tikzcd}
\bigoplus_0^{r-1} \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})\langle r-1-i\rangle \arrow[r] \arrow[d] & \bigoplus_0^{r} \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})\langle r-i\rangle \arrow[r] \arrow[d] & \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})\langle r\rangle \arrow[d] \\
\mathrm{C}^{\mathrm{BM}}_\bullet(P_{0/S}) \arrow[r] & \mathrm{C}^{\mathrm{BM}}_\bullet(P_{/S}) \arrow[r] & \mathrm{C}^{\mathrm{BM}}_\bullet(E_{/S})
\end{tikzcd}$$

where the upper row is split exact and the lower horizontal row is the localization triangle. The right-hand vertical arrow is Gysin along the projection $E \to X$ (which is invertible by homotopy invariance), and the left-hand and middle vertical arrows are (1.20) for $\mathscr{E} = \mathscr{E}_0$ and $\mathscr{E} = \mathscr{E}_0 \oplus \mathscr{O}$, respectively. Hence it will suffice to show that this diagram is commutative.

The inclusion $i : P_0 \hookrightarrow P$ exhibits P_0 as the derived zero locus on P of the cosection $\pi^*(\mathscr{O}_P(-1)) \oplus \mathscr{O}_P \to \mathscr{O}_P$ (the projection onto the second component), so we have a canonical homotopy

$$i_* i^!(-) \simeq (-) \cap e(\mathscr{O}_P(-1))$$

of maps $\mathrm{C}^{\mathrm{BM}}_\bullet(P_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(P_{/S})\langle 1\rangle$. The induced homotopy

$$i_* \pi_0^!(-) \simeq \pi^!(-) \cap e(\mathscr{O}_P(-1)),$$

where $\pi_0 : P_0 \to X$ and $\pi : P \to X$ are the projections, then gives rise to a homotopy up to which the left-hand square in the above diagram commutes. Since $\mathscr{O}_P(-1)$ on P restricts to the trivial line bundle on E, there are canonical homotopies $e(\mathscr{O}_P(-1))^{\cup i}|_E \simeq 0$ for all $i > 0$, whence a homotopy up to which the right-hand square also commutes. The claim follows. ☐

1.5. **Proper (co)descent.** The assignment $X \mapsto \mathbf{D}(X)$ determines two presheaves of ∞-categories $\mathbf{D}^*$ and $\mathbf{D}^!$ on the ∞-category dStk_k, where $\mathbf{D}^?$ is given by $X \mapsto \mathbf{D}(X)$ on objects and $f \mapsto f^?$ on morphisms (for $? \in \{*, !\}$). In the following statement, *proper descent* means Čech descent with respect to proper 1-Artin surjective morphisms between Artin stacks.

Remark 1.21. Following [CD1, Def. 2.1.7], a constructible ∞-category $\mathbf{D}$ is *semi-separated* if f^* is conservative for every finite radicial surjection f, and *separated* if it is semi-separated and f^* is conservative also for every finite *étale* surjection f; note that these properties can be checked on schemes, by descent. For example, the derived ∞-category of étale sheaves (say with $\mathbf{Z}/n\mathbf{Z}$-coefficients where n is prime to the residue characteristics of k) is separated, and every (resp. oriented) constructible ∞-category with rational coefficients is semi-separated (resp. separated) (see [EK]).

Theorem 1.22. *If $\mathbf{D}$ is separated, then the presheaves $\mathbf{D}^*$ and $\mathbf{D}^!$ satisfy proper descent. Moreover, for $S \in \mathrm{dStk}_k$ over k and any $\mathcal{F} \in \mathbf{D}(S)$, we have:*

(i) *The assignment $(f : X \to S) \mapsto f_* f^*(\mathcal{F})$, regarded as a $\mathbf{D}(S)$-valued presheaf on the ∞-category dStk_S, satisfies proper descent.*

(ii) *The assignment $(f : X \to S) \mapsto f_! f^!(\mathcal{F})$, regarded as a $\mathbf{D}(S)$-valued presheaf on the ∞-category dStk_S, satisfies proper co-descent.*

See [Kha5] for the proof of Theorem 1.22.

Corollary 1.23 (Proper codescent)**.** *Assume $\mathbf{D}$ is separated. Let $S \in \mathrm{dStk}_k$ and $p : X \twoheadrightarrow Y$ a proper 1-Artin surjective morphism between $X, Y \in \mathrm{dStk}_S$. Then $\mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S})$ is the homotopy colimit of the Čech nerve*

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{C}^{\mathrm{BM}}_\bullet(X \underset{Y}{\times} X_{/S}) \rightrightarrows \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}),$$

regarded as a simplicial diagram in $\mathbf{D}(S)$.

1.6. **Invariance for torsors under finite groups.**

Theorem 1.24. *Let $\mathbf{D}$ be separated (see Remark 1.21). Let $S \in \mathrm{dStk}_k$, $X, Y \in \mathrm{dStk}_S$, and G a finite group scheme*[4] *of multiplicative type over X. Then for every BG-torsor $f : Y \to X$ over S, the direct image map*

$$f_* : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})$$

is invertible.

Proof. By étale descent for $\mathbf{D}$ we may assume that X is affine, G is diagonalizable, and f is trivial, i.e. $Y = X \times BG$. Note that if G is a product $H \times H'$ (of group schemes over X), then by base change it is enough to show the claim for $G = H$ and $G = H'$. Since G is diagonalizable it is enough to treat

[4] i.e., a group stack which is finite schematic over X

$G = \mu_{n,X}$, hence G is either finite étale over X or $G_{\mathrm{red}} \simeq X$. In the latter case the claim follows from nilpotent invariance of $\mathrm{C}^{\mathrm{BM}}_\bullet$ and in the former case we may use étale descent again to reduce to the case where G is finite discrete.

Thus let G be a finite discrete group scheme and let us show that $f_* : \mathrm{C}^{\mathrm{BM}}_\bullet(BG_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})$ is invertible. Let $s : X \twoheadrightarrow BG$ denote the quotient map. Consider the simplicial diagram of direct image maps

$$\mathrm{C}^{\mathrm{BM}}_\bullet(G^{\bullet+1}_{/S})_{hG} \to \mathrm{C}^{\mathrm{BM}}_\bullet(G^\bullet_{/S})$$

where $G^\bullet$, resp. $G^{\bullet+1}$ is the Čech nerve of $X \twoheadrightarrow BG$, resp. $G \twoheadrightarrow X$. This is an isomorphism in every degree by finite Galois codescent (since G acts freely on G and its iterated fibre powers over S). By proper codescent, passing to the colimit gives rise to the isomorphism

$$s_* : \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S}) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})_{hG} \to \mathrm{C}^{\mathrm{BM}}_\bullet(BG_{/S})$$

where the isomorphism on the left is because G acts trivially on X. Since $f_* s_* \simeq \mathrm{id}$, it follows that f_* is also invertible as claimed. ☐

1.7. **Reduction to the subtorus.**

Theorem 1.25. *Let $S \in \mathrm{dStk}_k$ and $X \in \mathrm{dStk}_S$. Suppose $G = \mathrm{GL}_n$ acts on X and denote by $T \subseteq G$ the subgroup of diagonal matrices. Then Gysin pull-back along the smooth morphism $p : [X/T] \to [X/G]$*

$$p^! : \mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet([X/T]_{/S})\langle -d\rangle,$$

where $d = \dim(G/T)$, admits a $\mathrm{C}^\bullet([X/G])$-module retract. Moreover, after rationalization it induces an isomorphism

$$p^! : \mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S})_{\mathbf{Q}} \to \mathrm{C}^{\mathrm{BM}}_\bullet([X/T]_{/S})^{hW}_{\mathbf{Q}},$$

where $(-)^{hW}$ denotes homotopy invariants with respect to the W-action induced by the canonical W-action on $[X/T]$.

Proof. Since $[X/T] \to [X/B]$ is an affine bundle, where $B \subseteq G$ is the Borel subgroup, the Gysin map

$$\mathrm{C}^{\mathrm{BM}}_\bullet([X/B]_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet([X/T]_{/S})$$

is invertible by homotopy invariance. Therefore, it is enough to show that the Gysin map

$$\mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet([X/B]_{/S})$$

admits a retract. Under the Morita isomorphism $[X/B] \simeq [(X \times G/B)/G]$ (with G acting on G/B by conjugation), this is identified with the Gysin map

$$f^! : \mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet([(X \times G/B)/G]_{/S})$$

where $f : [(X \times G/B)/G] \to [X/G]$ is the canonical morphism. Note that f is proper (since G/B is proper over k), so that there is a proper push-forward f_*. We claim that this provides the desired retract. Indeed, since G/B is the scheme of complete flags in $\mathbf{A}^n_X$ over X, the projection f factors as a sequence of iterated projective bundles. Thus the claim follows from the fact that for a projective bundle π, the pullback $\pi^!$ admits a retract by the projective bundle formula (Theorem 1.19), cf. [Ful, Ex. 3.3.5]. The second part follows similarly from the projective bundle formula; see again *loc. cit.* and also recall that with rational coefficients, the homotopy groups of the homotopy invariants can be computed as the ordinary invariants of the homotopy groups. □

1.8. **The Chern character.** Let R be an oriented motivic commutative ring spectrum over k, e.g., a commutative ring object in $\mathbf{D}_{\mathrm{MGL}}(\mathrm{Spec}(k))$. Then its rationalization $R_{\mathbf{Q}}$ is canonically an algebra over $\mathbf{Q}^{\mathrm{mot}}\langle * \rangle$ (see [CD1, Cor. 14.2.16]), and hence over KGL via the Chern character map

$$\mathrm{KGL} \to \mathrm{KGL}_{\mathbf{Q}}^{\text{ét}} \simeq \prod_{n} \mathbf{Q}^{\mathrm{mot}}\langle n \rangle.$$

Here is KGL is Voevodsky's algebraic K-theory spectrum. For example, for any fppf group algebraic space G over k, there is a ring homomorphism

$$\mathrm{R}(G) \simeq \mathrm{K}_0(BG) \to \pi_0 \widehat{\mathrm{C}}^{\bullet}(BG; R_{\mathbf{Q}})$$

where $\mathrm{R}(G)$ is the representation ring.

In particular, if $\mathbf{D}$ is *rational* in the sense that $\mathbf{SH}(k) \to \mathbf{D}(k)$ factors through $\mathbf{SH}(k)_{\mathbf{Q},+}$ (cf. [CD1, Cor. 14.2.16]), then for any object $\Lambda \in \mathbf{D}(k)$ and $X \in \mathrm{dStk}_k$ with G-action it follows that $\widehat{\mathrm{C}}^{\bullet}_G(X; \Lambda)$ is $\mathrm{K}(BG)$-module.

2. Concentration

In this section we assume the base ring k is noetherian. Let $\Lambda \in \mathbf{D}(k)$ be an object, fixed throughout the section. As per our conventions recalled in Subsect. 1.1, we will leave Λ implicit in the notation.

2.1. **The master theorem.** Given any set Σ of line bundles on an Artin stack $\mathcal{X}$, our main result is a stabilizer-wise criterion for torsionness of Borel–Moore chains on $\mathcal{X}$ with respect to the first Chern classes of the line bundles in Σ.

Notation 2.1. Let $\mathcal{X} \in \mathrm{dStk}_k$ be quasi-compact and let $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$ be a subset. We denote by

$$\mathrm{C}^{\bullet}(\mathcal{X})_{\mathrm{loc}} \coloneqq \mathrm{C}^{\bullet}(\mathcal{X})_{\Sigma\text{-loc}} \coloneqq \mathrm{C}^{\bullet}(\mathcal{X})\langle * \rangle[c_1(\Sigma)^{-1}]$$

the localization, and similarly for $\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X})_{\mathrm{loc}} \coloneqq \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X})_{\Sigma\text{-loc}}$.

Definition 2.2. Let $\mathcal{X} \in \mathrm{dStk}_k$ and $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$ a subset. We introduce the following condition on Σ:

(L) For every geometric point x of $\mathcal{X}$, there exists an invertible sheaf $\mathscr{L}(x) \in \Sigma$ whose restriction $\mathscr{L}(x)|_{B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)}$ is trivial.

Remark 2.3. If k is a field, we may also consider variants of condition (L) where we look at k-rational points or $\bar{k}$-rational points for an algebraic closure $\bar{k}$ of k. The distinction between these versions will play no role in our proofs, so we may sometimes abuse language by referring to any of them as "condition (L)" when there is no risk of confusion. The same applies for other variants of this condition that we will consider below, such as conditions (K) and (L_G).

The main result of this section is as follows:

Theorem 2.4. *Let $S \in \mathrm{dStk}_k$, $\mathcal{X} \in \mathrm{dStk}_S$ quasi-compact 1-Artin with affine stabilizers, and $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$ a subset. If Σ satisfies condition (L) for $\mathcal{X}$, then we have*

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\Sigma\text{-loc}} = 0.$$

Corollary 2.5. *Let $\mathcal{X} \in \mathrm{dStk}_k$ be quasi-compact 1-Artin with affine stabilizers. If Σ satisfies condition (L) for $\mathcal{X}$, then we have $\mathrm{C}^{\bullet}(\mathcal{X})_{\Sigma\text{-loc}} = 0$.*

Proof. Take $S = \mathcal{X}$ in Theorem 2.4. ☐

Corollary 2.6. *Let $S \in \mathrm{dStk}_k$, $\mathcal{Z}, \mathcal{X} \in \mathrm{dStk}_k$ quasi-compact 1-Artin with affine stabilizers, and $i : \mathcal{Z} \to \mathcal{X}$ a closed immersion over S. Let $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$ be a subset satisfying condition (L) for $\mathcal{X} \smallsetminus \mathcal{Z}$. Then the direct image map*

$$i_* : \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}_{/S})_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\Sigma\text{-loc}}$$

is invertible.

Proof. Follows by the localization triangle. ☐

Remark 2.7. Taking homotopy groups, we recover the statements about $\mathrm{H}^{\mathrm{BM}}_{*}(-)_{\mathrm{loc}}$ made in the introduction. See [Lur1, Prop. 7.2.3.25(2)].

We will also consider the following variant where Σ is a set of K-theory classes.

Variant 2.8. Let $\Sigma \subseteq \mathrm{K}_0(\mathcal{X})$ be a set of *K-theory* classes on $\mathcal{X}$. We may consider the following variant of condition (L):

(K) For every geometric point x of $\mathcal{X}$, we have

$$\mathrm{K}_0(B\underline{\mathrm{Aut}}_{\mathcal{X}}(x))[\Sigma^{-1}] = 0,$$

where the action of Σ is given by inverse image along $B\underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to \mathcal{X}$.

Remark 2.9. Condition (K) can be reformulated as follows: for every geometric point x of $\mathcal{X}$, there exists a K-theory class $\alpha(x) \in \mathrm{K}_0(\mathcal{X})$ which belongs to the multiplicative closure of Σ such that $\alpha(x)|_{B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)} = 0$.

Recall from Subsect. 1.8 that when we work with rational coefficients, there is also canonical action of K-theory on Borel–Moore chains via the Chern character. As in Notation 2.1 we write $(-)_{\Sigma\text{-loc}}$ for the localization $(-)\langle * \rangle[\Sigma^{-1}]$ We have the following analogue of Theorem 2.4:

Theorem 2.10. *Suppose that* $\mathbf{D}$ *is rational in the sense of Subsect. 1.8. Let $S \in \mathrm{dStk}_k$, $\mathcal{X} \in \mathrm{dStk}_S$ quasi-compact 1-Artin with affine stabilizers, and $\Sigma \subseteq \mathrm{K}_0(\mathcal{X})$ a subset. If Σ satisfies condition (K) for $\mathcal{X}$, then we have*

$$\widehat{\mathrm{C}}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\Sigma\text{-loc}} := \widehat{\mathrm{C}}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})\langle * \rangle[\Sigma^{-1}] = 0.$$

Recall the notation $\widehat{\mathrm{C}}^{\mathrm{BM}}_{\bullet}(-)\langle * \rangle$ from Notation 1.5.

2.2. **Generic slices.** The proof of Theorem 2.4 will require a useful result of Thomason (see [Tho2, Thm. 4.10, Rem. 4.11]) which we reformulate here in stacky terms for the reader's convenience.

Proposition 2.11. *Let T be a diagonalizable torus of finite type over a noetherian scheme S. Let X be a reduced algebraic space of finite type over S with T-action. Then there exists a T-invariant nonempty affine open subspace $U \subseteq X$ such that there is a canonical isomorphism*

$$[U/T] \simeq BT' \underset{S}{\times} V$$

of stacks over BT, for some subgroup $T' \subseteq T$ and affine scheme V over S.

Proof. Recall that the schematic locus $X_0 \subseteq X$ is a nonempty open (see [SP, Tags 03JH, 03JG]). We claim that X_0 is T-invariant, i.e., the action morphism $T \times_S X_0 \to X$ factors through X_0. In other words, the open immersion $T \times_S X_0 \times_X X_0 \to T \times_S X_0$ (base change of the inclusion $X_0 \to X$) is invertible. We may assume that S is the spectrum of an algebraically closed field k, and it will suffice to check this on k-points. Let $t : \mathrm{Spec}(k) \to T$ be a k-point. Then t defines an isomorphism $t : X \to X$. Then $t(X_0) \subseteq X$ is a scheme and hence $t(X_0) \subseteq X_0$ (since X_0 is the largest open which is a scheme). Since t was arbitrary, this shows that X_0 is T-invariant.

Replacing X by X_0, we may assume that X is a scheme. Now the claim follows by combining [Tho2, Lem. 4.3][5] and [Tho2, Thm. 4.10, Rem. 4.11]. ☐

2.3. **Proof of Theorem 2.4.** Let the notation be as in Theorem 2.4. By derived and nil-invariance of Borel–Moore homology, we may replace all derived stacks by their reduced classical truncations. Since $\mathcal{X}$ has affine stabilizers, it admits a stratification by global quotient stacks (see [HR, Prop. 2.6]), so again using the localization triangle we may assume that $\mathcal{X} = [X/G]$, where X is a reduced quasi-affine scheme of finite type over k and $G = \mathrm{GL}_n$, $n \geqslant 0$. Using Theorem 1.25, we may replace G by its maximal subtorus $\mathbf{G}_m^{\times n}$ and thereby assume G is a split torus. (Note that the condition on Σ is clearly preserved under this change of notation.)

By Proposition 2.11, there is a nonempty G-invariant affine open U of X and a diagonalizable subgroup H of G such that $[U/G] \simeq Y \times BH$ with Y an affine scheme. By noetherian induction and the localization triangle, we may replace X by U, and then further replace X by Y and G by H so that $\mathcal{X} = X \times BG$. (Again, note that the condition on Σ is preserved.)

We can assume X is nonempty, so that by condition (L) there exists a geometric point x of X and an invertible sheaf $\mathscr{L} \in \mathrm{Pic}(X)$ such that $\mathscr{L}|_{BG_{k(x)}}$ is trivial. It is enough to show that $c = c_1(\mathscr{L})$ is nilpotent as an element of the ring $\pi_0 \mathrm{C}^\bullet(\mathcal{X})\langle * \rangle$.

Since G is diagonalizable, we may write $\mathscr{L} = \mathscr{L}_m|_{\mathcal{X}} \otimes m\mathscr{O}_{BG}|_{\mathcal{X}}$ where m is a character of G and $\mathscr{L}_m$ is an invertible sheaf on X (see [SGA3, Exp. I, 4.7.3]).

[5] One could skip the reductions above by simply observing that the argument of *loc. cit.* goes through without the locally separatedness assumptions.

The equality $c|_{BG_{k(x)}} = 0$ thus implies that the invertible sheaf $\mathscr{L}|_{BG_{k(x)}}$ is trivial, i.e., $m = 1$ and $\mathscr{L} \simeq \mathscr{L}_1|_{\mathfrak{X}}$. Thus $c = c_1(\mathscr{L}_1)|_{\mathfrak{X}}$. Since X is a scheme, $c_1(\mathscr{L}_m)$ is nilpotent (see e.g. [Dég, Prop. 2.1.22(1)]), hence so is c.

Remark 2.12. One can always find a *finite* subset $\Sigma_0 \subseteq \Sigma$ such that the conclusion of Theorem 2.4 still holds. Indeed, note that after the reductions we only need a *single* element of Σ, and in each reduction step we only need to use finitely many more elements.

2.4. **Proof of Theorem 2.10.** After the same reductions as in the proof of Theorem 2.4, we reduce to the case where $\mathfrak{X} = X \times BG$ with X a (nonempty) reduced quasi-affine scheme and G a diagonalizable group scheme. By condition (K), there exists a geometric point x of $\mathfrak{X}$ and a class $\alpha \in \Sigma$ with $\alpha|_{BG_{k(x)}}$ trivial, where $G_{k(x)} \simeq \underline{\mathrm{Aut}}_{\mathfrak{X}}(x)$ is the base change of G to the residue field $k(x)$. It is enough to show that α is nilpotent as an element of $\mathrm{K}_0(\mathfrak{X})$. If N denotes the group of characters of G, then we have $\mathrm{K}_0(\mathfrak{X}) \simeq \mathrm{K}_0(X) \otimes_{\mathbf{Z}} \mathbf{Z}[N]$ (see [SGA3, Exp. I, 4.7.3], as in [Tho2, Lem 4.14]). Thus we may write $\alpha = \sum_i a_i \otimes b_i$, where $a_i \in \mathrm{K}_0(X)$ and b_i are pairwise distinct elements of N. Restricting α along $BG_{k(x)} \to \mathfrak{X}$ and identifying $\mathrm{K}_0(BG_{k(x)}) \simeq \mathbf{Z}[N]$, we get $\sum_i \mathrm{rk}(a_i) \cdot b_i = 0$ in $\mathbf{Z}[N]$, and hence $\mathrm{rk}(a_i) = 0$ for all i. Thus by [SGA6, Exp. VI, Prop. 6.1], a_i are all nilpotent as elements of $\mathrm{K}_0(X)$. In particular, α is also nilpotent as claimed.

2.5. **The equivariant master theorem.** We formulate a G-equivariant version of condition (L):

Definition 2.13. Let G be an fppf group algebraic space over k acting on $X \in \mathrm{dStk}_k$, and $\Sigma \subseteq \mathrm{Pic}(BG)$ a subset. We introduce the following condition on Σ:

(L_G) For every geometric point of X there exists a rank one G-representation $\mathscr{L}(x) \in \Sigma$ such that the $\mathrm{St}^G_X(x)$-representation $\mathscr{L}(x)|_{B\mathrm{St}^G_X(x)}$ is trivial.

Here $\mathrm{St}^G_X(x)$ is the G-stabilizer at the point x (see Definition A.4).

Specializing Theorem 2.4 to the case of quotient stacks yields the following equivariant version of the master theorem. As in Notation 2.1, $(-)_{\Sigma\text{-loc}}$ denotes $(-)\langle * \rangle [c_1(\Sigma^{-1})]$.

Corollary 2.14. *Let $S \in \mathrm{dStk}_k$ and G an fppf group algebraic space over k acting on a 1-Artin $X \in \mathrm{dStk}_S$ with affine stabilizers. Let $\Sigma \subseteq \mathrm{Pic}(BG)$ be a subset which satisfies condition (L_G) for X. Then we have*

$$\mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S})_{\Sigma\text{-loc}} = 0$$

and in particular $\mathrm{C}^{\mathrm{BM},G}_\bullet(X)_{\Sigma\text{-loc}} = 0$.

Proof. Apply Theorem 2.4 to the quotient stack $\mathfrak{X} = [X/G]$ (with $\Sigma \subseteq \mathrm{Pic}(\mathfrak{X})$ the inverse image of Σ along $\mathfrak{X} = [X/G] \to BG$). Since there is a commutative square

$$\begin{tikzcd} B\underline{\mathrm{Aut}}_{\mathcal{X}}(x) \arrow[r] \arrow[d] & B\mathrm{St}^G_X(x) \arrow[d] \\ \mathcal{X} \arrow[r] & BG, \end{tikzcd}$$

triviality of a line bundle over $B\mathrm{St}^G_X(x)$ implies triviality over $B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$. □

Corollary 2.15. *Let G be an fppf group algebraic space over k. Let $S \in \mathrm{dStk}_k$ and $i : Z \to X$ a G-equivariant closed immersion over S where $Z, X \in \mathrm{dStk}_S$ are quasi-compact 1-Artin with affine stabilizers. Let $\Sigma \subseteq \mathrm{Pic}(BG)$ be a subset which satisfies condition (L$_G$) for $X \smallsetminus Z$. Then the direct image map*

$$i_* : \mathrm{C}^{\mathrm{BM}}_\bullet([Z/G]_{/S})_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S})_{\Sigma\text{-loc}}$$

is invertible.

We similarly have an equivariant version of conditions (K):

Variant 2.16. Let G act on $X \in \mathrm{dStk}_k$ and $\Sigma \subseteq \mathrm{R}(G) \simeq \mathrm{K}_0(BG)$ a subset.

(K$_G$) For every geometric point x of X we have

$$\mathrm{K}_0(B\mathrm{St}^G_X(x))[\Sigma^{-1}] = 0.$$

For Λ rational, we may then specialize Theorem 2.10 to the equivariant case as above.

2.6. Base change.

Lemma 2.17. *Let $\mathcal{X}, \mathcal{X}' \in \mathrm{dStk}_k$ be 1-Artin and let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism. Let $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$ be a subset and denote by $\Sigma' \subseteq \pi_0\mathrm{C}^\bullet(\mathcal{X}')\langle * \rangle$ its image by f^*. If Σ satisfies condition (L) for $\mathcal{X}$, then Σ' satisfies condition (L) for $\mathcal{X}'$.*

Proof. Let x' be a geometric point of $\mathcal{X}'$ and consider its image $x = f(x)$ in $\mathcal{X}$. By assumption, there exists an invertible sheaf $\mathscr{L}(x) \in \Sigma$ with $c_1(\mathscr{L}(x))|_{B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)} = 0$. Then its inverse image $\mathscr{L}(x') \coloneqq f^*\mathscr{L}(x)$ belongs to Σ'. Since there is a morphism of group schemes $\underline{\mathrm{Aut}}_{\mathcal{X}'}(x') \to \underline{\mathrm{Aut}}_{\mathcal{X}}(x)$, we also have $c_1(\mathscr{L}(x'))|_{B\underline{\mathrm{Aut}}_{\mathcal{X}'}(x')} = 0$. □

Corollary 2.18. *Let G be an fppf group algebraic space over k acting on $X, X' \in \mathrm{dStk}_k$ which are 1-Artin and let $f : X' \to X$ be a G-equivariant morphism. If $\Sigma \subseteq \mathrm{Pic}(BG)$ satisfies condition (L$_G$) for X, then it also satisfies condition (L$_G$) for X'.*

Combining this with equivariant concentration (Theorem 2.15) yields:

Corollary 2.19. *Let G be an fppf group algebraic space over k. Let $S \in \mathrm{dStk}_k$, $Z, X \in \mathrm{dStk}_S$ quasi-compact 1-Artin with affine stabilizers, and $i : Z \to X$ a G-equivariant closed immersion over S. Let $\Sigma \subseteq \mathrm{Pic}(BG)$ be a subset which satisfies condition (L$_G$) for $X \smallsetminus Z$. Then for every morphism $f : X' \to X$, direct image along the base change $i' : Z' = Z \times_X X' \to X'$ induces an isomorphism*

$$i'_* : \mathrm{C}^{\mathrm{BM},G}_\bullet(Z')_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM},G}_\bullet(X')_{\Sigma\text{-loc}}.$$

Construction 2.20. Let G be an fppf group algebraic space over k, acting on a finite type 1-Artin $X \in \mathrm{Stk}_k$ with affine stabilizers. Suppose we are given a G-equivariant morphism $\pi : X \to M$ to a finite type algebraic space $M \in \mathrm{Stk}_k$. Denote by M^G the fixed locus of M (see Proposition A.23), and form the cartesian square

$$\begin{tikzcd} M^G \times_M X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ M^G \arrow[r] & M. \end{tikzcd}$$

Corollary 2.21. *Let the notation be as in Construction 2.20. For any subset $\Sigma \subseteq \mathrm{Pic}(BG)$, condition ($L_G$) for $M \smallsetminus M^G$ implies condition (L_G) for $X \smallsetminus (M^G \times_M X)$. In particular, the direct image map*

$$i_* : \mathrm{C}^{\mathrm{BM},G}_\bullet(M^G \underset{M}{\times} X)_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM},G}_\bullet(X)_{\Sigma\text{-loc}}$$

is invertible.

Remark 2.22. For example, if $G = T$ is diagonalizable then we may take Σ to be the set of nontrivial rank one representations $\mathscr{L} \in \mathrm{Pic}(BT)$. Compare [Jos, Prop. 6.9] for G-theory of smooth Deligne–Mumford stacks with torus action over an algebraically closed field.

Remark 2.23. For example, if X has finite inertia (e.g. it is separated and Deligne–Mumford) then it admits a coarse moduli space M that is of finite type [KM]. Moreover, the G-action automatically descends to M by universal properties, in such a way that $\pi : X \to M$ is equivariant. Similarly, if X admits a good or adequate moduli space $\pi : X \to M$ in the sense of Alper [Alp1, Alp2] and is of finite type, then M is of finite type [Alp2, Thm. 6.3.3].

2.7. **Non-quasi-compact stacks.** Let $\mathcal{X}$ be a derived 1-Artin stack which has affine stabilizers but is only *locally* of finite type over k, and let $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$. Unfortunately, concentration in the form of Theorem 2.4 does not hold for such $\mathcal{X}$: we may have

$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X})_{\mathrm{loc}} \neq 0$$

even when Σ satisfies condition (L) for $\mathcal{X}$. In fact, we have simple counterexamples to Theorem 3.1 even for torus actions (and there are only finitely many T-stabilizer groups appearing).

Example 2.24. For every integer $n \geqslant 0$, consider the weight 1 scaling action of $T = \mathbf{G}_m$ on $X_n = \mathbf{A}^n \smallsetminus 0$ and the element $\alpha_n = t \cdot [X_n] \in \mathrm{C}^{\mathrm{BM},T}_\bullet(X_n)\langle -n+1\rangle$, where t is the first Chern class of the tautological line bundle on BT. Note that we have $t^i \cdot (\alpha_n) = 0$ if and only if $i \geqslant n-1$. Therefore, if $X = \coprod_n X_n$, the element

$$\alpha = (\alpha_n)_n \in \widehat{\mathrm{H}}^{\mathrm{BM},T}_*(X)_{\mathbf{Q}}$$

does not vanish after inverting t (see (0.1) for notation). Similarly, consider $Y = \coprod_n X_n \times W_n$ where W_n is smooth 1-Artin of finite type over k, of pure dimension $-n$ with trivial T-action. Then $\beta_n = t \cdot [X_n \times W_n] \in \mathrm{C}^{\mathrm{BM},T}_\bullet(X_n \times W_n)\langle 1\rangle$, and the element

$$\beta = (\beta_n)_n \in \mathrm{H}^{\mathrm{BM},T}_*(Y)_{\mathbf{Q}}$$

does not vanish after inverting t.

In this subsection we give a definition of "localized Borel–Moore homology"

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X})_{\mathrm{loc}}$$

which is the same as the localization of Borel–Moore homology when $\mathcal{X}$ is quasi-compact, but for which $\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X})_{\mathrm{loc}} = 0$ still holds under condition (L). If we write $\mathcal{X}$ as a filtered union of quasi-compact opens $\{\mathcal{X}_\alpha\}_\alpha$, then we will see $\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X})_{\mathrm{loc}} \simeq \varprojlim_\alpha \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_\alpha)_{\mathrm{loc}}$.

Definition 2.25. Let $S \in \mathrm{dStk}_k$, $\mathcal{X} \in \mathrm{dStk}_S$, and $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$. We define

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} \coloneqq \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\Sigma\text{-loc}} \coloneqq \varprojlim_{\mathcal{X}'\subseteq\mathcal{X}} \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}'_{/S})\langle *\rangle[c_1(\Sigma^{-1})]$$

where the homotopy limit is taken over quasi-compact opens $\mathcal{X}' \subseteq \mathcal{X}$ (with restriction maps as the transition arrows).

Since condition (L) is stable under restriction to opens, the following non-quasi-compact generalization of Theorem 2.4 is immediate from the definition:

Theorem 2.26. *Let $S \in \mathrm{dStk}_k$, $\mathcal{X} \in \mathrm{dStk}_S$ which is 1-Artin with affine stabilizers, and $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$. If Σ satisfies condition (L) for $\mathcal{X}$, then we have*

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} = 0.$$

We next note that this definition is compatible with Notation 2.1.

Proposition 2.27. *Let $S \in \mathrm{dStk}_k$, $\mathcal{X} \in \mathrm{dStk}_S$, and $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$. If $\mathcal{X}$ is a filtered union of quasi-compact opens $\{\mathcal{X}_\alpha\}_\alpha$, then there is a canonical isomorphism*

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} \simeq \varprojlim_{\alpha} \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{\alpha/S})_{\mathrm{loc}}.$$

Proof. Consider the following commutative square:

$$\begin{tikzcd}
\varprojlim_{\mathcal{X}'} \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}')\langle *\rangle[\Sigma^{-1}] \arrow[r] \arrow[d] & \varprojlim_{\alpha} \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_\alpha)\langle *\rangle[\Sigma^{-1}] \arrow[d] \\
\varprojlim_{\mathcal{X}'} \varprojlim_{\alpha} \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_\alpha \cap \mathcal{X}')\langle *\rangle[\Sigma^{-1}] \arrow[r, equal] & \varprojlim_{\alpha} \varprojlim_{\mathcal{X}'} \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_\alpha \cap \mathcal{X}')\langle *\rangle[\Sigma^{-1}]
\end{tikzcd}$$

where we omit the "$_{/S}$" from the notation for simplicity. The upper horizontal arrow comes from the fact that each $\mathcal{X}_\alpha$ is quasi-compact. The vertical arrows are induced by restriction. The right-hand one is invertible because $\mathcal{X}_\alpha \hookrightarrow \mathcal{X}$ is cofinal in $\{\mathcal{X} \cap \mathcal{X}_\alpha \hookrightarrow \mathcal{X}\}_\alpha$. The left-hand one is invertible because $\mathcal{X}'$ is quasi-compact: we may choose a finite subset of opens in $(\mathcal{X}_\alpha \cap \mathcal{X}')_\alpha$ which cover $\mathcal{X}'$, and then by filteredness there exists some large enough index β such that $\mathcal{X}_\beta \cap \mathcal{X}' = \mathcal{X}'$. ☐

Corollary 2.28. *Let $S \in \mathrm{dStk}_k$, $\mathcal{X} \in \mathrm{dStk}_S$, and $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$. If $\mathcal{X}$ is quasi-compact then there is a canonical isomorphism*

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} \simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})\langle *\rangle[c_1(\Sigma)^{-1}].$$

Note also that $\mathrm{C}^{\mathrm{BM}}_\bullet(-)_{\mathrm{loc}}$ still satisfies the localization triangle:

Proposition 2.29. *Let $S \in \mathrm{dStk}_k$, $\mathcal{X} \in \mathrm{dStk}_S$, and $\Sigma \subseteq \mathrm{Pic}(\mathcal{X})$. Then for every closed immersion $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ with open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$, we have a canonical exact triangle*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Z}_{/S})_{\mathrm{loc}} \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \xrightarrow{j^!} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{U}_{/S})_{\mathrm{loc}}.$$

Proof. Write $\mathcal{X}$ as a filtered union of quasi-compact opens $\{\mathcal{X}_\alpha\}_\alpha$ (e.g. take the partially ordered set of all quasi-compact opens). For each α we have the localization triangle

$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Z} \cap \mathcal{X}_{\alpha/S})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{\alpha/S})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{U} \cap \mathcal{X}_{\alpha/S})\langle * \rangle[\Sigma^{-1}]$$

since localization preserves exact triangles (as an exact functor). Passing to the homotopy limit (which is also exact) thus yields the claim by Corollary 2.27. $\square$

As before we may now specialize to the equivariant case. Let $S \in \mathrm{dStk}_k$, G be an fppf group algebraic space over k which acts on a 1-Artin $X \in \mathrm{dStk}_S$ with affine stabilizers, and $\Sigma \subseteq \mathrm{Pic}(BG)$ a subset. We have

$$\mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S})_{\mathrm{loc}} = \varprojlim_{X' \subseteq X} \mathrm{C}^{\mathrm{BM}}_\bullet([X'/G]_{/S})\langle * \rangle[c_1(\Sigma)^{-1}]$$

where the limit is taken over quasi-compact G-invariant opens $X' \subseteq X$. We get the following non-quasi-compact version of Corollary 2.14:

Corollary 2.30. *If $\Sigma \subseteq \mathrm{Pic}(BG)$ satisfies condition (L_G) for X, then we have*

$$\mathrm{C}^{\mathrm{BM}}_\bullet([X/G]_{/S})_{\Sigma\text{-loc}} = 0.$$

For $\mathbf{D}$ rational, we have a parallel version of the above story for any $\Sigma \subseteq \mathrm{K}_0(\mathcal{X})$, so that we get a non-quasi-compact version of Theorem 2.10 under condition (L).

3. Torus concentration

In this section we fix a split torus T over k, and we assume that k is noetherian with no nontrivial idempotents.

3.1. The statement.

Theorem 3.1. *Let T act on a 1-Artin $X \in \mathrm{dStk}_k$. Let $Z \subseteq X$ be a T-invariant closed substack such that for every point x of the complement $X \smallsetminus Z$, $\mathrm{St}^T_X(x)$ is properly contained in $T_{k(x)}$. Then we have:*

(i) *Direct image along the closed immersion $i : Z \to X$ induces an isomorphism*

$$i_* : \mathrm{C}^{\mathrm{BM},T}_\bullet(Z)_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X)_{\Sigma\text{-loc}}$$

of $\mathrm{C}^\bullet(BT)_{\Sigma\text{-loc}}$-modules, where $\Sigma \subseteq \mathrm{Pic}(BT)$ is the set of nontrivial line bundles on BT.

(ii) *If X is quasi-compact then there exists a* finite *subset $\Sigma_0 \subseteq \Sigma$ such that we may replace Σ by Σ_0 in claim (i).*

Remark 3.2. In Theorem 3.1 the condition that $\mathrm{St}^T_X(x) \subsetneq T_{k(x)}$ is equivalent to the condition that $\dim(\mathrm{St}^T_X(x)) < \dim(T_{k(x)})$, since $T_{k(x)}$ is irreducible.

Remark 3.3. Given a subgroup $H \subsetneq T$, consider the (diagonalizable) quotient $K = T/H$. Choose a nontrivial character of K, corresponding to a one-dimensional representation of K. The T-representation obtained by restriction may be regarded as a line bundle $\mathscr{L}$ on BT whose restriction $\mathscr{L}|_{BH}$ is trivial.

Example 3.4. Suppose that the fixed locus X^T (Definition A.11) is empty, i.e., for every geometric point x of X the T-stabilizer $\mathrm{St}^T_X(x)$ is not equal to $T_{k(x)}$. Consider the set $\mathfrak{G}$ of subgroups $H \subseteq T$ for which there exists a geometric point x of X such that the T-stabilizer $\mathrm{St}^T_X(x)$ is equal to the base change $H_{k(x)}$. For every $H \in \mathfrak{G}$ let $\mathscr{L}(H)$ be a nontrivial line bundle on BT such that $\mathscr{L}(H)|_{BH}$ is trivial (as in Remark 3.3). Then $\Sigma_0 = \{\mathscr{L}(H)\}_{H\in\mathfrak{G}}$ satisfies condition (L_T) for X.

Proposition 3.5. *Let T act on a quasi-compact 1-Artin $X \in \mathrm{dStk}_k$. Then there exists a nonempty T-invariant open of X whose T-stabilizers are constant. In particular, the set $\mathfrak{G}$ (Example 3.4) is finite.*

Proof. Note first of all that the second statement follows from the first by noetherian induction.

For the main statement, we may as well assume that X is reduced (since the statement only involves geometric points). For X an algebraic space, the statement follows immediately from Proposition 2.11. In general we argue as follows. By [SP, Tag 06QJ] and generic flatness of the morphism $[I_X/T] \to [X/T]$ (quotient of the projection $I_X \to X$), there exists a nonempty T-invariant open $X_0 \subseteq X$ such that X_0 is a gerbe (with respect to a flat finitely presented group algebraic space) over an algebraic space. Note also that this algebraic space is locally of finite type over k (since X is locally a trivial gerbe over it), hence of finite type over k since it is quasi-compact (because X is). Replacing X by X_0 we may therefore assume that X is a gerbe over an algebraic space M of finite type over k.

The T-action on X descends along $\pi : X \to M$, for example because it is a coarse moduli space. By the algebraic space case, it will suffice to show that $\pi : X \to M$ is T-stabilizer-preserving, i.e., for every geometric point x of X the canonical morphism

$$\mathrm{St}^T_X(x) \to \mathrm{St}^T_M(\pi(x))$$

is invertible. Since this is a homomorphism of subgroups of $T_{k(x)}$, it is enough to show surjectivity. A geometric point of $\mathrm{St}^T_M(\pi(x))$ is a geometric point t of T such that $t \cdot \pi(x) = \pi(x)$. But this means precisely that there is an identification $t \cdot x \simeq x$ as geometric points of X, since on geometric points π exhibits M as the set of connected components of X. $\square$

Proof of Theorem 3.1. Let $\Sigma_0 \subseteq \Sigma$ be as in Example 3.4; when X is quasi-compact this is finite by Proposition 3.5. Thus it will suffice to show that i_* becomes invertible after inverting $c_1(\Sigma_0)$. By construction, Σ_0 satisfies condition (L_T) for $X \smallsetminus Z$, so the claim follows directly from Corollary 2.30 at least when X has affine stabilizers.

Otherwise, by the localization triangle we may replace X with $X \smallsetminus Z$ and thereby reduce to showing that $\mathrm{C}^{\mathrm{BM},T}_\bullet(X)_{\mathrm{loc}} = 0$ when Σ_0 satisfies condition (L_T) for X. As in the proof of Proposition 3.5, using the localization triangle again, we may assume that there exists a T-equivariant stabilizer-preserving morphism $\pi : X \to M$ which exhibits X as a gerbe over an algebraic space M of finite type over k. Then Σ_0 satisfies condition (L_T) for M as it does for X and π is a surjection on geometric points. Since M is an algebraic space, it follows from Proposition 2.11 that there exists a nonempty T-invariant affine open U of M and an invertible sheaf $\mathscr{L} \in \Sigma_0$ such that $c_1(\mathscr{L})$ is nilpotent as an element of the ring $\pi_0\mathrm{C}^\bullet_T(U)\langle * \rangle$ (see proof of Theorem 2.4). Therefore $V = \pi^{-1}(U)$ is a nonempty T-invariant open in X such that $c_1(\mathscr{L})$ is nilpotent in $\pi_0\mathrm{C}^\bullet_T(V)\langle * \rangle$. In particular

$$\mathrm{C}^{\mathrm{BM},T}_\bullet(V)\langle * \rangle[c_1(\mathscr{L})^{-1}] = 0$$

as it is a module over $\mathrm{C}^\bullet_T(V)\langle * \rangle[c_1(\mathscr{L})^{-1}] = 0$. The claim now follows by noetherian induction using the localization triangle. □

3.2. **Example: fixed loci.** In Appendix A we define a T-fixed locus $X^T \subseteq X$ (Definition A.11). In terms of X^T, the condition in Theorem 3.1 is that the open complement $X \smallsetminus Z$ is contained in $X \smallsetminus X^T$. Although the substack $X^T \subseteq X$ is not, a priori, closed in general (see Question A.14), we obtain the following variants of concentration using the reduced fixed locus (Definition A.16) or homotopy fixed point stack (Definition A.17) when X has finite stabilizers or is Deligne–Mumford, respectively.

Corollary 3.6. *Let the notation be as in Theorem 3.1 and suppose X has finite stabilizers. Denote by X^T_{red} the reduced T-fixed locus (Definition A.16). Then direct image along the closed immersion $i : X^T_{\mathrm{red}} \to X$ (Proposition A.15) induces an isomorphism*

$$i_* : \mathrm{C}^{\mathrm{BM},T}_\bullet(X^T_{\mathrm{red}})_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X)_{\Sigma\text{-loc}}$$

of $\mathrm{C}^\bullet(BT)_{\Sigma\text{-loc}}$-modules.

In the Deligne–Mumford case we can alternatively use the homotopy fixed point stack.

Corollary 3.7. *Let the notation be as in Theorem 3.1 and suppose $X \in \mathrm{dStk}_k$ is quasi-compact Deligne–Mumford. Choose a reparametrization $T' \twoheadrightarrow T$ as in Corollary A.49 and denote by $X^{hT'}$ the homotopy fixed point stack with respect to the induced T'-action (Definition A.17). Then direct image along the closed immersion $\varepsilon : X^{hT'} \to X$ (Proposition A.27) induces an isomorphism*

$$\varepsilon_* : \mathrm{C}^{\mathrm{BM},T}_\bullet(X^{hT'})_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X)_{\Sigma\text{-loc}}$$

of $\mathrm{C}^\bullet(BT)_{\Sigma\text{-loc}}$-modules.

3.3. **Example: vector bundles and abelian cones.** For future use we record the following example:

Proposition 3.8. *Let $X \in \mathrm{dStk}_k$ be 1-Artin with finite stabilizers. Let $\mathscr{E} \in \mathrm{Coh}^T(X) \simeq \mathrm{Coh}(X \times BT)$ be a T-equivariant coherent sheaf on X. Write $E = \mathbf{V}_X(\mathscr{E})$ for the associated cone over X with T-action, and $E \smallsetminus X$ for the complement of the zero section. If $\mathscr{E}$ has no fixed part, i.e., $\mathscr{E}^{\mathrm{fix}} \simeq 0$, then for every point v of E, we have $\mathrm{St}^T_E(v) = T_{k(v)}$ if and only v belongs to the zero section.*

Proof. Suppose first that X is the spectrum of a field. Since T acts trivially on X, the T-representation $\mathscr{E}$ splits as a direct sum of 1-dimensional representations. Since $\mathscr{E}$ has no fixed part, the latter have nonzero weights, so the claim is clear in this case.

Now consider the general case, i.e., X is 1-Artin with finite stabilizers. Let v be a field-valued point of E. It is clear that if v belongs to the image of the zero section then it has $\mathrm{St}^T_E(v) = T_{k(v)}$. Conversely, suppose that $v \in E \smallsetminus X$. To show that the inclusion $\mathrm{St}^T_E(v) \subseteq T_{k(v)}$ is proper it will suffice to show that $\dim(\mathrm{St}^T_E(v)) < \dim(T_{k(v)})$ (since the scheme $T_{k(v)}$ is irreducible). Note that E also has finite stabilizers, since it is affine over X. By the short exact sequence of group schemes over $k(v)$ (A.5)

$$1 \to \underline{\mathrm{Aut}}_E(v) \to \underline{\mathrm{Aut}}_{\mathcal{E}}(v) \to \mathrm{St}^T_E(v) \to 1,$$

where $\mathcal{E} = [E/T]$ is the quotient, it will moreover suffice to show that $\dim(\underline{\mathrm{Aut}}_{\mathcal{E}}(v)) < \dim(T_{k(v)})$.

Let x denote the projection of v in X, E_x the fibre of E over x, and $\mathcal{E}_x = [E_x/T]$ the quotient. We have another short exact sequence

$$1 \to \underline{\mathrm{Aut}}_{\mathcal{E}_x}(v) \to \underline{\mathrm{Aut}}_{\mathcal{E}}(v) \to \underline{\mathrm{Aut}}_X(x),$$

by applying Remark A.1 twice, to the representable morphism $\mathcal{E}_x \to \mathcal{E}$ and to the morphism $\mathcal{E} \to X \times BT \to X$. By the special case of our claim where the base X is $\mathrm{Spec}(k(x))$, we have $\dim(\underline{\mathrm{Aut}}_{\mathcal{E}_x}(v)) < \dim(T_{k(v)})$. Since $\underline{\mathrm{Aut}}_X(x)$ is finite, it follows that $\dim(\underline{\mathrm{Aut}}_{\mathcal{U}}(v)) < \dim(T_{k(v)})$ as claimed. □

By Theorem 3.1 we get:

Corollary 3.9. *Let the notation be as in Proposition 3.8. Let $\Sigma \subseteq \mathrm{Pic}(BT)$ be the subset of nontrivial line bundles. Then Σ satisfies condition (L_T) for $E \smallsetminus X$.*

Corollary 3.10. *Let $S \in \mathrm{dStk}_k$, $X \in \mathrm{dStk}_S$ with finite stabilizers. Let $\mathscr{E}$ be a T-equivariant connective coherent complex on X with no fixed part, i.e., $\mathscr{E} \in \mathbf{D}_{\mathrm{coh}}(X \times BT)^{\leqslant 0}$ with $\mathscr{E}^{\mathrm{fix}} \simeq 0$, and write $E = \mathbf{V}_X(\mathscr{E})$. Then direct image along the zero section $0 : X \to E$ induces an isomorphism of $\mathrm{C}^\bullet(BT)_{\Sigma\text{-loc}}$-modules*

$$0_* : \mathrm{C}^{\mathrm{BM}}_\bullet(X \times BT_{/S})_{\Sigma\text{-loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet([E/T]_{/S})_{\Sigma\text{-loc}}, \tag{3.11}$$

where Σ is as in Corollary 3.9.

Proof. By derived invariance, we may assume that X is classical and $\mathscr{E}$ is 0-truncated (i.e., a coherent sheaf). Now the claim follows from Corollary 3.9 and Corollary 2.15. □

3.4. **Counterexample with infinite stabilizers.** We warn the reader that the finite stabilizers assumption in Corollary 3.10 is necessary, as we can already see in the "simplest" example of an Artin stack with infinite stabilizers, i.e. $B\mathbf{G}_m$. Indeed, even the conclusion of Proposition 3.8 does not hold in this example.

Example 3.12. Let $Z = B\mathbf{G}_m$, $X = [\mathbf{A}^1/\mathbf{G}_m]$, and $i : Z \to X$ the closed immersion induced by $0 : \mathrm{Spec}(k) \to \mathbf{A}^1$. Let $T = \mathbf{G}_m$ act on X with weight 1, so that i is T-equivariant and induces a closed immersion

$$[Z/T] \simeq B\mathbf{G}_m \times BT \to [\mathbf{A}^1/\mathbf{G}_m \times T] \simeq [X/T].$$

Then the induced direct image map

$$\mathrm{C}^{\mathrm{BM},T}_\bullet(B\mathbf{G}_m)_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet([\mathbf{A}^1/\mathbf{G}_m])_{\mathrm{loc}}$$

is *not* an isomorphism.

4. Equivariant concentration via stabilizer rank

In Sect. 2 we proved an abstract concentration theorem for the inclusion of a G-invariant closed substack (Corollary 2.15). Unlike the case of torus actions (Proposition 3.6), this need not apply to the inclusion of the fixed locus in the case of G a general algebraic group (see Example 4.13). In this section we will demonstrate one way of getting around this problem. Roughly speaking, the idea is to replace the G-locus by the locus of points whose G-stabilizer is of *reductive rank* equal to that of G. Recall that the reductive rank is equal to the rank of any maximal subtorus.

In this section, the base ring k is assumed noetherian.

4.1. **Rank functions for algebraic groups.** We begin with some general considerations about "rank functions" for algebraic groups.

Notation 4.1. Let G an fppf affine group scheme over k. Given a geometric point x of $\mathrm{Spec}(k)$, denote by $\mathrm{Sub}_G(x)$ denote the set of subgroups of $G_{k(x)} = G \times_k k(x)$. Denote by Sub_G the union of the set $\{G\}$ and the sets $\mathrm{Sub}_G(x)$ over x.

Definition 4.2. Let $\Gamma \subseteq \mathrm{Sub}_G$ be a subset. A *rank function* for G, defined on Γ and valued in a totally ordered set V, is a function $r : \Gamma \cup \{G\} \to V$.

(i) We say that r is *admissible* if for every $H \in \Gamma$ with $r(H) < r(G)$, the restriction homomorphism $X(G) \to X(H)$ is not injective (where $X(G)$ and $X(H)$ are the character groups of G and H, respectively).

(ii) Given a subset $\Sigma \subseteq \mathrm{R}(G)$, we say that r is Σ-*admissible* if for every $H \in \Gamma$ with $r(H) < r(G)$, the kernel of $\mathrm{R}(G) \to \mathrm{R}(H)$ contains some element of Σ.

Remark 4.3. If r is admissible, then it is Σ-admissible if Σ contains the elements $\lambda_{-1}(\mathscr{L}) = 1 - [\mathscr{L}]$ for all nontrivial line bundles $\mathscr{L} \in \mathrm{Pic}(BG)$.

Example 4.4. Let G be an fppf affine group scheme over k. The *character rank* of G is the rank of its character group $X(G)$, i.e., the dimension of the $\mathbf{Q}$-vector space $X(G) \otimes_{\mathbf{Z}} \mathbf{Q}$. This defines a rank function for G.

Example 4.5. Let G be an fppf affine group scheme over k. The *representation rank* of G is the transcendence degree[6] of the $\mathbf{Q}$-algebra $\mathrm{R}(G)_{\mathbf{Q}}$. This defines a rank function for G.

Lemma 4.6. *Suppose k is a field and let G be an algebraic group over k.*

(i) *If G is connected reductive, or k is of characteristic zero, then the representation rank of G is finite.*

(ii) *If G is connected reductive, or k is of characteristic zero and G is connected, then the representation rank of G agrees with its* reductive rank *(in the sense of* [SGA3, Exp. X, Rem. 8.7]*).*

Proof. First note that if $G = T$ is a torus of rank r, then $\mathrm{R}(T)_{\mathbf{Q}} \simeq \mathbf{Q}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ is of transcendence degree r. For G connected reductive, restriction to a maximal subtorus $T \subseteq G$ induces an isomorphism $\mathrm{R}(G)_{\mathbf{Q}} \simeq \mathrm{R}(T)_{\mathbf{Q}}^W$ by [Ser, Thm. 4], where W denotes the Weyl group. Since W is a finite group, it follows that the restriction morphism $\mathrm{R}(G)_{\mathbf{Q}} \to \mathrm{R}(T)_{\mathbf{Q}}$ is finite, and hence $\mathrm{trdeg}(\mathrm{R}(G)_{\mathbf{Q}}) = \mathrm{trdeg}(\mathrm{R}(T)_{\mathbf{Q}})$. By above, the latter is equal to the reductive rank of G.

Assume k is of characteristic zero and G is an arbitrary algebraic group over k. Then $\mathrm{R}(G)$ is a finite type $\mathbf{Z}$-algebra by [CG, 5.2.1], so the representation rank is finite. If G is connected then we may choose a Levi decomposition $G \simeq H.U$, where U is the unipotent radical and $H \subseteq G$ is connected reductive (see [Bor2, 11.22]). By [CG, 5.2.18], the restriction homomorphism $\mathrm{R}(G) \to \mathrm{R}(H)$ is bijective, so the representation rank of G is equal to that of H. The reductive rank of G is also equal to that of $G/U \simeq H$. Thus it follows from the connected reductive case that the representation rank is equal to the reductive rank. □

Lemma 4.7. *Let G be an fppf affine group scheme over k.*

(i) *The character rank is admissible if G is diagonalizable. Moreover, it is invariant under base change in k.*

(ii) *The character rank is admissible if k is an algebraically closed field.*

(iii) *Let $\Sigma \subseteq \mathrm{R}(G)$ be the subset of nonzero elements. Then the representation rank is Σ-admissible if k is an algebraically closed field.*

Proof. (i): Follows from e.g. [SGA3, Exp. IX, Rem. 1.4.1].

[6] Recall that for a $\mathbf{Q}$-algebra A, not necessarily of finite type, the transcendence degree is defined as the supremum of the cardinalities of all subsets of algebraically independent elements of A.

(ii): Suppose $H \subseteq G$ is a subgroup that is of strictly lesser character rank. This means that there exists some nontrivial character χ of G whose restriction to H is trivial. Then the tautological line bundle γ on $B\mathbf{G}_m$ pulls back along $B\chi : BG \to B\mathbf{G}_m$ to a nontrivial line bundle on BG whose restriction to BH is trivial.

(iii): If $H \subseteq G$ is a subgroup such that $\mathrm{R}(G) \to \mathrm{R}(H)$ is injective, then we have $\mathrm{trdeg}(\mathrm{R}(H)_{\mathbf{Q}}) \geqslant \mathrm{trdeg}(\mathrm{R}(G)_{\mathbf{Q}})$ from the definitions. □

4.2. **Condition (L_G) via ranks of stabilizers.**

Proposition 4.8. *Let G an fppf affine group scheme over k and $\Sigma \subseteq \mathrm{Pic}(BG)$ the subset of nontrivial line bundles on BG. Let X be a G-equivariant derived Artin stack with affine stabilizers. Let r be an admissible rank function defined on a subset $\Gamma \subseteq \mathrm{Sub}_G$ containing the G-stabilizer $\mathrm{St}^G_{k(x)}$ for every geometric point x of X. If for every geometric point x of X we have $r(\mathrm{St}^G_X(x)) < r(G)$, then Σ satisfies condition (L_G) for X.*

Proof. Suppose that x is a geometric point such that $r(\mathrm{St}^G_X(x)) < r(G)$. Since r is admissible, the restriction map $X(G) \to X(H)$ contains some nontrivial character χ. Then $B\chi^*(\gamma)$, where γ is the tautological line bundle on $B\mathbf{G}_m$ and $B\chi : BG \to B\mathbf{G}_m$ is the induced morphism, is a nontrivial line bundle on BG whose restriction to BH is trivial. This shows condition (L_G). □

Combining Proposition 4.8 with Lemma 4.7, we get the following examples:

Corollary 4.9. *Let T be a split torus over k acting on a 1-Artin $X \in \mathrm{dStk}_k$ such that for every geometric point x of X, the T-stabilizer $\mathrm{St}^T_X(x) \subsetneq T$ is a proper subgroup. Then the subset $\Sigma \subseteq \mathrm{Pic}(BT)$ of nontrivial line bundles on BT satisfies condition (L_T) for X.*

Corollary 4.10. *Let G be an affine algebraic group over an algebraically closed field k acting on a 1-Artin $X \in \mathrm{dStk}_k$ such that for every k-rational point x of X, the G-stabilizer $\mathrm{St}^G_X(x)$ is of character rank strictly less than that of G. Then the subset $\Sigma \subseteq \mathrm{Pic}(BG)$ of nontrivial line bundles on BG satisfies condition (L_G) for X.*

If Σ is instead a subset of $\mathrm{R}(G)$, we similarly have:

Proposition 4.11. *Let G an fppf affine group scheme over k acting on a 1-Artin $X \in \mathrm{dStk}_k$. Let $\Sigma \subseteq \mathrm{R}(G)$ be a subset and r a Σ-admissible rank function defined on a subset $\Gamma \subseteq \mathrm{Sub}_G$ containing the G-stabilizer $\mathrm{St}^G_{k(x)}$ for every geometric point x of X. If for every geometric point x of X we have $r(\mathrm{St}^G_X(x)) < r(G)$, then Σ satisfies condition (K_G) for X.*

Corollary 4.12. *Let G be an affine algebraic group over an algebraically closed field k acting on a 1-Artin $X \in \mathrm{dStk}_k$ such that for every k-rational point x of X, the G-stabilizer $\mathrm{St}^G_X(x)$ is of representation rank strictly less than that of G. Then the subset $\Sigma \subseteq \mathrm{R}(G)$ of nonzero elements satisfies condition (K_G) for X.*

4.3. **Counterexample: fixed loci.** Unlike the torus case (Subsect. 3.2), the G-fixed locus does not usually satisfy condition (L_G) (even for schemes). Indeed, we show that concentration does not even hold in this setting:

Example 4.13. Let k be a field and let $G = \mathrm{SL}_{2,k}$. Consider the three-dimensional irreducible G-representation V. Let X be the total space of V regarded as a k-scheme with G-action. Then the G-fixed locus X^G (Definition A.11) is the origin and the direct image map

$$0_* : \pi_0\mathrm{C}^{\mathrm{BM},G}_\bullet(\mathrm{Spec}(k))_{\mathbf{Q}}\langle *\rangle \to \pi_0\mathrm{C}^{\mathrm{BM},G}_\bullet(X)_{\mathbf{Q}}\langle *\rangle$$

is zero. However, the source

$$\pi_0\,\mathrm{C}^{\mathrm{BM},G}(\mathrm{Spec}(k))_{\mathbf{Q}}\langle *\rangle \simeq \pi_0\mathrm{C}^\bullet(BG)_{\mathbf{Q}}\langle *\rangle \simeq \mathbf{Q}[t^2],$$

is a nonzero integral domain (where t^2 is the top Chern class of the standard 2-dimensional representation of G). Thus the localization of 0_* is not an isomorphism.

5. Localization and integration formulas

For this section we assume the base ring k is noetherian with no nontrivial idempotents. We fix a split torus T over k.

Given a derived Artin stack with T-action, we will denote the quotient stack in script font (e.g. $\mathcal{X} = [X/T]$, $\mathcal{Y} = [Y/T]$, etc.). If T acts on X, we write

$$\mathrm{C}^\bullet_T(X)_{\mathrm{loc}} \coloneqq \mathrm{C}^\bullet(\mathcal{X})_{\mathrm{loc}} \coloneqq \mathrm{C}^\bullet(\mathcal{X})\langle *\rangle[c_1(\Sigma)^{-1}]$$

where Σ is the set of nontrivial line bundles on BT. Similarly, we write

$$\mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/Y})_{\mathrm{loc}} \coloneqq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/\mathcal{Y}})_{\mathrm{loc}} \coloneqq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/\mathcal{Y}}) \otimes_{\mathrm{C}^\bullet(BT)} \mathrm{C}^\bullet(BT)_{\mathrm{loc}}$$

for any T-equivariant morphism $X \to Y$, and more generally

$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \coloneqq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S}) \otimes_{\mathrm{C}^\bullet(\mathcal{X})} \mathrm{C}^\bullet(\mathcal{X})_{\mathrm{loc}}$$

when $X \to S$ is a locally of finite type morphism which is T-equivariant with respect to the trivial action on S.

5.1. **Fixed and moving parts of complexes.** Let X be a derived stack over k. Let G be a diagonalizable group scheme of finite type over k. Recall the following standard ∞-categorical version of [SGA3, Exp. I, 4.7.3]:

Proposition 5.1. *There is a canonical equivalence of stable ∞-categories*

$$\mathbf{D}_{\mathrm{qc}}(X \times BG) \to \prod_\chi \mathbf{D}_{\mathrm{qc}}(X),$$

where the product is taken over characters $\chi : G \to \mathbf{G}_m$.

More generally, let G be an fppf group scheme acting trivially on X. Then Čech descent along the cover $X \twoheadrightarrow [X/G] \simeq X \times BG$ yields a canonical equivalence between $\mathbf{D}_{\mathrm{qc}}(X \times BG)$ and the ∞-category of quasi-coherent $\mathcal{O}_{G_X}$-comodules, where $G_X = G \times X$. Now suppose that G is *diagonalizable*, so that $\mathcal{O}_{T_X} \simeq \mathcal{O}_X[M]$ is the group algebra of an abelian group M (= the group of characters of G). In this case one can argue as in the proof of [Mou, Prop. 4.2] to show that the ∞-category of quasi-coherent $\mathscr{O}_{G_X}$-comodules is equivalent to the ∞-category

$$\mathrm{Fun}(M, \mathbf{D}_{\mathrm{qc}}(X)) \simeq \prod_{\chi \in M} \mathbf{D}_{\mathrm{qc}}(X),$$

where M is regarded as a discrete set.

Given a quasi-coherent complex $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X \times BG)$, write $\mathscr{F}^{(\chi)}$ for the χ-eigenspace ($\chi \in M$), so that there are canonical isomorphisms

$$\bigoplus_{\chi} \mathscr{F}^{(\chi)} \to \mathscr{F},$$
$$\mathscr{F} \to \prod_{\chi} \mathscr{F}^{(\chi)}.$$

Indeed, the equivalence F of Proposition 5.1 admits left and right adjoints F^L and F^R, respectively, and these isomorphisms are the counit of (F^L, F) and unit of (F, F^R), respectively.

Definition 5.2. The *fixed part* of $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X \times BG)$ is its weight zero eigenspace and its *moving part* is the direct sum of its nonzero weight eigenspaces. We write

$$\mathscr{F}^{\mathrm{fix}} := \mathscr{F}^{(0)}, \quad \mathscr{F}^{\mathrm{mov}} := \bigoplus_{\chi \neq 0} \mathscr{F}^{(\chi)}.$$

5.2. **Euler classes.** In this subsection we define Euler classes of certain 2-term perfect complexes (Construction 5.9), using a generalized homotopy invariance property (Theorem 5.6).

Recall the following self-intersection formula:

Proposition 5.3. *Let $X \in \mathrm{dStk}_k$ and $\mathscr{E}$ a locally free sheaf of rank r on X. Let $E = \mathbf{V}(\mathscr{E})$ denote the total space and $0 : X \to E$ the zero section. Then the Euler class $e(\mathscr{E}) \in \mathrm{C}^\bullet(X)$ is canonically identified with the image of the unit $1 \in \mathrm{C}^\bullet(X)$ by the composite*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X_{/X}) \xrightarrow{0_*} \mathrm{C}^{\mathrm{BM}}_\bullet(E_{/X}) \xrightarrow{0^!} \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/X})\langle r\rangle \tag{5.4}$$

under the isomorphism $\mathrm{C}^\bullet(X) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/X})$.

Proof. See [Kha2, Cor. 3.17]. □

Proposition 5.5. *Let $S \in \mathrm{dStk}_k$, $X \in \mathrm{dStk}_S$ which is 1-Artin with finite stabilizers. Let $\mathscr{E}$ be a locally free sheaf of rank r on $X \times BT$ with no fixed part, i.e., $\mathscr{E}^{\mathrm{fix}} \simeq 0$. Then we have:*

(i) *The Euler class $e(\mathscr{E}) \in \mathrm{C}^\bullet(\mathcal{X})$ is invertible in $\mathrm{C}^\bullet(\mathcal{X})_{\mathrm{loc}}$.*

(ii) *Let $\pi : \mathcal{E} = \mathbf{V}(\mathscr{E}) \to \mathcal{X}$ be the projection of the total space and $0 : \mathcal{X} \to \mathcal{E}$ the zero section. Then we have a canonical homotopy*

$$\pi^! \simeq 0_*(- \cap e(\mathscr{E})^{-1})$$

of maps $\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{E}_{/S})_{\mathrm{loc}}$.

Proof.

(i) By Proposition 5.3 it suffices to show that both maps in (5.4) become invertible after localization. But $0^!$ is an isomorphism by homotopy invariance (even before localization) and 0_* becomes an isomorphism by Corollary 3.10.

(ii) From the formula

$$0_{E,*}0_E^! = (-) \cap e(\pi_E^*(\mathscr{E}))$$

we get

$$0_{E,*} = \pi_E^!(- \cap e(\mathscr{E}))$$

by applying $\pi_E^!$ on the right.

□

To define Euler classes for 2-term complexes, we will need the following generalized homotopy invariance property:

Theorem 5.6. *Let $S \in \mathrm{dStk}_k$, and $X \in \mathrm{dStk}_S$ which is 1-Artin with finite stabilizers. Regard X with trivial T-action. Let $\mathscr{E} \in \mathbf{D}^{T,\geqslant -1}_{\mathrm{perf}}(X)$ be a T-equivariant perfect complex whose fixed part $\mathscr{E}^{\mathrm{fix}}$ belongs to $\mathbf{D}^{T,\geqslant 0}_{\mathrm{perf}}(X)$. Let $E = \mathbf{V}_X(\mathscr{E})$ be the total space and $\pi : E \to X$ be the projection. Then π is quasi-smooth and the Gysin pull-back induces an isomorphism*

$$\pi^! : \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(E_{/S})_{\mathrm{loc}}$$

of $\mathrm{C}^\bullet_T(X)_{\mathrm{loc}}$-modules.

Proof. Using the localization triangle and stratifying X by global quotient stacks, we may assume that X has the resolution property.

Note that $\pi : E \to X$ factors through $\pi^{\mathrm{mov}} : E^{\mathrm{mov}} \to X$ and $E \to E^{\mathrm{mov}}$, which is a torsor under the vector bundle stack $\pi^{\mathrm{fix}} : E^{\mathrm{fix}} \to X$. By homotopy invariance for vector bundle stacks [Kha2, Prop. 2.20], we may therefore replace $\mathscr{E}$ by $\mathscr{E}^{\mathrm{mov}}$ and assume that $\mathscr{E}$ has no fixed part.

Since X has the resolution property, we may argue as in the proof of [Kha2, Prop. A.10] by induction on the Tor-amplitude of the perfect complex $\mathscr{E}$ to reduce to the case where

$$\mathscr{E} = \mathrm{Cofib}(\mathscr{E}^{-1} \to \mathscr{E}^0) \in \mathbf{D}^{T,[-1,0]}_{\mathrm{perf}}(X)$$

with $\mathscr{E}^{-1}, \mathscr{E}^0 \in \mathbf{D}^{T,[0,0]}_{\mathrm{perf}}(X)$. In this case we claim that

$$\pi^!(-) = 0_{E,*}(-) \cap \pi^*\big(e(\mathscr{E}^{-1}) \cap e(\mathscr{E}^0)^{-1}\big). \tag{5.7}$$

Recall that $e(\mathscr{E}^0)$ and $e(\mathscr{E}^{-1})$ are invertible by Proposition 5.5 and $0_{E,*}$ is invertible by Corollary 3.10, so this will imply that $\pi^!$ is invertible.

Let us prove (5.7). Note that the total space $E = \mathbf{V}(\mathscr{E})$ fits in a homotopy cartesian square

$$\begin{tikzcd} E \arrow[r, "s"] \arrow[d, "\pi"] & E_0 \arrow[d, "p"] \\ X \arrow[r, "0_{E_1}"] & E_1, \end{tikzcd}$$

where $E_0 = \mathbf{V}(\mathscr{E}^0)$ and $E_1 = \mathbf{V}(\mathscr{E}^{-1})$. Recall the formulas

$$0_{E_0,*} = \pi^!_{E_0}(-) \cap \pi^*_{E_0} e(\mathscr{E}^0)$$
$$0_{E_1,*} = \pi^!_{E_1}(-) \cap \pi^*_{E_1} e(\mathscr{E}^{-1})$$

from Proposition 5.5. The second implies

$$s_*\pi^! = \pi^!_{E_0}(-) \cap \pi^*_{E_0} e(\mathscr{E}^{-1})$$

by applying $p^!$ on the left and using the base change formula $p^! \circ 0_{E_1,*} \simeq s_* \circ \pi^!$. Since s_* is an isomorphism by Corollary 2.6, (5.7) now follows from the above identities. □

Notation 5.8. In the situation of Theorem 5.6, we write $0^!_T$ for the homotopy invariance isomorphism

$$0^!_T : \mathrm{C}^{\mathrm{BM},T}_\bullet(E_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}},$$

inverse to $\pi^!$.

We can now define Euler classes of certain "quasi-smooth cones":

Construction 5.9. Let $S \in \mathrm{dStk}_k$ and $X \in \mathrm{dStk}_S$ which is 1-Artin with finite stabilizers. Regard X with trivial T-action. Let $\mathscr{E} \in \mathbf{D}^{T,\geqslant -1}_{\mathrm{perf}}(X)$ be a T-equivariant perfect complex whose fixed part $\mathscr{E}^{\mathrm{fix}}$ belongs to $\mathbf{D}^{T,\geqslant 0}_{\mathrm{perf}}(X)$. Let $E = \mathbf{V}_X(\mathscr{E})$ be the total space and let $0 : X \to E$ denote the zero section. The *Euler class*

$$e(\mathscr{E}) \in \mathrm{C}^\bullet_T(X)_{\mathrm{loc}} = \mathrm{C}^\bullet(\mathcal{X})_{\mathrm{loc}}$$

is the image by the $\mathrm{C}^\bullet_T(\mathrm{Spec}(k))_{\mathrm{loc}}$-linear map

$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/\mathcal{X}})_{\mathrm{loc}} \xrightarrow{0_*} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{E}_{/\mathcal{X}})_{\mathrm{loc}} \xrightarrow{0^!_T} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/\mathcal{X}})_{\mathrm{loc}} \tag{5.10}$$

of the unit $1 \in \mathrm{C}^\bullet(\mathcal{X})_{\mathrm{loc}}$, under the isomorphism $\mathrm{C}^\bullet(\mathcal{X})_{\mathrm{loc}} \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/\mathcal{X}})_{\mathrm{loc}}$.

Remark 5.11. Let $\mathscr{E}$ be as in Construction 5.9 and suppose moreover that it has no fixed part, i.e., $\mathscr{E}^{\mathrm{fix}} \simeq 0$. In this case, the Euler class $e(\mathscr{E}) \in \mathrm{C}^\bullet(\mathcal{X})_{\mathrm{loc}}$ is invertible. Since this is equivalent to invertibility of the map (5.10), we may use the localization triangle in Borel–Moore homology (and a stratification by quotient stacks) to reduce to the case where X is classical and admits the resolution property. Then we may represent $\mathscr{E}$ as a bounded complex of finite rank locally free sheaves $\mathscr{E}^i$, in which case $e(\mathscr{E})$ is the cup product of $e(\mathscr{E}^i)^{(-1)^i}$, as in the proof of Theorem 5.6.

Remark 5.12. With notation as in Construction 5.9, let $s : X \to E$ be any T-equivariant section. Then we may similarly define the *localized Euler class*

$$e(\mathcal{E}, s) \in \mathrm{C}^\bullet_T(X)_{\mathrm{loc}}$$

as the image of 1 by $0^!_T \circ s_* : \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/\mathcal{X}})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{E}_{/\mathcal{X}})_{\mathrm{loc}}$. This may also be regarded as a class in $\mathrm{C}^\bullet_{T,Z}(X)_{\mathrm{loc}}$, (localized equivariant) cohomology with support in the zero locus Z of s.

5.3. Gysin pull-backs.

Definition 5.13. Let $f : X \to Y$ be a T-equivariant morphism of derived Artin stacks over k. Assume that the action on X is trivial. We say that f is *quasi-smooth in weight* 0 if the relative cotangent complex $L_{X/Y}$ lies in $\mathbf{D}^{T,\geqslant -2}_{\mathrm{perf}}(X)$ and has fixed part $L^{\mathrm{fix}}_{X/Y}$ in $\mathbf{D}^{T,\geqslant -1}_{\mathrm{perf}}(X)$.

Remark 5.14. Let $i : Z \to X$ be a closed immersion which is quasi-smooth in weight 0. Then the conormal complex $\mathscr{N}_{Z/X} := L_{Z/X}[-1]$ is as above, so we can form the Euler class $e(\mathscr{N}_{Z/X}) \in \mathrm{C}^\bullet_T(Z)_{\mathrm{loc}}$. When Z has finite stabilizers and $\mathscr{N}_{Z/X}$ has no fixed part, then this is invertible (Remark 5.11).

Example 5.15. Let $X \in \mathrm{dStk}_k$ be quasi-compact Deligne–Mumford with T-action, and denote by $Z = X^{hT}$ the homotopy fixed point stack. Then the canonical morphism $\varepsilon : Z \to X$ is a closed immersion (Proposition A.27) and $\mathscr{N}_{Z/X}$ has no fixed part (Corollary A.36).

Construction 5.16. Let $S \in \mathrm{dStk}_k$, $X, Y \in \mathrm{dStk}_S$ with T-action, and $f : X \to Y$ a T-equivariant morphism. Suppose that X is 1-Artin with finite stabilizers and f is quasi-smooth in weight 0.

(i) The T-equivariant localized *Gysin pull-back*
$$f^!_T : \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \tag{5.17}$$
is defined as follows. Recall that there is a specialization map
$$\mathrm{sp}^T_{X/Y} : \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet([N_{X/Y}/T]_{/S})_{\mathrm{loc}}.$$
Then $f^!_T$ is the composite
$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \xrightarrow{\mathrm{sp}_{\mathcal{X}/\mathcal{Y}}} \mathrm{C}^{\mathrm{BM}}_\bullet([N_{X/Y}/T]_{/S})_{\mathrm{loc}} \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}},$$
where the isomorphism is Theorem 5.6.

(ii) The *T-equivariant fundamental class* of $X \to Y$ is
$$[X/Y]^T := [\mathcal{X}/\mathcal{Y}] := f^!(1) \in \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/Y})_{\mathrm{loc}} \tag{5.18}$$
where $f^!$ is the Gysin map of Construction 5.16 with $S = [Y/T]$ and $1 \in \mathrm{C}^\bullet_T(Y)_{\mathrm{loc}}$.

Remark 5.19. Note that if f is in fact *quasi-smooth* (not just in weight 0), then $f^!_T$ is just the usual quasi-smooth Gysin pull-back $f^!$ (see [Kha2]) by definition.

5.4. Functoriality.

Fix $S \in \mathrm{dStk}_k$.

Proposition 5.20. *Suppose given a homotopy cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of derived Artin stacks locally of finite type over S with T-action, where T acts trivially on X and X', X and X' are 1-Artin with finite stabilizers, f is quasi-smooth in weight 0, and q is quasi-smooth. Then the square

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \arrow[r, "f^!_T"] \arrow[d, "q^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \arrow[d, "p^!"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}'_{/S})_{\mathrm{loc}} \arrow[r, "f'^!_T"] & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}'_{/S})_{\mathrm{loc}} \end{tikzcd}$$

commutes, i.e., there is a canonical homotopy

$$p^! \circ f^!_T \simeq f'^!_T \circ q^!$$

of maps $\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}'_{/S})_{\mathrm{loc}}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \arrow[r, "\mathrm{sp}_{\mathcal{X}/\mathcal{Y}}"] \arrow[d, "q^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{\mathcal{X}/\mathcal{Y}_{/S}})_{\mathrm{loc}} \arrow[d, "N^!_p"] & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \arrow[l, "\pi^!"'] \arrow[d, "p^!"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}'_{/S})_{\mathrm{loc}} \arrow[r, "\mathrm{sp}_{\mathcal{X}'/\mathcal{Y}'}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{\mathcal{X}'/\mathcal{Y}'_{/S}})_{\mathrm{loc}} & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}'_{/S})_{\mathrm{loc}} \arrow[l, "\pi'^!"'] \end{tikzcd}$$

where $N_p : N_{\mathcal{X}/\mathcal{Y}} \to N_{\mathcal{X}'/\mathcal{Y}'}$ is the induced morphism, and $\pi : N_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$, $\pi' : N_{\mathcal{X}'/\mathcal{Y}'} \to \mathcal{X}'$ are the projections. The left-hand square commutes by Proposition 1.17 and the right-hand square commutes by functoriality of quasi-smooth Gysin pull-backs [Kha2, Thm. 3.12]. The claim thus follows by construction of the Gysin maps $f^!_T$ and $f'^!_T$. □

Proposition 5.21. *Let $f : X \to Y$ and $g : Y \to Z$ be T-equivariant morphisms of derived 1-Artin stacks locally of finite type over S. Suppose that X has finite stabilizers and trivial T-action. Assume that f and $g \circ f$ are quasi-smooth in weight 0, and g is quasi-smooth. Then we have:*

(i) *There is a canonical identification*

$$[X/Z]^T \simeq [X/Y]^T \circ [Y/Z]^T \in \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/Z})_{\mathrm{loc}}.$$

(ii) *There is a commutative square*

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Z}_{/S})_{\mathrm{loc}} \arrow[r, "g^!"] \arrow[d, equal] & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \arrow[d, "f^!_T"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Z}_{/S})_{\mathrm{loc}} \arrow[r, "(g\circ f)^!_T"] & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \end{tikzcd}$$

or in other words, a canonical homotopy

$$(g \circ f)^!_T \simeq f^!_T \circ g^!$$

of maps $\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Z}_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}}$.

Proof. The first claim follows from the second by taking $S = \mathcal{Z}$ and evaluating on $1 \in \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}_{/\mathcal{Z}})$. For the second, consider the following square:

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}_{/S})_{\mathrm{loc}} \arrow[rr, "(g\circ f)^!_T"] \arrow[d, equal] & & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} \arrow[d, equal] \\
\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}_{/S})_{\mathrm{loc}} \arrow[r, "g^!"] \arrow[d, "\mathrm{sp}_{Y/Z}"] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Y}_{/S})_{\mathrm{loc}} \arrow[r, "f^!_T"] \arrow[d, equal] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} \arrow[d, equal] \\
\mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{\mathcal{Y}/\mathcal{Z}_{/S}})_{\mathrm{loc}} \arrow[r, "0^!_{N_{\mathcal{Y}/\mathcal{Z}}}"] \arrow[d, equal] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Y}_{/S})_{\mathrm{loc}} \arrow[r, "f^!_T"] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} \arrow[d, equal] \\
\mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{\mathcal{Y}/\mathcal{Z}_{/S}})_{\mathrm{loc}} \arrow[rr, "(0_{N_{Y/Z}}\circ f)^!_T"] & & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}}
\end{tikzcd}$$

The middle left-hand square commutes by definition of the Gysin map $g^!$, and the middle right-hand square commutes tautologically. Therefore, to show that the upper rectangle commutes it is enough to show that the total outer composite square commutes, i.e.,

$$(0_{N_{Y/Z}} \circ f)^!_T \circ \mathrm{sp}^T_{Y/Z} \simeq (g \circ f)^!_T, \tag{5.22}$$

and that the lower rectangle commutes, i.e.,

$$(0_{N_{Y/Z}} \circ f)^!_T \simeq f^!_T \circ 0^!_{N_{Y/Z}}. \tag{5.23}$$

Let us show (5.22). Consider the following diagram of T-equivariant derived stacks over S:

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, "f"] & X \times \mathbf{A}^1 \arrow[d, "f\times\mathrm{id}"] & X \arrow[l, "1"'] \arrow[d, "f"] \\
Y \arrow[r, "0"] \arrow[d, "0_{N_{Y/Z}}"] & Y \times \mathbf{A}^1 \arrow[d, "\hat{g}"] & Y \arrow[l, "1"'] \arrow[d, "g"] \\
N_{Y/Z} \arrow[r] \arrow[d, "u"] & D_{Y/Z} \arrow[d, "v"] & Z \arrow[l] \arrow[d, equal] \\
Z \arrow[r, "0"] & Z \times \mathbf{A}^1 & Z \arrow[l, "1"']
\end{tikzcd}$$

where each square is homotopy cartesian and $D_{Y/Z}$ is the derived deformation space (Theorem 1.11). Note that the morphism $h = \hat{g}\circ(f\times\mathrm{id}) : X\times\mathbf{A}^1 \to D_{Y/Z}$ is quasi-smooth in weight 0, since the conditions on Tor-amplitude can be checked on the derived fibres. Thus we have the following diagram

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}_{/S})_{\mathrm{loc}} \arrow[d, "u^!"] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}\times\mathbf{A}^1_{/S})_{\mathrm{loc}} \arrow[l, "0^!"'] \arrow[r, "1^!"] \arrow[d, "v^!"] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}_{/S})_{\mathrm{loc}} \arrow[d, equal] \\
\mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{\mathcal{Y}/\mathcal{Z}_{/S}})_{\mathrm{loc}} \arrow[d, "(0_{N_{Y/Z}}\circ f)^!_T"] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(D_{\mathcal{Y}/\mathcal{Z}_{/S}})_{\mathrm{loc}} \arrow[l] \arrow[r] \arrow[d, "h^!_T"] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}_{/S})_{\mathrm{loc}} \arrow[d, "(g\circ f)^!_T"] \\
\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}} & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}\times\mathbf{A}^1_{/S})_{\mathrm{loc}} \arrow[l, "0^!"'] \arrow[r, "1^!"] & \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}_{/S})_{\mathrm{loc}}.
\end{tikzcd}$$

The two upper squares commute by functoriality of quasi-smooth Gysin maps, and the two lower squares commute by Proposition 5.20. Moreover, by $\mathbf{A}^1$-homotopy invariance, the two upper and lower horizontal arrows $0^!$ and $1^!$ are isomorphisms and $0^! \simeq 1^!$. It follows that the left-hand and right-hand vertical composites are identified. Since $u^! \simeq \mathrm{sp}_{Y/Z}$ (by construction of quasi-smooth Gysin maps), this yields the desired homotopy (5.22).

Let us show (5.23). By homotopy invariance for the vector bundle stack $\pi : N_{Y/Z} \to Y$, it is enough to show the claim after applying $\pi^!$ on the right, i.e.,

$$(0_{N_{Y/Z}} \circ f)^!_T \circ \pi^! \simeq f^!_T.$$

By definition, $f^!_T$ and $(0_{N_{Y/Z}} \circ f)^!_T$ are the upper and lower composite arrows, respectively, in the following diagram:

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Y}_{/S})_{\mathrm{loc}} \arrow[r, "\mathrm{sp}_{\mathcal{Y}/\mathcal{Z}}"] \arrow[d, "\pi^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{\mathcal{Y}/\mathcal{Z}_{/S}})_{\mathrm{loc}} \arrow[d, "q^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \arrow[l, "p^!"'] \arrow[d, equal] \\
\mathrm{C}^{\mathrm{BM}}_\bullet(N_{\mathcal{Y}/\mathcal{Z}_{/S}})_{\mathrm{loc}} \arrow[r, "\mathrm{sp}^T_{0_{N_{Y/Z}}\circ f}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{\mathcal{X}/\mathcal{Y}} \oplus N_{\mathcal{Y}/\mathcal{Z}_{/S}})_{\mathrm{loc}} & \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}} \arrow[l, "r^!"']
\end{tikzcd}$$

where p, q and r are the projections (so that $p^!$, $q^!$, $r^!$ are invertible). The right-hand square commutes by functoriality of quasi-smooth Gysin maps for the composition

$$r : N_{0_{N_{Y/Z}}\circ f} \simeq N_{X/Y} \oplus N_{Y/Z} \xrightarrow{q} N_{X/Y} \xrightarrow{p} X,$$

and the left-hand square commutes by Proposition 1.17 applied to the square

$$\begin{tikzcd}
X \arrow[r, "0_{N_{Y/Z}}\circ f"] \arrow[d, equal] & N_{Y/Z} \arrow[d, "\pi"] \\
X \arrow[r, "f"] & Y,
\end{tikzcd}$$

where π and $q : N_{X/Y} \oplus N_{Y/Z} \to N_{X/Y}$ are both smooth (the latter because $N_{Y/Z} \to Y$ is smooth, as $Y \to Z$ is quasi-smooth). $\square$

5.5. The virtual localization formula.

Fix $S \in \mathrm{dStk}_k$.

Theorem 5.24. *Let $i : Z \to X$ be a closed immersion of T-equivariant derived Artin stacks over S. Suppose that X is quasi-smooth over S, Z has finite stabilizers and trivial T-action, and the conormal sheaf $\mathcal{N}_{Z/X}$ has no fixed part. If*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X} \smallsetminus \mathcal{Z}_{/S})_{\mathrm{loc}} \simeq 0 \tag{5.25}$$

then we have

$$[\mathcal{X}_{/S}] \simeq i_*([\mathcal{Z}_{/S}] \cap e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})^{-1}) \tag{5.26}$$

in $\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X}_{/S})_{\mathrm{loc}}$.

Corollary 5.27. *Let $X \in \mathrm{dStk}_S$, Z a T-invariant closed derived substack which is 1-Artin with finite stabilizers and trivial T-action. Suppose that* *the conormal sheaf $\mathcal{N}_{Z/X}$ has no fixed part and for every geometric point of $X \smallsetminus Z$ we have $\mathrm{St}^T_X(x) \subsetneq T_{k(x)}$. If X is quasi-smooth over S, then*

$$[\mathfrak{X}_{/S}] \simeq i_*([\mathfrak{Z}_{/S}] \cap e(\mathcal{N}_{\mathfrak{Z}/\mathfrak{X}})^{-1}) \tag{5.28}$$

in $\mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{X}_{/S})_{\mathrm{loc}}$, where $i : Z \to X$ is the inclusion.

Proof. The additional assumption on $X \smallsetminus Z$ guarantees the vanishing (5.25) by the concentration theorem (Theorem 3.1). □

Proof of Theorem 5.24. First note that the assumptions imply that Z is also quasi-smooth over S (see Remark A.50). By (5.25) and the localization triangle, the direct image

$$i_* : \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{Z}_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{X}_{/S})_{\mathrm{loc}}$$

is invertible. Therefore, there exists a unique (up to contractible choice) point $\alpha \in \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{Z}_{/S})_{\mathrm{loc}}$ such that $i_*(\alpha) \simeq [\mathfrak{X}_{/S}]$. Let $p : \mathfrak{X} \to S$ and $q : \mathfrak{Z} \to S$ denote the projections. By Proposition 5.21, we have a canonical homotopy

$$i^!_T \circ p^! \simeq q^!_T,$$

where $q^!_T = q^!$ since q is quasi-smooth (see Remark 5.19). Hence

$$[\mathfrak{Z}_{/S}] = q^!_T(1) \simeq i^!_T \circ p^!(1) \simeq i^!_T([\mathfrak{X}_{/S}]) \simeq i^!_T i_*(\alpha).$$

The right-hand side can be further computed as

$$i^!_T \circ i_* = 0^!_T \circ \mathrm{sp}_{\mathfrak{Z}/\mathfrak{X}} \circ i_* \simeq 0^!_T \circ 0_* = e(\mathcal{N}_{\mathfrak{Z}/\mathfrak{X}}) \cap (-)$$

since by Corollary 1.16 we have $\mathrm{sp}_{\mathfrak{Z}/\mathfrak{X}} \circ i_* \simeq 0_*$. Combining the two displayed formulas we get

$$i_*([\mathfrak{Z}_{/S}] \cap e(\mathcal{N}_{\mathfrak{Z}/\mathfrak{X}})^{-1}) \simeq i_*(\alpha) \simeq [\mathfrak{X}_{/S}].$$

□

Corollary 5.29. *Let $X \in \mathrm{dStk}_k$ be a quasi-smooth derived algebraic space with T-action. Let $i : X^T \to X$ denote the inclusion of the fixed locus (see Proposition A.23). Then we have a canonical identification*

$$[X] \simeq i_*([X^T] \cap e(\mathcal{N}_{X^T/X})^{-1})$$

in $\mathrm{C}^{\mathrm{BM},T}_\bullet(X)_{\mathrm{loc}}$.

Proof. By Proposition A.23, X^T is identified with the homotopy fixed point stack X^{hT}. The assumptions of Corollary 5.27 thus hold by definition of X^T and X^{hT}. □

Corollary 5.30. *Let $X \in \mathrm{dStk}_k$ be quasi-compact quasi-smooth Deligne–Mumford with T-action. Let $T' \twoheadrightarrow T$ be a reparametrization such that the canonical morphism $X^{hT'} \to X^T$ is surjective (Corollary A.49). Then we have*

$$[X] \simeq \varepsilon_*([X^{hT'}] \cap e(\mathcal{N}_{X^{hT'}/X})^{-1})$$

in $\mathrm{C}^{\mathrm{BM},T}_\bullet(X)_{\mathrm{loc}}$, where $\varepsilon : X^{hT'} \to X$ is the canonical morphism (Definition A.17).

Proof. By Corollary 5.27 and Remark A.50 it remains to show that the reparametrization induces an isomorphism $\mathrm{C}^{\mathrm{BM},T'}_\bullet(X)_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T'}_\bullet(X)_{\mathrm{loc}}$, which follows from Theorem 1.24 applied to the BG-torsor $[X/T'] \to [X/T]$ (where G is the kernel of $T' \twoheadrightarrow T$). ☐

5.6. **Simple wall-crossing formula.** We prove a wall-crossing formula for simple $\mathbf{G}_m$-wall crossings as in [KL2, §2.1, App. A], [CKL, §4], [Joy, Cor. 2.21]. In particular, we remove the global resolution assumptions in *op. cit.* (and generalize to Deligne–Mumford stacks over general base fields).

Let X be a derived Deligne–Mumford stack of finite type over k with $T = \mathbf{G}_m$-action and quotient $\mathfrak{X} = [X/T]$. Let X_+ and X_- be open substacks of X such that $M_\pm = [X_\pm/T] \subseteq \mathfrak{X}$ are Deligne–Mumford.

Definition 5.31. The *master space* associated to (X, M_+, M_-) is the quotient stack

$$\mathfrak{M} = [X \times \mathbf{P}^1 \smallsetminus (U_- \times \{0\} \cup U_+ \times \{\infty\})/\mathbf{G}_m]$$

where $\mathbf{G}_m$ acts diagonally on $X \times \mathbf{P}^1$ and U_- and U_+ are the respective complements of X_+ and X_- in X. Note that $\mathfrak{M}$ is Deligne–Mumford and that the T-action on X induces a T-action on $\mathfrak{M}$.

Theorem 5.32. *Let $Z = X^{hT'}$ where $T' \twoheadrightarrow T$ is a reparametrization as in Corollary A.49. If X is quasi-smooth, then M_+, M_- and Z are quasi-smooth and we have*

$$[M_+]^{\mathrm{vir}} - [M_-]^{\mathrm{vir}} \simeq \mathrm{res}_{t=0}\Big(\frac{[Z]^{\mathrm{vir}}}{e(\mathcal{N}_{Z/X})}\Big)$$

in $\mathrm{C}^{\mathrm{BM},T}_\bullet(\mathfrak{M}^{hT'})_{\mathrm{loc}} \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{M}^{hT'}) \otimes \mathrm{C}^\bullet(BT)_{\mathrm{loc}}$.

Corollary 5.33. *Suppose M_+, M_- and Z are moreover proper. Given $\alpha \in \pi_0\mathrm{C}^\bullet_T(X)\langle d\rangle$, where X is of virtual dimension $d+1$, let*

$$\alpha_\pm \in \pi_0\mathrm{C}^\bullet_T(M_\pm)\langle d\rangle$$

correspond to $\alpha|_{X_\pm} \in \mathrm{C}^\bullet_T(X_\pm)\langle d\rangle \simeq \mathrm{C}^\bullet_T(M_\pm)\langle d\rangle$, where $X_\pm = M_\pm \times_{\mathfrak{X}} X \subseteq X$. Then we have

$$\alpha_+ \cdot [M_+]^{\mathrm{vir}} - \alpha_- \cdot [M_-]^{\mathrm{vir}} = \mathrm{res}_{t=0}\Big(\alpha \cdot \frac{[Z]^{\mathrm{vir}}}{e(\mathcal{N}_{Z/X})}\Big).$$

Here $t \in \pi_0\mathrm{C}^\bullet(BT)\langle 1\rangle \simeq \pi_0\mathrm{C}^\bullet(\mathrm{Spec}(k))[t, t^{-1}]$ is the first Chern class of the tautological line bundle, and $\mathrm{res}_{t=0}$ denotes the residue of a Laurent series at $t = 0$.

Proof of Theorem 5.32. There is a canonical morphism

$$Z \coprod M_+ \coprod M_- \to \mathfrak{M}^{hT'} \tag{5.34}$$

which is clearly bijective (on field-valued points). It is also formally étale, as one can see immediately from Corollary A.36. Indeed, the relative cotangent complexes over $\mathfrak{M}$ are given by

$$\begin{aligned}
L_{Z/\mathfrak{M}} &\simeq L_{Z/X} \simeq L_X|_Z^{\mathrm{mov}}[1],\\
L_{M_+/\mathfrak{M}} &\simeq L_{M_+/[X_+\times(\mathbf{P}^1\smallsetminus\{\infty\})/\mathbf{G}_m]} \simeq \mathscr{O}^{(1)}[1],\\
L_{M_-/\mathfrak{M}} &\simeq L_{M_-/[X_-\times(\mathbf{P}^1\smallsetminus\{0\})/\mathbf{G}_m]} \simeq \mathscr{O}^{(-1)}[1],\\
L_{\mathfrak{M}^{hT'}/\mathfrak{M}} &\simeq L_{\mathfrak{M}}|_{\mathfrak{M}^{hT'}}^{\mathrm{mov}}[1],
\end{aligned}$$

where $(-)^{(i)}$ indicates the weight of the $\mathbf{G}_m$-action. As a radicial étale surjection, (5.34) an isomorphism. Now it follows from Corollary 5.30 that we have

$$[\mathfrak{M}/T]^{\mathrm{vir}} = \frac{[M_+]^{\mathrm{vir}}}{-t} + \frac{[M_-]^{\mathrm{vir}}}{t} + \frac{[X^{hT'}]^{\mathrm{vir}}}{e(\mathscr{N}_{X^{hT'}/X})}.$$

The claim follows by applying $\mathrm{res}_{t=0}$. □

Remark 5.35. Following Subsect. 5.7, one could also formulate the above result using the language of perfect obstruction theories. In particular, one may remove the global embeddability or global resolution hypotheses in [KL2, Thm. A.2] and [CKL, Thm. 4.2]. See also [Joy, Cor. 2.21, Rem. 2.20].

Remark 5.36. In particular, this proves the non-symmetric analogue of [KL2, Conj. 1.2].

5.7. Perfect obstruction theories.

We describe some analogous results in the language of [BF, AP]. We first recall the notion of perfect obstruction theory in the setting of Artin stacks. We stick to 1-Artin stacks for simplicity, and we assume that the six functor formalism $\mathbf{D}$ satisfies étale descent (e.g. it is the ∞-category of Betti sheaves, étale sheaves, or rational motives).

Definition 5.37. Let $f : X \to Y$ be a morphism of 1-Artin stacks in Stk_k, and let $\phi : \mathscr{E} \to L_{X/Y}^{\geq -1} \in \mathbf{D}_{\mathrm{coh}}(X)$. We say that ϕ is an *obstruction theory* for f if $h^i(\phi)$ are isomorphisms for all $i \geq 0$ and $h^{-1}(\phi)$ is surjective.

We say that ϕ is a *perfect obstruction theory* if it is an obstruction theory and $\mathscr{E} \in \mathbf{D}_{\mathrm{perf}}(X)^{\geq -1}$.

We now introduce an analog of the notion of quasi-smooth in weight zero.

Definition 5.38. Let T act on 1-Artin stacks $X, Y \in \mathrm{Stk}_k$ and let $f : X \to Y$ be a T-equivariant morphism. Assume that the T-action on X is trivial. We say that a morphism $\phi : \mathscr{E} \to L_{X/Y}^{\geq -1}$ in $\mathbf{D}_{\mathrm{coh}}([X/T])$ is a *T-equivariant good obstruction theory* for $f : X \to Y$ if

(i) ϕ is an obstruction theory, and

(ii) $\mathscr{E}^{\mathrm{fix}} \in \mathbf{D}_{\mathrm{perf}}^{\geq -1}(X \times BT)$ and $\mathscr{E}^{\mathrm{mov}} \in \mathbf{D}_{\mathrm{perf}}^{\geq -2}(X \times BT)$.

The construction of T-equivariant Gysin pull-back in Construction 5.16 for quasi-smooth in weight zero morphisms can be generalized to T-equivariant good obstruction theories.

Construction 5.39. Let T act on 1-Artin stacks $S \in \mathrm{Stk}_k$ and $X, Y \in \mathrm{Stk}_S$. Let $f : X \to Y$ be a T-equivariant morphism over S. Assume that X has finite stabilizers and the T-action on X is trivial. Let $\phi : \mathscr{E} \to L^{\geq -1}_{X/Y}$ be a T-equivariant good obstruction theory for $f : X \to Y$. Then we have a closed immersion $i : \mathfrak{C}_{X/Y} \hookrightarrow N^{\mathrm{vir}}_{X/Y} := \mathbf{V}(\mathscr{E}[-1])$ by [AP, Prop. 8.2(2)]. We define the *T-equivariant virtual pullback*

$$f^!_T : \mathrm{C}^{\mathrm{BM},T}_\bullet(Y_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}} \tag{5.40}$$

as the composition

$$\mathrm{C}^{\mathrm{BM},T}_\bullet(Y_{/S})_{\mathrm{loc}} \xrightarrow{\mathrm{sp}_{X/Y}} \mathrm{C}^{\mathrm{BM},T}_\bullet((\mathfrak{C}_{X/Y})_{/S})_{\mathrm{loc}} \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM},T}_\bullet((N^{\mathrm{vir}}_{X/Y})_{/S})_{\mathrm{loc}} \simeq \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}},$$

where the specialization map $\mathrm{sp}_{X/Y}$ is constructed from the deformation space $M^\circ_{X/Y}$ in [AP, Thm. 7.2] and the isomorphism is Theorem 5.6.

T-equivariant virtual pullbacks commute with proper pushforwards and ordinary virtual pullbacks.

Proposition 5.41. *Let $S \in \mathrm{Stk}_S$ be 1-Artin and suppose given a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p"] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of T-equivariant morphisms of 1-Artin stacks in Stk_S. Assume that X and X' have finite stabilizers, and the T-actions on X and X' are trivial. Let $\phi : \mathscr{E} \to L^{\geq -1}_{X/Y}$ be a T-equivariant good obstruction theory for f. Then the composition $\mathscr{E}|_{X'} \to L^{\geq -1}_{X/Y}|_{X'} \to L^{\geq -1}_{X'/Y'}$ is also a T-equivariant good obstruction theory for f'.

(i) *If q is proper, then there is a canonical homotopy*
$$f^!_T \circ q_* \simeq p_* \circ f'^!_T : \mathrm{C}^{\mathrm{BM},T}_\bullet(Y'_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}}.$$

(ii) *If q is equipped with a perfect obstruction theory, then there is a canonical homotopy*
$$q^! \circ f^!_T \simeq f'^!_T \circ q^! : \mathrm{C}^{\mathrm{BM},T}_\bullet(Y_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X'_{/S})_{\mathrm{loc}}.$$

We omit the proof of Proposition 5.41 since it follows by the arguments in section Sect. 1 by replacing the derived deformation space $D_{X/Y}$ of [HKR] with the classical deformation space $M^\circ_{X/Y}$ of [AP] (cf. [Man, Thm. 4.1]).

Theorem 5.42. *Let $S \in \mathrm{Stk}_k$ be 1-Artin and let $f : X \to Y$ and $g : Y \to Z$ be T-equivariant morphisms of 1-Artin stacks in Stk_S. Assume that X is has finite stabilizers and the T-action on X is trivial. Let $\phi_{X/Y} : \mathscr{E}_{X/Y} \to L^{\geq -1}_{X/Y}$, $\phi_{X/Z} : \mathscr{E}_{X/Z} \to L^{\geq -1}_{X/Z}$ be T-equivariant good obstruction theories and*

$\phi_{Y/Z} : \mathscr{E}_{Y/Z} \to L^{\geq -1}_{Y/Z}$ be a T-equivariant perfect obstruction theory. Assume that there exists a morphism of homotopy cofiber sequences

$$\begin{tikzcd} f^*\mathscr{E}_{Y/Z} \arrow[r] \arrow[d, "f^*\phi_{Y/Z}"] & \mathscr{E}_{X/Z} \arrow[r] \arrow[d, "\phi_{X/Z}"] & \mathscr{E}_{X/Y} \arrow[d, "\phi'_{X/Y}"] \\ (f^*L^{\geq -1}_{Y/Z})^{\geq -1} \arrow[r, "a"] & L^{\geq -1}_{X/Z} \arrow[r] & \mathrm{Cofib}(a) \end{tikzcd}$$

with an equivalence

$$\phi_{X/Y} \simeq r \circ \phi'_{X/Y} : \mathscr{E}_{X/Y} \to L^{\geq -1}_{X/Y}$$

where $r : \mathrm{Cofib}(a) \to \mathrm{Cofib}(a)^{\geq -1} \simeq L^{\geq -1}_{X/Y}$ is the canonical map. Then we have a canonical homotopy

$$(g \circ f)^!_T \simeq f^!_T \circ g^! : \mathrm{C}^{\mathrm{BM},T}_\bullet(Z_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}}. \tag{5.43}$$

Since the arguments are almost the same as in Proposition 5.21, we will only give a sketch proof of Theorem 5.42.

Sketch of the proof. Consider the composition

$$h : X \times \mathbf{A}^1 \to Y \times \mathbf{A}^1 \to M^\circ_{Y/Z}.$$

We claim that there is a canonical isomorphism

$$L^{\geq -1}_{X\times\mathbf{A}^1/M^\circ_{Y/Z}} \simeq \mathrm{Cofib}(f^*L_{Y/Z} \boxtimes \mathscr{O}_{\mathbf{A}^1} \xrightarrow{(T,a)} (f^*L_{Y/Z} \oplus L_{X/Z}) \boxtimes \mathscr{O}_{\mathbf{A}^1})^{\geq -1}$$

where $T \in \Gamma(\mathbf{A}^1, \mathscr{O}_{\mathbf{A}^1})$ is the coordinate function. Indeed, when f and g are DM morphisms of 1-Artin stacks, the claim is shown in [KKP]. The general case follows from descent.

Form a morphism of homotopy cofiber sequences

$$\begin{tikzcd} f^*\mathscr{E}_{Y/Z} \arrow[r] \arrow[d, "f^*\phi_{Y/Z}"] & f^*\mathscr{E}_{Y/Z} \oplus \mathscr{E}_{X/Z} \arrow[r] \arrow[d, "f^*\phi_{Y/Z}\oplus\phi_{X/Z}"] & \mathscr{E}_h \arrow[d, "\phi'_h"] \\ (f^*L_{Y/Z})^{\geq -1} \arrow[r, "{(T,a)}"] & (f^*L_{Y/Z})^{\geq -1} \oplus L^{\geq -1}_{X/Z} \arrow[r] & \mathrm{Cofib}(T,a). \end{tikzcd}$$

Then the composition

$$\phi_h : \mathscr{E}_h \xrightarrow{\phi'_h} \mathrm{Cofib}(T,a) \to \mathrm{Cofib}(T,a)^{\geq -1} \simeq L^{\geq -1}_h$$

is a T-equivariant good obstruction theory for h.

Consider the composition

$$k := 0_{N^{\mathrm{vir}}_{Y/Z}} \circ f : X \to N^{\mathrm{vir}}_{Y/Z}$$

where $N^{\mathrm{vir}}_{Y/Z} := \mathbf{V}(\mathscr{E}_{Y/Z}[-1])$. Then k has a T-equivariant good obstruction theory

$$\phi_k := f^*\phi_{Y/Z} \oplus \phi_{X/Y} : f^*\mathscr{E}_{Y/Z} \oplus \mathscr{E}_{X/Y} \to \tau^{\geq -1}(f^*L_{Y/Z} \oplus L_{X/Y}).$$

By Proposition 5.41(ii), we have a canonical homotopy

$$(g \circ f)^!_T \simeq k^!_T \circ \mathrm{sp}_{Y/Z} : \mathrm{C}^{\mathrm{BM},T}_\bullet(Z_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}}.$$

Hence it suffices to show the proposition for

$$X \to Y \to N^{\mathrm{vir}}_{Y/Z}.$$

By the homotopy property of $\mathrm{C}^{\mathrm{BM}}_\bullet((-)_{/S})$, it suffices to show the proposition for

$$X \to N^{\mathrm{vir}}_{Y/Z} \to Z.$$

Then an analog of Proposition 1.17 for classical specialization maps [AP] and smooth pullbacks completes the proof. ☐

Proposition 5.44. *Let $S \in \mathrm{Stk}_k$ be 1-Artin and $f : X \to Y$ be a T-equivariant morphism of 1-Artin stacks in Stk_S. Assume that X has finite stabilizers and the T-action on X is trivial. Let $\phi : \mathscr{E} \to L^{\geq -1}_{X/Y}$ be a T-equivariant good obstruction theory for $f : X \to Y$. If $f : X \to Y$ is a closed immersion, then we have a canonical homotopy*

$$f^!_T \circ f_* \simeq e(N^{\mathrm{vir}}_{\mathcal{X}/\mathcal{Y}}) \cap (-) : \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X_{/S})_{\mathrm{loc}}$$

where $N^{\mathrm{vir}}_{\mathcal{X}/\mathcal{Y}} := \mathbf{V}(\mathscr{E}[-1])$.

Proposition 5.44 follows immediately from Corollary 1.16.

Corollary 5.45. *Let $X \in \mathrm{Stk}_k$ be a Deligne–Mumford stack with a T-action. Let $\phi : \mathscr{E} \to L^{\geq -1}_X$ be a T-equivariant perfect obstruction theory. Choose a reparametrization $\rho : T' \to T$ such that $X^{rhT} = X^{hT'}$. Then the composition $\mathscr{E}|^{\mathrm{fix}}_{X^{hT'}} \to L_X|^{\geq -1}_{X^{hT'}} \to L^{\geq -1}_{X^{hT'}} = L^{\geq -1}_{X^{rhT}}$ is a perfect obstruction theory for X^{rhT} and is independent of the choice of T'. Moreover, we have*

$$[X]^{\mathrm{vir}} = i_*([X^{rhT}]^{\mathrm{vir}} \cap e(N^{\mathrm{vir}}))^{-1} \in \mathrm{C}^{\mathrm{BM},T}_\bullet(X)_{\mathrm{loc}}$$

where $i : X^{rhT} \hookrightarrow X$ denotes the inclusion map and $N^{\mathrm{vir}} := \mathbf{V}(\mathscr{E}|^{\mathrm{mov}}_{X^{hT'}})$.

6. Localization by cosections

In this section, the base ring k is arbitrary; all derived stacks are assumed locally of finite type over k.

6.1. **Cohomological reduction.**

Let $f : X \to Y$ be a homotopically finitely presented morphism in dStk_k.

Notation 6.1. Let $\alpha : \mathscr{O}_X \to \mathcal{L}_{X/Y}[-1]$ be a (-1)-shifted 1-form (or equivalently, a cosection $\alpha^\vee : \mathcal{L}^\vee_{X/Y}[1] \to \mathscr{O}_X$).

(i) We write $X(\alpha)$ for the derived zero locus of α, so that there is a homotopy cartesian square

$$\begin{tikzcd} X(\alpha) \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\ X \arrow[r, "0"] & N^*_{X/Y} \end{tikzcd}$$

where $N^*_{X/Y} \coloneqq \mathbf{V}_X(\mathcal{L}^\vee_{X/Y}[1])$ is the relative conormal bundle (= (−1)-shifted cotangent bundle).

(ii) Let $\mathcal{K}^\alpha$ denote the cofibre of α, so that there is an exact triangle

$$\mathcal{O}_X \xrightarrow{\alpha} L_{X/Y}[-1] \to \mathcal{K}^\alpha.$$

Set $K^\alpha \coloneqq \mathbf{V}_X(\mathcal{K}^\alpha)$, and denote by $i_K : K^\alpha \to N_{X/Y}$ the canonical closed immersion.

We say that X is *cohomologically reduced* over Y with respect to α if the specialization map

$$\mathrm{sp}_{X/Y} : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/Y})$$

factors through $\mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y})$. More precisely:

Definition 6.2. Let $f : X \to Y$ be a homotopically of finite presentation morphism between derived Artin stacks. A *cohomological reduction* ρ of $f : X \to Y$ consists of the data of a (−1)-shifted 1-form $\alpha : \mathcal{O}_X \to \mathcal{L}_{X/Y}[-1]$ and a null-homotopy of the composite

$$\mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \xrightarrow{\mathrm{sp}_{X/Y}} \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/Y}) \xrightarrow{\mathrm{res}} \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y} \smallsetminus K^\alpha_{/Y}).$$

By the localization triangle, the latter gives rise to the *ρ-reduced specialization map*

$$\mathrm{sp}^\rho_{X/Y} : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y})$$

and an identification

$$\mathrm{sp}_{X/Y} \simeq i_{K,*} \circ \mathrm{sp}^\rho_{X/Y}.$$

Given such ρ and α, we will also say that ρ is a *cohomological reduction of f with respect to α.*

Here is one source of examples of cohomological reduction:

Proposition 6.3. *Let $f : X \to Y$ be a homotopically of finite presentation morphism of derived Artin stacks. Let $\phi : \mathcal{O}_X \to \mathcal{O}_X[-1]$ be a (−1)-shifted function on X, and let*

$$\alpha : \mathcal{O}_X \xrightarrow{d_{\mathrm{dR}}(\phi)} \mathcal{L}_X[-1] \to \mathcal{L}_{X/Y}[-1]$$

be induced by the de Rham differential of ϕ. Then there exists a cohomological reduction ρ of f by α.

Proof. Form the derived zero locus of the (−1)-shifted function ϕ:

$$\begin{tikzcd} X^\phi \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\ \mathrm{Spec}(k) \arrow[r, "0"] & \mathbf{A}^1_k[-1]. \end{tikzcd}$$

Note that $X^\phi \to X$ induces an isomorphism on classical truncations.

By definition of $\mathcal{K}^\alpha$ and the transitivity exact triangle

$$\mathscr{O}_{X^\phi} \xrightarrow{\alpha} \mathcal{L}_{X/Y}|_{X^\phi}[-1] \to \mathcal{L}_{X^\phi/Y}[-1]$$

we get a canonical identification $\mathcal{K}^\alpha|_{X^\phi} \simeq \mathcal{L}_{X^\phi/Y}[-1]$. By Proposition 1.15 we have the commutative square of specialization maps

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \arrow[r, "\mathrm{sp}_{X^\phi/Y}"] \arrow[d, equal] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X^\phi/Y}{}_{/Y}) \arrow[r, equal] \arrow[d] & \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha \times_X X^\phi_{/Y}) \arrow[d] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \arrow[r, "\mathrm{sp}_{X/Y}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/Y}) \arrow[r, equal] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/Y}) \end{tikzcd}$$

where the right-hand vertical arrow is direct image along the closed immersion $K^\alpha \times_X X^\phi \to K^\alpha \to N_{X/Y}$. The claim follows. □

We also have a "cone reduction" criterion, inspired by [KL].

Definition 6.4. Let $f : X \to Y$ be a homotopically finitely presented morphism of derived Artin stacks and let α be a (-1)-shifted 1-form. Let $\mathfrak{C}$ be the relative intrinsic normal cone of the classical truncation $f_{\mathrm{cl}} : X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ (see [AP, Thm. 6.2]). We say that f satisfies *cone reduction* with respect to α if there is an inclusion $\mathfrak{C}_{\mathrm{red}} \subseteq K^\alpha$ of closed substacks of $N_{X/Y}$.

Proposition 6.5. *Let $f : X \to Y$ be a quasi-smooth morphism of derived Artin stacks and let α be a (-1)-shifted 1-form. If f satisfies cone reduction with respect to α, then there exists a canonical cohomological reduction ρ of f by α, and moreover a factorization*

$$\mathrm{sp}^\rho_{X/Y} : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{C}_{/Y}) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{C}_{\mathrm{red}/Y}) \xrightarrow{i_{\mathfrak{C}/K,*}} \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y})$$

where $i_{\mathfrak{C}/K} : \mathfrak{C}_{\mathrm{red}} \to K^\alpha$ is the inclusion.

Proof. In the following, we will identify the Borel–Moore homology of all derived stacks with those of their classical truncations (using the derived invariance property). First recall that the specialization map factors through $\mathfrak{C}$:

$$\mathrm{sp}_{X/Y} : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \xrightarrow{\mathrm{sp}^{\mathrm{cl}}_{X_{\mathrm{cl}}/Y_{\mathrm{cl}}}} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{C}_{/Y}) \xrightarrow{i_{\mathfrak{C}/N,*}} \mathrm{C}^{\mathrm{BM}}_\bullet(N_{/Y}). \tag{6.6}$$

where $i_{\mathfrak{C}/N} : \mathfrak{C} \to N$ is the inclusion into the derived normal bundle $N := N_{X/Y}$, and the classical specialization $\mathrm{sp}^{\mathrm{cl}}_{X_{\mathrm{cl}}/Y_{\mathrm{cl}}}$ is defined using [AP, Thm. 7.2] (see [Kha5, Rem. 1.13]).

By the assumption we have a commutative diagram

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{C}_{\mathrm{red}/Y}) \arrow[r] \arrow[d, "i_{\mathfrak{C}/K,*}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{/Y}) \arrow[r, "\mathrm{res}"] \arrow[d, equal] & \mathrm{C}^{\mathrm{BM}}_\bullet(N \smallsetminus \mathfrak{C}_{/Y}) \arrow[d, "\mathrm{res}"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y}) \arrow[r] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{/Y}) \arrow[r, "\mathrm{res}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N \smallsetminus K^\alpha_{/Y}). \end{tikzcd}$$

Therefore, it is enough to exhibit a null-homotopy of the composite

$$\mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \xrightarrow{\mathrm{sp}_{X/Y}} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathfrak{C}_{/Y}) \xrightarrow{\mathrm{res}} \mathrm{C}^{\mathrm{BM}}_\bullet(N \smallsetminus \mathfrak{C}_{/Y}),$$

which exists by the factorization (6.6) and the canonical null-homotopy of the upper row above. ☐

Thus, there are two orthogonal types of examples in which cone reduction (and hence cohomological reduction) is known to hold. First, we can restrict to (-1)-shifted 1-forms arising from a (-1)-shifted function as in Proposition 6.3 (that the cone reduction condition holds in that case is implicit in the proof). On the other hand, we can have the condition for arbitrary (-1)-shifted 1-forms, but at the cost of restricting to complex Deligne–Mumford stacks:

Example 6.7 (Kiem–Li)**.** Suppose that the base ring k is the field of complex numbers. Let X be a quasi-smooth derived Deligne–Mumford stack over $Y = \mathrm{Spec}(k)$ and let α be a (-1)-shifted 1-form. Then it follows from the cone reduction lemma of Kiem–Li that there is an inclusion $\mathfrak{C}_{\mathrm{red}} \subseteq K^{\alpha}$, where $\mathfrak{C}$ is the intrinsic normal cone of X_{cl} (see [KL, Lem. 4.4, Cor. 4.5]). In particular, Proposition 6.5 gives a cohomological reduction of X with respect to α; we may regard this as a cohomological variant of the cycle-theoretic statement of [KL, Prop. 4.3].

We believe that Example 6.7 holds for $Y = \mathrm{Spec}(k)$ with k an arbitrary field of characteristic zero. On the other hand, the next two examples show that the cone reduction condition for general (-1)-shifted 1-forms cannot hold for general bases Y.

Example 6.8. Let $Y = \mathrm{Spec}(k)$, with k a field of characteristic $p > 0$. Let $X \subseteq \mathbf{A}^1_k$ be the zero locus of the function $\mathbf{A}^1_k \to \mathbf{A}^1_k$ sending $x \mapsto x^p$, and let $\alpha : \mathscr{O}_X \to \mathcal{L}_X[-1]$ be the (-1)-shifted 1-form corresponding under the isomorphism $\mathcal{L}_X[-1] \simeq \mathscr{O}_X \oplus \mathscr{O}_X[-1]$ to the inclusion. Then α is nowhere zero, but the fundamental class of X (which is underived quasi-smooth, i.e., lci) is nonzero (it is the generator of $\pi_0 \mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle * \rangle$). Moreover, there is no inclusion of $\mathfrak{C}_{X,\mathrm{red}} \simeq [\mathbf{A}^1_k/\mathbf{A}^1_k]$ (trivial action) in $K^{\alpha}_{\mathrm{red}} \simeq [\mathrm{Spec}(k)/\mathbf{A}^1_k]$.

Example 6.9. When Y is higher dimensional, cone reduction rarely holds e.g. for regular closed immersions $X \to Y$. For example, take f the inclusion $0 : \mathrm{Spec}(k) \to \mathbf{A}^1_k$.

We also record the following corollary of Proposition 6.5:

Corollary 6.10. *Let $f : X \to Y$ be a quasi-smooth morphism of derived Artin stacks and let α be a (-1)-shifted 1-form. Suppose there exists a smooth surjection $U \twoheadrightarrow X$ of derived Artin stacks, a (-1)-shifted function $\phi : \mathscr{O}_U \to \mathscr{O}_U[-1]$, and a homotopy $\alpha|_U \simeq d_{\mathrm{dR}}(\phi)$ of 1-forms. Then f satisfies cone reduction with respect to α. In particular, there exists a canonical cohomological reduction ρ of f by α.*

Proof. First note that cone reduction holds for $f|_U$ with respect to $d_{\mathrm{dR}}(\phi)$ (as is implicit in the proof of Proposition 6.3). By smooth descent of both the intrinsic normal cone and the derived normal bundle (see [AP, Thm. 6.2] and [Kha2, Prop. 1.2] respectively), the cone reduction property is smooth-local on X. Hence the claim follows. ☐

By the following lemma, any (-1)-shifted closed 1-form (in the sense of shifted symplectic geometry [PTVV]) is locally exact, and thus admits a cohomological reduction by Corollary 6.10.

Lemma 6.11. *Suppose the base field k is algebraically closed of characteristic zero. Let $X \in \mathrm{dStk}_k$ be quasi-compact and α a closed (-1)-shifted 1-form on X. Then there exists a smooth surjection $U \twoheadrightarrow X$ of derived Artin stacks, a (-1)-shifted function $\phi : \mathscr{O}_U \to \mathscr{O}_U[-1]$, and a homotopy $\alpha|_U \simeq d_{\mathrm{dR}}(\phi)$ of closed 1-forms (where d_{dR} is as in [PTVV, Rem. 1.17]).*

Proof. This is a variation of the proof of [BBJ, Prop. 5.7(a)]. Choose any smooth surjection $U \twoheadrightarrow X$ where $U = \mathrm{Spec}(A)$ is an affine derived scheme of finite type over k. By [BBJ, Prop. 5.6(a)], α lifts to a class in $\mathrm{HC}^{-1}(A)(0)$ and is hence represented by an A-linear morphism $\phi : A \to A[-1]$ with $d_{\mathrm{dR}}(\phi) \simeq \alpha|_U$. □

6.2. Localized fundamental classes.

Theorem 6.12. *Let $f : X \to Y$ be a quasi-smooth morphism of derived Artin stacks of relative virtual dimension d over a derived Artin stack S. Let $\alpha : \mathscr{O}_X \to \mathcal{L}_{X/Y}[-1]$ be a (-1)-shifted 1-form. Then for any cohomological reduction ρ of f with respect to α, there is a ρ-localized Gysin pull-back*

$$f^!_\rho : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X(\alpha)_{/S})\langle -d\rangle$$

and a commutative diagram

$$\begin{tikzcd} \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \arrow[r, "f^!_\rho"] \arrow[d, equal] & \mathrm{C}^{\mathrm{BM}}_\bullet(X(\alpha)_{/S})\langle -d\rangle \arrow[d, "i_*"] \\ \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \arrow[r, "f^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})\langle -d\rangle, \end{tikzcd}$$

where $i : X(\alpha) \to X$ is the inclusion.

In particular:

Corollary 6.13. *In the situation of Theorem 6.12, there is a ρ-localized (relative) fundamental class*

$$[X_{/Y}]_\rho \in \mathrm{C}^{\mathrm{BM}}_\bullet(X(\alpha)_{/Y})\langle -d\rangle$$

with a canonical identification in $\mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})$

$$i_*[X_{/Y}]_\rho \simeq [X_{/Y}],$$

and a localized Gysin pull-back

$$f^!_\rho : \mathrm{C}^{\mathrm{BM}}_\bullet(Y) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X(\alpha))\langle -d\rangle$$

with $i_* \circ f^!_\rho \simeq f^!$, where $i : X(\alpha) \to X$ is the inclusion.

Proof. For the fundamental class, define $[X_{/Y}]_\rho$ as the image of 1 by

$$f^!_\rho : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X(\alpha)_{/Y})\langle -d\rangle.$$

For the second claim, take $S = \mathrm{Spec}(k)$. □

Notation 6.14. We will usually abuse notation and leave ρ implicit in the notation when there is no risk of ambiguity. However, we warn the reader that different choices of ρ can lead to non-homotopic $f^!_\alpha$ and $[X_{/Y}]_\alpha$.

Remark 6.15. Our ρ-localized Gysin pull-back is comparable with the construction of [CKL, Def. 2.9] (in Chow homology), rather than that of [KL, Prop. 1.3].

Proof of Theorem 6.12. Let $i : X(\alpha) \to X$ denote the inclusion and $j : X \smallsetminus X(\alpha) \to X$ the complementary open immersion. Consider the localization triangle

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X(\alpha)_{/Y}) \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y}) \xrightarrow{j^!} \mathrm{C}^{\mathrm{BM}}_\bullet(X \smallsetminus X(\alpha)_{/Y}). \tag{6.16}$$

It will suffice to construct a null-homotopy of the composite

$$j^! \circ f^! : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X \smallsetminus X(\alpha)_{/Y}),$$

which by functoriality is identified with $(f \circ j)^!$. By replacing X with $X \smallsetminus X(\alpha)$, we may therefore assume that α is nowhere zero (i.e., that the cosection $\alpha^\vee$ is surjection) and construct a null-homotopy of $f^!$.

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \arrow[r, "\mathrm{sp}^\rho"] \arrow[d, equal] & \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y}) \arrow[r, equal] \arrow[d, "i_{K,*}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y}) \arrow[d, "\cap e(\mathscr{O})"] \\
\mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \arrow[r, "\mathrm{sp}"] & \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/Y}) \arrow[r, "i_K^!"] & \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y})\langle 1\rangle
\end{tikzcd}$$

where d is the relative virtual dimension of $f : X \to Y$ (= the opposite of the virtual rank of $N_{X/Y}$). The left-hand square comes from the definition of the cohomological reduction ρ. The middle square is the self-intersection formula for the closed immersion i_K, which is quasi-smooth with trivial conormal sheaf. Since $e(\mathscr{O})$ is canonically null-homotopic, this provides a null-homotopy of $i_K^! \circ \mathrm{sp}_{X/Y}$.

Finally, note the condition that α is nowhere zero translates to the fact that $\mathcal{K}^\alpha \in \mathbf{D}_{\mathrm{perf}}(X)^{\geqslant 0}$, i.e., that K^α is smooth (a vector bundle stack), and hence the zero section $0_K : X \to K^\alpha$ is quasi-smooth. Thus by functoriality of Gysin pull-back and the definition of the Gysin pull-back along f, we get the chain of identifications

$$f^! \simeq 0_N^! \circ \mathrm{sp}_{X/Y} \simeq 0_K^! \circ i_K^! \circ \mathrm{sp}_{X/Y} \simeq 0$$

as desired. □

6.3. **Reduced fundamental classes.** For (−1)-shifted 1-forms that are nowhere zero (so that the dual cosection is surjective), we deduce from Corollary 6.13 that the fundamental class vanishes. In this subsection, we construct a *reduced fundamental class* in this situation.

Construction 6.17. Let $f : X \to Y$ be a quasi-smooth morphism of derived Artin stacks of relative virtual dimension d over a derived Artin stack S. Let $\alpha : \mathscr{O}_X \to \mathcal{L}_{X/Y}[-1]$ be a (-1)-shifted 1-form. If α is *nowhere zero*, then $X(\alpha)$ is empty and $\mathcal{K}^\alpha$ is of Tor-amplitude $\leqslant 0$ (Notation 6.1). For any cohomological reduction ρ of f with respect to α, the *ρ-reduced Gysin pull-back* along f is the composite

$$f^!_{\rho\text{-red}} : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Y}) \xrightarrow{\mathrm{sp}^\rho_{X/Y}} \mathrm{C}^{\mathrm{BM}}_\bullet(K^\alpha_{/Y}) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})\langle -d-1\rangle$$

where the second isomorphism is homotopy invariance for the vector bundle stack K^α. The *ρ-reduced fundamental class* of X over Y, denoted $[X_{/Y}] \in \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})\langle -d-1\rangle$ is the image of 1 by $f^!_{\rho\text{-red}}$.

When α comes from a (-1)-shifted function, we show that the reduced fundamental class can be realized by a quasi-smooth derived stack:

Construction 6.18. Let $f : X \to Y$ be a quasi-smooth morphism of derived Artin stacks of relative virtual dimension d over a derived Artin stack S. Let $\phi : \mathscr{O}_X \to \mathscr{O}_X[-1]$ be a (-1)-shifted function on X. The *derived reduction* of the derived stack X with respect to ϕ is the derived fibred product

$$\begin{tikzcd} X^{\phi\text{-red}} \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\ \mathrm{Spec}(k) \arrow[r, "0"] & \mathbf{A}^1_k[-1]. \end{tikzcd}$$

Note that $X^{\phi\text{-red}}$ has the same classical truncation as X, and is of virtual dimension $d+1$ over Y (where d is the virtual dimension of X over Y). (Note that this construction already appeared in the proof of Proposition 6.3.)

Proposition 6.19. *Let the notation be as in Construction 6.18. Suppose that the de Rham differential α of ϕ is nowhere zero. Let ρ be the canonical cohomological reduction of f by α as in Proposition 6.3. Then the derived reduction $X^{\phi\text{-red}}$ is quasi-smooth over Y, and its fundamental class is canonically identified with the ρ-reduced fundamental class of X over Y:*

$$[X^{\phi\text{-red}}_{/Y}] \simeq [X_{/Y}]^{\rho\text{-red}}$$

under the derived invariance isomorphism $\mathrm{C}^{\mathrm{BM}}_\bullet(X^{\phi\text{-red}}_{/Y}) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})$.

Proof. Easy. □

7. Examples from moduli theory

In this section we assume that the base ring k is regular noetherian.

7.1. **Perfect complexes and pairs.**

7.1.1. *Perfect complexes.* Let $\underline{\mathrm{Perf}}$ denote the derived moduli stack of perfect complexes. By construction, we have the universal perfect complex $\mathscr{E}^{\mathrm{univ}}$ on $\underline{\mathrm{Perf}}$. Its cotangent complex is given by

$$L_{\underline{\mathrm{Perf}}} = \mathscr{E}^{\mathrm{univ}} \otimes \mathscr{E}^{\mathrm{univ},\vee}[-1], \tag{7.1}$$

see [Lur2, Rem. 19.2.2.4].

Given a derived stack X, let $\underline{\mathrm{Perf}}(X)$ denote the derived moduli stack of perfect complexes on X, i.e. the derived mapping stack

$$\underline{\mathrm{Perf}}(X) = \underline{\mathrm{Map}}(X, \underline{\mathrm{Perf}}).$$

There is an evaluation map

$$\mathrm{ev} : X \times \underline{\mathrm{Perf}}(X) = X \times \underline{\mathrm{Map}}(X, \underline{\mathrm{Perf}}) \to \underline{\mathrm{Perf}}.$$

Let $\mathscr{E}_X = \mathrm{ev}^*(\mathscr{E}^{\mathrm{univ}})$. When X is smooth proper and Deligne–Mumford, $\underline{\mathrm{Perf}}(X)$ is a union of n-Artin derived stacks (for varying n) which are homotopically of finite presentation, and its cotangent complex is given by

$$L_{\underline{\mathrm{Perf}}(X)} = \mathrm{pr}_{2,*}(\underline{\mathrm{Hom}}(\mathscr{E}_X, \mathscr{E}_X) \otimes \mathrm{pr}_1^*(\omega_X))[-1] \tag{7.2}$$

where pr_i are the respective projections of $X \times \underline{\mathrm{Perf}}(X)$, and $\omega_X = \Lambda^d L_X[d]$ is the derived dualizing complex (with d the dimension of X). See [HLP, Thm. 5.1.1], [TV, Cor. 0.4].

7.1.2. *Determinant.* Given a derived stack X, denote by

$$\underline{\mathrm{Pic}}(X) = \underline{\mathrm{Map}}(X, B\mathbf{G}_m)$$

the (derived) Picard stack of X. If X is smooth proper and Deligne–Mumford, then $\underline{\mathrm{Pic}}(X)$ is Artin by [HLP, Thm. 5.1.1]. Recall that there is a canonical determinant morphism

$$\det : \underline{\mathrm{Perf}}(X) \to \underline{\mathrm{Pic}}(X),$$

see e.g. [STV, §3.1].

7.1.3. *Pairs.* Consider the derived moduli stack $\underline{\mathrm{Pair}}$ of pairs $(\mathscr{E}, \sigma)$ where $\mathscr{E}$ is a perfect complex and $\sigma : \mathscr{O} \to \mathscr{E}$ is a section. This is equivalently $\mathbf{V}(\mathscr{E}^{\mathrm{univ},\vee})$ where $\mathscr{E}^{\mathrm{univ}}$ is the universal perfect complex on $\underline{\mathrm{Perf}}$. Given a derived Artin stack X, we let $\underline{\mathrm{Pair}}(X)$ denote the derived mapping stack $\underline{\mathrm{Map}}(X, \underline{\mathrm{Pair}})$.

Let $\mathscr{F} \in \mathbf{D}_{\mathrm{perf}}(\underline{\mathrm{Pair}})$ denote the fibre of the universal section σ^{univ}. The cotangent complex of $\underline{\mathrm{Pair}}$ is then

$$L_{\underline{\mathrm{Pair}}} = \mathscr{F} \otimes \pi^*(\mathscr{E}^{\mathrm{univ},\vee}) \simeq \underline{\mathrm{Hom}}_{\underline{\mathrm{Pair}}}(\mathscr{F}, \pi^*(\mathscr{E}^{\mathrm{univ}})),$$

where $\pi : \underline{\mathrm{Pair}} \to \underline{\mathrm{Perf}}$ is the projection, since $L_{\underline{\mathrm{Perf}}} = \mathscr{E}^{\mathrm{univ}} \otimes \mathscr{E}^{\mathrm{univ},\vee}[-1]$.

Note that there is also a canonical morphism

$$\underline{\mathrm{Pair}} \to \underline{\mathrm{Perf}} \tag{7.3}$$

sending $(\mathscr{E}, \sigma) \mapsto \mathrm{Cofib}(\sigma)$.

7.2. **Perfect complexes on surfaces.** In this section, we will construct (-1)-shifted 1-forms on the moduli of perfect complexes on a smooth projective surface with $h^1 = 0$ (see Construction 7.6). We begin with some preliminaries about the Picard stack.

Proposition 7.4. *Let k be a field and S be a smooth projective surface with $h^1(S) = 0$. Then there is a canonical isomorphism*

$$\underline{\mathrm{Pic}}(S) \simeq \underline{\mathrm{Pic}}(S)_{\mathrm{cl}} \times \mathbf{V}(H^0(S, K_S))[-1].$$

We will make use of the following lemma in the proof.

Lemma 7.5. *Let B be a derived Artin stack and $f : X \to Y$ a morphism between derived Artin stacks over B. If X and Y are flat over B and f_{cl} is an isomorphism, then f is an isomorphism.*

Proof. It will suffice to show that f is formally étale, i.e., the relative cotangent complex $L_{X/Y}$ vanishes. Let $s : \mathrm{Spec}(k) \to B$ be a morphism and denote by $X_s = \mathrm{Spec}(k) \times_B X$ and $Y_s = \mathrm{Spec}(k) \times_B Y$ the fibres. Since X and Y are flat over B, X_s and Y_s are classical. Therefore, we have a commutative diagram

$$\begin{tikzcd} X_s \arrow[r] \arrow[d, "f_s"] & X_{\mathrm{cl}} \arrow[r] \arrow[d, "f_{\mathrm{cl}}"] & X \arrow[d, "f"] \\ Y_s \arrow[r] & Y_{\mathrm{cl}} \arrow[r] & Y \end{tikzcd}$$

where the right-hand square is homotopy cartesian (since f is flat) and the outer square is homotopy cartesian (by definition), hence so is the left-hand square. In particular, since f_{cl} is an isomorphism, so is $f_s : X_s \to Y_s$. Thus we have $L_{X/Y}|_{X_s} \simeq L_{X_s/Y_s} \simeq 0$. Since s was arbitrary, the claim follows. ☐

Proof of Proposition 7.4. Consider the quotient $\underline{\mathrm{Pic}}^{\mathrm{rig}}(S)$ of $\underline{\mathrm{Pic}}(S)$ by the $B\mathbf{G}_m$-action. Consider the following diagram:

$$\begin{tikzcd} \underline{\mathrm{Pic}}(S)_{\mathrm{cl}} \arrow[r] \arrow[d] & \underline{\mathrm{Pic}}(S) \arrow[r, dashed, "r"] \arrow[d] & \underline{\mathrm{Pic}}(S)_{\mathrm{cl}} \arrow[d] \\ \underline{\mathrm{Pic}}^{\mathrm{rig}}(S)_{\mathrm{cl}} \arrow[r] & \underline{\mathrm{Pic}}^{\mathrm{rig}}(S) \arrow[r, "r_{\mathrm{cl}}"] & \underline{\mathrm{Pic}}^{\mathrm{rig}}(S)_{\mathrm{cl}}, \end{tikzcd}$$

where $\underline{\mathrm{Pic}}^{\mathrm{rig}}(S)_{\mathrm{cl}}$ is equivalently the rigidification of the classical Picard stack $\underline{\mathrm{Pic}}(S)_{\mathrm{cl}}$. In the left-hand square, the horizontal arrows are the inclusions of the classical truncations, and the square is homotopy cartesian because the quotient maps are smooth. Note that the assumptions on S imply that $\underline{\mathrm{Pic}}^{\mathrm{rig}}(S)_{\mathrm{cl}}$ is a disjoint union of spectra of fields (see e.g. [Mum, Lect. 24]). Therefore, we have a retraction r_{cl} as in the right-hand square. Since the left-hand square is homotopy cartesian, it lifts to a retraction r of the upper morphism. We now claim that the right-hand square is also homotopy cartesian, or equivalently, that the morphism $\underline{\mathrm{Pic}}(S) \to \underline{\mathrm{Pic}}^{\mathrm{rig}}(S) \times_{\underline{\mathrm{Pic}}^{\mathrm{rig}}(S)_{\mathrm{cl}}} \underline{\mathrm{Pic}}(S)_{\mathrm{cl}}$ is invertible. Indeed, since both source and target are smooth over $\underline{\mathrm{Pic}}^{\mathrm{rig}}(S)$, this follows from Lemma 7.5. By [MR, Prop. C.0.1], it follows that there is an isomorphism $\underline{\mathrm{Pic}}^{\mathrm{rig}}(S) \simeq N_{\underline{\mathrm{Pic}}^{\mathrm{rig}}(S)_{\mathrm{cl}}/\underline{\mathrm{Pic}}^{\mathrm{rig}}(S)}$. By base change we thus deduce

$$\underline{\mathrm{Pic}}(S) \simeq N_{\underline{\mathrm{Pic}}(S)_{\mathrm{cl}}/\underline{\mathrm{Pic}}(S)}.$$

Note that the cotangent complex of $\underline{\mathrm{Pic}}(S)$ is given (by its definition as a mapping stack) by

$$L_{\underline{\mathrm{Pic}}(S)} \simeq \mathscr{O}_{\underline{\mathrm{Pic}}(S)} \otimes R\Gamma(S, K_S)[1]$$

since S is a smooth surface. Since $\underline{\mathrm{Pic}}(S)_{\mathrm{cl}}$ is smooth, we have $L_{\underline{\mathrm{Pic}}(S)_{\mathrm{cl}}} \simeq (L_{\underline{\mathrm{Pic}}(S)})^{\geqslant 0}|_{\underline{\mathrm{Pic}}(S)_{\mathrm{cl}}}$. Thus $L_{\underline{\mathrm{Pic}}(S)_{\mathrm{cl}}/\underline{\mathrm{Pic}}(S)}[-1] \simeq L^{<0}_{\underline{\mathrm{Pic}}(S)} \simeq \mathscr{O}_{\underline{\mathrm{Pic}}(S)} \otimes \mathrm{H}^0(S, K_S)[1]$. Finally, we get

$$\underline{\mathrm{Pic}}(S) \simeq \underline{\mathrm{Pic}}(S)_{\mathrm{cl}} \times \mathbf{V}(\mathrm{H}^0(S, K_S)[1])$$

as claimed. □

Construction 7.6. Let k be a field and S be a smooth projective surface with $h^1(S) = 0$. Given a canonical divisor on S, i.e., an element $\Theta \in \mathrm{H}^0(S, K_S)$, consider the composite

$$\underline{\mathrm{Perf}}(S) \xrightarrow{\det} \underline{\mathrm{Pic}}(S) \simeq \underline{\mathrm{Pic}}(S)_{\mathrm{cl}} \times \mathbf{V}(H^0(S, K_S))[-1] \\ \to \mathbf{V}(H^0(S, K_S))[-1] \xrightarrow{\Theta} \mathbf{A}^1_k[-1].$$

This is a (-1)-shifted function on $\underline{\mathrm{Perf}}(S)$ and its de Rham differential defines a (-1)-shifted 1-form. By restriction along the projection $\underline{\mathrm{Pair}}(S) \to \underline{\mathrm{Perf}}(S)$, we also get a (-1)-shifted function on $\underline{\mathrm{Pair}}(S)$.

7.3. **Hilbert schemes of divisors and stable maps on surfaces.** Let k be a field and S be a smooth projective surface with $h^1(S) = 0$. We apply Construction 7.6 to the Hilbert scheme of divisors on S and to the moduli of stable maps on S. In particular, we construct a reduced virtual fundamental class for stable maps on K3 surfaces over arbitrary base fields (even though the moduli stack is not Deligne–Mumford in positive characteristic).

Example 7.7. Let $\underline{\mathrm{Div}}(S)$ denote the derived Hilbert scheme of divisors, i.e., the open substack of $\underline{\mathrm{Pair}}(S)$ classifying pairs $(\mathscr{O}_D, \sigma)$ where D is a divisor on S and $\sigma : \mathscr{O}_S \twoheadrightarrow \mathscr{O}_D$ is the canonical surjection. Construction 7.6 gives rise to a canonical (-1)-shifted 1-form on $\underline{\mathrm{Div}}(S)$. This is dual to the cosection used in [CK] (over $k = \mathbf{C}$).

Example 7.8. Let S be as in Construction 7.6. Let $\overline{\mathcal{M}}_{g,n}(S)$ denote the derived moduli stack of stable maps to S of genus g with n marked points. By pull-back along the canonical morphism

$$\overline{\mathcal{M}}_{g,n}(S) \to \underline{\mathrm{Perf}}(S),$$

sending $(f : C \to X)$ for a curve C to $Rf_*(\mathscr{O}_C) \in \mathbf{D}_{\mathrm{perf}}(X)$ (see e.g. [STV, Def. 3.9]), we get a canonical (-1)-shifted 1-form on $\overline{\mathcal{M}}_{g,n}(S)$. In particular, there is a corresponding derived reduction $\overline{\mathcal{M}}^{\mathrm{red}}_{g,n}(S)$ (see Construction 6.18). When S is a K3 surface, this coincides with [STV, Def. 4.7]. Since the (-1)-shifted 1-form is nowhere zero in the K3 case (see [STV, Thm. 4.8], which holds over general base fields), we find that the fundamental class of

$\overline{\mathcal{M}}^{\mathrm{red}}_{g,n}(S)$ coincides (in the case of motivic cohomology) with the Chow cycle constructed in [MPT] by Proposition 6.19.

7.4. **Pairs on threefolds.** We now consider the moduli stack of pairs on a threefold, whose stable loci give rise to the Pandharipande–Thomas theory of stable pairs [PT].

Example 7.9. Let X be a smooth projective threefold. Let $\theta \in \mathrm{H}^0(X, \Omega^2_X)$ be a holomorphic 2-form. We define a canonical (-1)-shifted 1-form on $\underline{\mathrm{Perf}}(X)$ as follows.

Let $\mathscr{E}_X$ be the perfect complex on $X \times \underline{\mathrm{Perf}}(X)$ defined in 7.1.1. Its Atiyah class (see e.g. [Lur2, §19.2.2]) is a morphism $\mathrm{At} : \mathscr{E}_X \to L_{X\times\underline{\mathrm{Perf}}(X)} \otimes \mathscr{E}_X[1]$, which by post-composition with $L_{X\times\underline{\mathrm{Perf}}(X)} \to L_{X\times\underline{\mathrm{Perf}}(X)/\underline{\mathrm{Perf}}(X)}$ gives

$$\mathrm{At} : \mathscr{E}_X \to L_{X\times\underline{\mathrm{Perf}}(X)/\underline{\mathrm{Perf}}(X)} \otimes \mathscr{E}_X[1] \simeq \mathrm{pr}_1^*\Omega_X \otimes \mathscr{E}_X[1],$$

where $\mathrm{pr}_1 : X \times \underline{\mathrm{Perf}}(X) \to X$ is the projection. This gives rise to

$$\underline{\mathrm{Hom}}(\mathscr{E}_X, \mathscr{E}_X) \xrightarrow{\mathrm{At}} \underline{\mathrm{Hom}}(\mathscr{E}_X, \mathrm{pr}_1^*\Omega_X \otimes \mathscr{E}_X) \xrightarrow{\mathrm{tr}} \mathrm{pr}_1^*\Omega_X[1]. \tag{7.10}$$

Composing (7.10) with the morphism

$$\Omega_X \xrightarrow{\cup\theta} \Omega_X \otimes \Lambda^2\Omega_X \to \Lambda^3\Omega_X$$

yields

$$\underline{\mathrm{Hom}}(\mathscr{E}_X, \mathscr{E}_X)[2] \to \mathrm{pr}_1^*\Omega_X[3] \to \mathrm{pr}_1^*\Lambda^3\Omega_X[3] = \mathrm{pr}_1^*(\omega_X).$$

Since $\underline{\mathrm{Hom}}(\mathscr{E}_X, \mathscr{E}_X) \simeq \mathscr{E}_X \otimes \mathscr{E}_X^\vee$ is self-dual this corresponds to a canonical map

$$\mathscr{O}_{X\times\underline{\mathrm{Perf}}(X)} \to \underline{\mathrm{Hom}}(\mathscr{E}_X, \mathscr{E}_X)[-2] \otimes \mathrm{pr}_1^*(\omega_X).$$

Finally by adjunction, this is identified with

$$\mathscr{O}_{\underline{\mathrm{Perf}}(X)} \to \mathrm{pr}_{2,*}\big(\underline{\mathrm{Hom}}(\mathscr{E}_X, \mathscr{E}_X) \otimes \mathrm{pr}_1^*(\omega_X)\big)[-2] \simeq L_{\underline{\mathrm{Perf}}(X)}[-1]$$

which is the desired (-1)-shifted 1-form.

Remark 7.11. We expect that the 1-form in Example 7.9 is closed (or more precisely, it admits a closing structure in the sense of [PTVV]). This would imply (at least over algebraically closed fields of characteristic zero) that it admits a cohomological reduction (combine Lemma 6.11 and Corollary 6.10).

Remark 7.12. By inverse image along the cofibre morphism $\underline{\mathrm{Pair}} \to \underline{\mathrm{Perf}}$ (7.3), Example 7.9 gives a (-1)-shifted 1-form on the moduli of pairs. Consider its restriction to the locus of Pandharipande–Thomas stable pairs with fixed curve class $\beta \in \mathrm{H}_2(X, \mathbf{Z})$ and Euler characteristic $n \in \mathbf{Z}$ (see [PT]). If this restriction is nowhere zero, then by Corollary 6.13 the virtual fundamental class vanishes, and we get a reduced fundamental class (Construction 6.17) for the moduli of stable pairs. Otherwise, assuming the conjecture in Remark 7.11 (and assuming k is algebraically closed of characteristic zero), the (-1)-shifted 1-form gives rise to a localized virtual fundamental class for the moduli of stable pairs. When X is a local surface, i.e. the canonical bundle K_S of a smooth projective surface S over k (or its projective completion), these constructions are considered in [KT, Eq. (28)] and [KT2, Eq. (16)], respectively.

7.5. **Higgs bundles/sheaves on curves.** Let k be a field and C a smooth proper and geometrically connected curve over k. Consider the moduli stack $\underline{\mathrm{Coh}}(C)$ of coherent sheaves on C. This is a smooth 1-Artin stack locally of finite type over k.

Denote by $\underline{\mathrm{Higgs}}$ the moduli stack of Higgs sheaves on C, i.e., the cotangent bundle of $\underline{\mathrm{Coh}}(C)$ (which for us means the total space of the cotangent *complex*). This admits a canonical scaling action by the torus $T = \mathbf{G}_m$. Let $\mathbf{\Lambda}$ denote the closed substack parametrizing *nilpotent* Higgs sheaves, i.e., pairs $(\mathscr{F}, \theta)$ where $\mathscr{F} \in \mathrm{Coh}$ and $\theta : \mathscr{F} \to \mathscr{F} \otimes K_C$ is nilpotent (with K_C the canonical bundle of C).

Theorem 7.13. *Let $\Sigma \subseteq \mathrm{Pic}(BT)$ denote the set of nontrivial invertible sheaves on BT. Then Σ satisfies condition (L_T) for $\underline{\mathrm{Higgs}} \smallsetminus \mathbf{\Lambda}$. In particular, the direct image map*

$$\mathrm{C}^{\mathrm{BM},T}_{\bullet}(\mathbf{\Lambda})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM},T}_{\bullet}(\underline{\mathrm{Higgs}})_{\mathrm{loc}}$$

is invertible (where the localization is as in Corollary 2.30).

Proof. Let $\mathscr{F}$ be a Higgs sheaf over an algebraically closed extension field κ of k. By Corollary 4.9 it is enough to show that, if $\mathscr{F}$ is not nilpotent, then the T-stabilizer at the corresponding geometric point of $\underline{\mathrm{Higgs}}$ is a proper subgroup of T_κ. We are grateful to A. Minets for providing the following argument.

First note that if $\mathscr{F}$ is not nilpotent, then either its maximal torsion Higgs subsheaf $\mathscr{F}_0$ is nilpotent or the quotient $\mathscr{F}' = \mathscr{F}/\mathscr{F}_0$ is nilpotent. Thus we may assume that $\mathscr{F}$ is either torsion or locally free.

Recall that the moduli stack of locally free Higgs sheaves (= Higgs bundles) of rank r admits a canonical T-equivariant map (the Hitchin fibration) to the scheme $A_r = \bigoplus_{i=1}^r \mathrm{H}^0(C, K_C^{\otimes i})$, where T acts on $\mathrm{H}^0(C, K_C^{\otimes i})$ with weight i, by sending $\mathscr{F}$ to the coefficients of the characteristic polynomial of its Higgs field. Moreover, a locally free Higgs sheaf $\mathscr{F}$ is nilpotent if and only if its image in the Hitchin base is trivial. By Corollary 2.21, this implies the claim in the locally free case.

In the torsion case, we can argue similarly using the canonical map from the moduli stack of torsion Higgs sheaves to $\mathrm{Sym}^d(T^*_C)$, sending a Higgs sheaf to the support of the corresponding coherent sheaf on T^*_C (under the "BNR correspondence", see [BNR] or [Sim, Lem. 6.8]). □

Remark 7.14. For the substack of torsion Higgs sheaves, Theorem 7.13 recovers [Min, Cor. 4.3] by taking π_0. In other words, we can regard the result as a generalization of *loc. cit.* to arbitrary rank Higgs sheaves[7] and to "higher" oriented Borel–Moore homology theories.

Remark 7.15. Theorem 7.13 admits an analogous statement for Higgs G-bundles, for a connected reductive group G, using the same Hitchin fibration

[7]A similar claim is made in [SS, Prop. 3.7], but the proof is not correct because the localization does not commute with cofiltered limits. This is related to the reason why the discussion in Subsect. 2.7 is necessary.

argument. Similarly, there is a parabolic variant using the parabolic Hitchin fibration [Yok].

Appendix A. Fixed loci of group actions on algebraic stacks

In this appendix we fix a scheme S and an fppf group algebraic space G over S. All Artin stacks will be assumed to have separated diagonal.

For an Artin stack X over S with G-action, we will define a focus locus $X^G \subseteq X$ and study its properties. We will also introduce a homotopy fixed point stack X^{hG}, which usually is not a substack of X but has better deformation-theoretic properties. In the case of torus actions, we will prove a certain relation between the two constructions (Theorem A.48).

A.1. **Stabilizers of group actions.** Given an action of an algebraic group G on an Artin stack X, we define the stabilizer of the action ("G-stabilizer") at any point x of X. When the stabilizer at x of X itself is trivial, this coincides with the stabilizer of the quotient stack $[X/G]$ at x.

Remark A.1. Let $f : X \to Y$ be a morphism of Artin stacks over S. The relative inertia stack $I_{X/Y}$ is a group Artin stack over X which fits into a cartesian square

$$\begin{tikzcd} I_{X/Y} \arrow[r] \arrow[d] & I_{X/S} \arrow[d] \\ X \arrow[r] & I_{Y/S} \times_Y X \end{tikzcd}$$

of group stacks over X. The lower horizontal arrow is the base change of the identity section $e : Y \to I_{Y/S}$. When f is representable, $I_{X/Y} \to X$ is an isomorphism, i.e., $I_{X/S} \to I_{Y/S} \times_Y X$ is a monomorphism of group stacks. See e.g. [SP, Tag 050P].

Remark A.2. Let X be an Artin stack over S with G-action and denote by $\mathfrak{X} = [X/G]$ the quotient stack. Applying Remark A.1 to the morphisms $X \twoheadrightarrow \mathfrak{X}$ and $\mathfrak{X} \to BG$, we get the cartesian squares of group stacks over X

$$\begin{tikzcd} X \arrow[r] \arrow[d] & I_{X/S} \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & I_{\mathfrak{X}/S} \times_{\mathfrak{X}} X \arrow[r] & G \times_S X, \end{tikzcd}$$

where $I_{X/\mathfrak{X}} \simeq X$ since $X \twoheadrightarrow \mathfrak{X}$ is representable and the right-hand vertical arrow is the identity section. For every S-scheme A and every A-valued point x of X, this gives rise to an exact sequence of group algebraic spaces over A

$$1 \to \underline{\mathrm{Aut}}_X(x) \to \underline{\mathrm{Aut}}_{\mathfrak{X}}(x) \xrightarrow{\alpha_A} G_A \tag{A.3}$$

where $G_A \coloneqq G \times_S A$ denotes the fibre of G over x.

Definition A.4. Let X be an Artin stack over S with G-action. For any scheme A and every A-valued point x of X, the *G-stabilizer* (or *stabilizer of the G-action*) at x is an fppf sheaf of groups $\mathrm{St}^G_X(x)$ defined as the cokernel of the homomorphism $\underline{\mathrm{Aut}}_X(x) \hookrightarrow \underline{\mathrm{Aut}}_{\mathcal{X}}(x)$. Thus we have a short exact sequence

$$1 \to \underline{\mathrm{Aut}}_X(x) \to \underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to \mathrm{St}_X^G(x) \to 1 \tag{A.5}$$

of sheaves of groups over A. Note that $\mathrm{St}_X^G(x)$ can be regarded as a subgroup of G_A, since it is the image of $\alpha_A : \underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to G_A$.

Remark A.6. For a field-valued point $x : \mathrm{Spec}(k(x)) \to X$, the G-stabilizer $\mathrm{St}_X^G(x)$ is a group algebraic space. This follows from [SGA3, Exp. V, Cor. 10.1.3], since in this case $\underline{\mathrm{Aut}}_X(x)$ is flat over $\mathrm{Spec}(k(x))$. Since X has separated diagonal, $\mathrm{St}_X^G(x)$ is moreover a group *scheme* by [SP, 0B8F].

Remark A.7. When X has trivial stabilizers (i.e., is an algebraic space), then the G-stabilizer $\mathrm{St}_X^G(x)$ at a point x is nothing else than the automorphism group $\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$ of the quotient stack $\mathcal{X} = [X/G]$.

Remark A.8. Let X be a *derived* Artin stack locally of finite type over k with G-action. For any field-valued point x of X, the *G-stabilizer* of X at x is defined to be the G-stabilizer of the classical truncation X_{cl} at x.

Remark A.9. Let X be an Artin stack over S with G-action. Let A be an S-scheme and x an A-valued point of X. From the short exact sequence (A.5) we see that the induced morphism $B\underline{\mathrm{Aut}}_X(x) \to B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$ is a $\mathrm{St}_X^G(x)$-torsor, where $\mathcal{X} = [X/G]$. Moreover, there is a commutative diagram

$$\begin{tikzcd} B\underline{\mathrm{Aut}}_X(x) \arrow[r] \arrow[d] & X \arrow[d] \\ B\underline{\mathrm{Aut}}_{\mathcal{X}}(x) \arrow[r] & \mathcal{X} \end{tikzcd}$$

where the right-hand arrow is a G-torsor. In particular, we find that there is a canonical $\mathrm{St}_X^G(x)$-action on the group algebraic space $\underline{\mathrm{Aut}}_X(x)$, and the canonical monomorphism

$$B\underline{\mathrm{Aut}}_X(x) \hookrightarrow X$$

is equivariant with respect to the $\mathrm{St}_X^G(x)$-action on the source and G-action on the target.

Notation A.10. We denote by $\mathfrak{a}(x)^\vee$ the dual Lie algebra of the group algebraic space $\underline{\mathrm{Aut}}_X(x)$ over A, i.e.,

$$\mathfrak{a}(x)^\vee = e^* \Omega^1_{\underline{\mathrm{Aut}}_X(x)/A}$$

where $e : A \to \underline{\mathrm{Aut}}_X(x)$ is the identity section. The $\mathrm{St}_X^G(x)$-action on $\underline{\mathrm{Aut}}_X(x)$ (Remark A.9) descends to $\mathfrak{a}(x)^\vee$.

A.2. **Fixed points.**

Definition A.11. Let X be an Artin stack over S with G-action. The *G-fixed locus* $X^G \subseteq X$ is the locus where the canonical homomorphism of group algebraic spaces over X

$$I_{\mathcal{X}} \underset{\mathcal{X}/S}{\times} X \to G \underset{S}{\times} X \tag{A.12}$$

is surjective[8], where $\mathcal{X} = [X/G]$. In other words, let A be an S-scheme and $x \in X(A)$ an A-point, and consider the homomorphism of group algebraic spaces over A (A.3)

$$\alpha_A : \underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to G_A$$

obtained by base changing (A.12) along $x : A \to X$. Then x belongs to the fixed locus X^G if and only if α_A admits a section after base change along some fppf cover $A' \twoheadrightarrow A$.

Remark A.13. For an A-point x of X, recall that the image of α_A is the G-stabilizer $\mathrm{St}^G_X(x)$ at x (Definition A.4). Thus, the fixed locus X^G is the locus of points x where the inclusion $\mathrm{St}^G_X(x) \subseteq G_A$ is an equality.

Question A.14. Is the inclusion $X^G \to X$ a closed immersion?

We will see that this holds for algebraic spaces (Proposition A.23). It also "almost" holds for split torus actions on 1-Artin stacks with finite stabilizers:

Proposition A.15. *Suppose G acts on a 1-Artin stack X locally of finite type over k with finite stabilizers. If G has connected fibres over S, then the subset $|X^G| \subseteq |X|$ is closed. In particular, there exists a reduced closed substack X^G_{red} of X such that $|X^G_{\mathrm{red}}| = |X^G|$.*

Proof. The subset $|X^G|$ is the locus of points $x \in |X|$ for which $\mathrm{St}^G_X(x)$ is equal to $G_{k(x)} = G \times_S \mathrm{Spec}(k(x))$. Since $G_{k(x)}$ is connected (hence irreducible, see [SGA3, Exp. VI$_\mathrm{A}$, Cor. 2.4.1]), this is equivalent to the condition that $\dim(\mathrm{St}^G_X(x)) = \dim(G_{k(x)})$. Since X has finite stabilizers, the short-exact sequence (A.5) shows that $\dim(\mathrm{St}^G_X(x)) = \dim(\underline{\mathrm{Aut}}_{\mathcal{X}}(x))$. Note that $\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$ is the fibre $\pi^{-1}(x)$ of the projection of the inertia stack $\pi : I_{\mathcal{X}} \to \mathcal{X}$ of $\mathcal{X} = [X/G]$, so closedness of the locus where $\dim(\underline{\mathrm{Aut}}_{\mathcal{X}}(x)) = \dim(G_{k(x)})$ follows from the upper semi-continuity of the function $x \mapsto \dim(\pi^{-1}(x))$ on $|\mathcal{X}|$ (see [Ryd2], [SP, Tag 0DRQ]). ☐

Definition A.16. Let the notation be as in Proposition A.15. The Artin stack X^G_{red} is called the *reduced G-fixed locus* of X. If X is a *derived* 1-Artin stack with finite stabilizers and G-action, then its reduced G-fixed locus is the reduced G-fixed locus of X_{cl} (with the induced G-action).

A.3. **Homotopy fixed points.** For classical stacks, the following definition is studied in [Rom2, Rom3, Rom4].

Definition A.17. Let X be a derived Artin stack over S with G-action. The *homotopy fixed point stack* of X is the stack of G-equivariant morphisms $S \to X$, i.e.,

$$X^{hG} := \underline{\mathrm{Map}}^G_S(S, X),$$

where S is regarded with trivial G-action. We write $\varepsilon := \varepsilon^G_X : X^{hG} \to X$ for the canonical morphism.

[8] i.e., an effective epimorphism of fppf sheaves

Remark A.18. Equivalently, X^{hG} can be described as the Weil restriction of $[X/G] \to BG$ along $BG \to S$. More explicitly, it fits into the following homotopy cartesian square:

$$\begin{tikzcd} X^{hG} \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_S(BG, [X/G]) \arrow[d] \\ S \arrow[r, "\mathrm{id}_{BG}"] & \underline{\mathrm{Map}}_S(BG, BG). \end{tikzcd}$$

In other words, X^{hG} classifies sections of $[X/G] \to BG$.

Remark A.19. Yet another way to describe X^{hG} is that it classifies group-theoretic sections of the homomorphism of group algebraic spaces over X

$$I_{\mathcal{X}} \underset{\mathcal{X}}{\times} X \to G \underset{S}{\times} X$$

where $\mathcal{X} = [X/G]$. Indeed, there is a cartesian square

$$\begin{tikzcd} X^{hG} \arrow[r] \arrow[d, "\varepsilon"] & \underline{\mathrm{Grp}}_X(G \times_S X, I_{\mathcal{X}} \times_{\mathcal{X}} X) \arrow[d] \\ X \arrow[r, "\mathrm{id}"] & \underline{\mathrm{Grp}}_X(G \times_S X, G \times_S X) \end{tikzcd}$$

where $\underline{\mathrm{Grp}}_X(-,-)$ denotes the sheaf of group homomorphisms over X. See e.g. [Rom4, Lem. 4.1.2].

In terms of points, we see that for any S-scheme A a lift of an A-point x of X along $\varepsilon : X^{hG} \to X$ amounts to a *group-theoretic section* of the homomorphism (A.3)

$$\alpha_A : \underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to G_A$$

of group algebraic spaces over A. Comparing with Definition A.11, we find in particular that $\varepsilon : X^{hG} \to X$ factors through the fixed locus $X^G \subseteq X$.

Remark A.20. Informally speaking, we can think of a point of X^{hG} as a point x of X together with a collection of "fixings", i.e., for every point g of G an isomorphism $g \cdot x \simeq x$, together with a homotopy coherent system of compatibilities between them (with respect to the group operation).

For the next statement, we recall the notion of *formal properness* from [HLP].

Remark A.21. The classifying stack BG is formally proper over S when either (a) S is the spectrum of a field and G is reductive (see [HLP, Ex. 4.3.5]); or (b) S is noetherian and G is linearly reductive (see Prop. 4.2.3 and Thm. 4.2.1 in [HLP]).

Theorem A.22 (Halpern-Leistner–Preygel)**.** *Let X be a derived Artin stack locally of finite type over S with G-action. If BG is formally proper over S and X is 1-Artin with affine stabilizers, then the derived stack X^{hG} is 1-Artin with affine stabilizers.*

Proof. Follows from [HLP, Thm. 5.1.1, Rmk. 5.1.3] in view of Remark A.18. □

For G of multiplicative type (and X classical), a different proof of Theorem A.22 was given in [Rom3, Thm. 1].

A.4. **Properties.** We record some general properties of the constructions X^G and X^{hG}. Our main interest is in the properties of the canonical morphisms $X^G \hookrightarrow X$ and $\varepsilon : X^{hG} \to X$.

We begin by comparing X^G and X^{hG} in the case of algebraic spaces. This is probably well-known.

Proposition A.23. *Let X be an algebraic space over S with G-action. Assume either that S is the spectrum of a field, or that G is of multiplicative type. Then there is a canonical isomorphism $X^G \simeq X^{hG}$ over X. Moreover, the morphisms $X^G \to X$ and $\varepsilon : X^{hG} \to X$ are closed immersions.*

Proof. By [CGP, Prop. A.8.10], $\varepsilon : X^{hG} \to X$ is a closed immersion. Therefore, it will suffice to show that the canonical morphism (Remark A.19)

$$X^{hG} \to X^G \tag{A.24}$$

is invertible. Since $X^G \to X$ and $X^{hG} \to X$ are both monomorphisms, it will suffice to show that (A.24) is surjective on A-valued points (for all S-schemes A). But since X has trivial stabilizers, for every A-valued point $x : A \to X$ the canonical homomorphism $\underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to G_A$ is surjective if and only if it is invertible (see Remark A.7). In particular, if x belongs to X^G then $\underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to G_A$ already admits a section over A. ☐

Let us now turn our attention to the morphism $\varepsilon : X^{hG} \to X$. In general, it is not a closed immersion or even a monomorphism (Example A.29). We begin with the following statement (we thank M. Romagny for providing the idea for the proof).

Proposition A.25. *Suppose S is locally noetherian and G has smooth and connected fibres over S. Let X be a derived 1-Artin stack with G-action, which is locally of finite type over S with finite stabilizers. Then the canonical morphism $\varepsilon : X^{hG} \to X$ is essentially proper*[9]*.*

Proof. Since $(-)^{hG}$ commutes with classical truncation, we may assume that X is classical. Since ε is separated and locally of finite presentation [Rom3, Rom4], it will suffice to show that for every discrete valuation ring R with fraction field K and every commutative solid arrow diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & X^{hG} \arrow[d, "\varepsilon"] \\
\mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & X,
\end{tikzcd}$$

[9] I.e., it is locally of finite type and satisfies the valuative criterion for properness. Thus essentially proper + quasi-compact ⇒ proper. See [EGA, IV_4, Rem. 18.10.20].

there exists a dashed arrow making the total diagram commute. This amounts to showing that for every R-point x of X and any group-theoretic section σ_K of the morphism $\alpha_K : \underline{\mathrm{Aut}}_{\mathfrak{X}}(x_K) \to G_K$ (A.3), there exists a section

$$\sigma : G \to \underline{\mathrm{Aut}}_{\mathfrak{X}}(x)$$

of $\alpha_R : \underline{\mathrm{Aut}}_{\mathfrak{X}}(x) \to G_R$ which lifts σ_K.

Since X has finite stabilizers, α_R is quasi-finite (since its kernel $\underline{\mathrm{Aut}}_X(x)$ is quasi-finite). Since α_R is affine (as a morphism between affine schemes, by [Ray, IX, Lem. 2.2]), the section σ_K is a closed immersion. Let $\Gamma \subseteq \underline{\mathrm{Aut}}_{\mathfrak{X}}(x)$ denote the schematic closure of $\sigma_K(G_K) \subseteq \underline{\mathrm{Aut}}_{\mathfrak{X}}(x_K)$ in $\underline{\mathrm{Aut}}_{\mathfrak{X}}(x)$. This is a closed subgroup of $\underline{\mathrm{Aut}}_{\mathfrak{X}}(x)$ (see [Rom1, §4.1]). We claim that the homomorphism of group R-schemes

$$\Gamma \subseteq \underline{\mathrm{Aut}}_{\mathfrak{X}}(x) \to G_R \tag{A.26}$$

is invertible. Since it is quasi-finite, separated and birational (because over K it is the isomorphism $\Gamma_K \subseteq \underline{\mathrm{Aut}}_{\mathfrak{X}}(x_K) \to G_K$) with normal target, Zariski's main theorem implies that (A.26) is an open immersion.

Let $\mathfrak{m} \subseteq R$ denote the maximal ideal. The base change $\alpha_{R/\mathfrak{m}}$ has *finite* kernel, hence is finite. Thus (A.26) base changes to a finite open immersion (i.e., an inclusion of connected components) over $R/\mathfrak{m}$. Since $G_{R/\mathfrak{m}}$ is irreducible and $\Gamma_{R/\mathfrak{m}}$ is nonempty, this shows that (A.26) is bijective over $R/\mathfrak{m}$. Since it is also an isomorphism over the fraction field K, it follows that (A.26) is invertible as claimed.

We now obtain the desired section σ by taking the composite

$$\sigma : G_R \xleftarrow{\sim} \Gamma \subseteq \underline{\mathrm{Aut}}_{\mathfrak{X}}(x)$$

which is a group homomorphism by construction. □

For torus actions on Deligne–Mumford stacks, we see that ε is a closed immersion:

Proposition A.27. *Assume that $G = T$ is a torus and S is locally noetherian. Let X be a derived Deligne–Mumford stack with T-action, quasi-separated and locally of finite type over S with separated diagonal. Then the canonical morphism $\varepsilon : X^{hT} \to X$ is a closed immersion.*

Proof. Since $(-)^{hT}$ commutes with classical truncation, we may assume that X is classical. By Proposition A.25, it is enough to show that ε is a locally closed immersion. For this we may argue as in the proof of [AHR, Thm. 5.16][10] to reduce to the case where S is the spectrum of an algebraically closed field and X is affine (note that the first statement of [AHR, Prop 5.20] only uses connectedness of the group). Then the claim follows from Proposition A.23. □

[10]Note that *loc. cit.* claims to show that ε is a closed immersion, but in fact the argument only shows it is locally closed due to the use of their generalized Sumihiro theorem. We thank A. Kresch for pointing this out to us.

Remark A.28. If G is of multiplicative type and X is 1-Artin with affine and finitely presented diagonal, then $\varepsilon : X^{hG} \to X$ is representable, separated and locally of finite presentation ([Rom3, Thm. 1]). This is generalized further in the forthcoming work [Rom4].

Example A.29 (Romagny)**.** If G acts on a 1-Artin stack X with finite stabilizers, the canonical morphism $\varepsilon : X^{hG} \to X$ is not generally a monomorphism or even unramified. The following example appears in [Rom4], who checks that ε is not a monomorphism. Here we show the stronger statement that, in the same example, ε is not even unramified.

Let S be the spectrum of an algebraically closed field k of characteristic $p > 0$ and let $G = T$ be the rank one torus $\mathbf{G}_{m,k}$. Let α_p denote the group k-scheme fitting in the short-exact sequence

$$0 \to \alpha_p \to \mathbf{G}_{a,k} \xrightarrow{F} \mathbf{G}_{a,k} \to 0$$

where F sends $x \mapsto x^p$. The latter is $\mathbf{G}_m$-equivariant with respect to the scaling action with weight 1 on the source and weight p on the target, so α_p inherits a $\mathbf{G}_m$-action by scaling. This gives rise to a $\mathbf{G}_m$-action on the classifying stack $X = B\alpha_p$, and we claim that the morphism $\varepsilon : X^{h\mathbf{G}_m} \to X$ is ramified.

By Corollary A.37 it will suffice to show that the dual Lie algebra $\mathfrak{a}(x)^\vee$ of $\underline{\mathrm{Aut}}_X(x)$ has nonzero moving part, where $x : \mathrm{Spec}(k) \twoheadrightarrow X = B\alpha_p$ is the quotient morphism. But $\underline{\mathrm{Aut}}_X(x) = \alpha_p$ and $\mathfrak{a}(x)^\vee$ is the Lie algebra of α_p, which is H^0 of the cotangent complex of α_p (restricted along the identity section), which since $dF = 0$ we compute ($\mathbf{G}_m$-equivariantly) as

$$\mathscr{O}^{(-1)} \oplus \mathscr{O}^{(-p)}[1]$$

where $\mathscr{O}^{(i)}$ is the structure sheaf with weight i scaling action.

A.5. **Deformation theory of homotopy fixed points.**

Remark A.30. When BG is *formally proper* over S (see [HLP] and Remark A.21), the functor $f^* : \mathbf{D}_{\mathrm{qc}}(S) \to \mathbf{D}_{\mathrm{qc}}(BG)$ of inverse image along $f : BG \to S$ admits a *left* adjoint

$$f_+ : \mathbf{D}_{\mathrm{qc}}(BG) \to \mathbf{D}_{\mathrm{qc}}(S),$$

see [HLP, Prop. 5.1.6] which is computed by $f_+(\mathscr{F}) = f_*(\mathscr{F}^\vee)^\vee$ on perfect complexes $\mathscr{F}$. The same holds for any base change of f. Under the identification $\mathbf{D}_{\mathrm{qc}}(BG) \simeq \mathbf{D}_{\mathrm{qc}}^G(S)$, f_* and f_+ are the functors of (derived) G-invariants and coinvariants, respectively.

Corollary A.31. *Suppose that BG is formally proper over S. Then the cotangent complex of X^{hG} is given by*

$$L_{X^{hG}/S} \simeq \bar{f}_+ e^*(L_{[X/G]/BG}) \tag{A.32}$$

where $e : X^{hG} \times BG \simeq [X^{hG}/G] \to [X/G]$ is induced by $\varepsilon : X^{hG} \to X$ and $\bar{f} : X^{hG} \times_S BG \to X^{hG}$ is the projection.

Proof. This is the formula for the cotangent complex of a Weil restriction given in [Lur2, Prop. 19.1.4.3], which generalizes to our formally proper situation as in [HLP, Prop. 5.1.10]. ☐

In other words, Corollary A.31 states that $L_{X^{hG}/S}$ is given by the (derived) G-coinvariants of the pull-back to X^{hG} of the cotangent complex $L_{X/S}$ (regarded with its canonical G-action). Dually, the tangent complex is given by the (derived) G-invariants (= G-fixed part) of the pull-back to X^{hG} of the tangent complex $T_{X/S}$. In the linearly reductive case, we do not need to distinguish between invariants and coinvariants.

Lemma A.33. *Suppose G is linearly reductive. Let X be a locally noetherian Artin stack over S and write $f : X \times_S BG \to X$ for the projection. Then there is a canonical isomorphism*

$$f_*(\mathscr{F}) \to f_+(\mathscr{F})$$

for every quasi-coherent complex $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X \times_S BG)$.

Proof. We will show that the canonical morphism

$$f^* f_*(\mathscr{F}) \xrightarrow{\text{counit}} \mathscr{F} \xrightarrow{\text{unit}} f^* f_+(\mathscr{F})$$

is invertible; the claimed isomorphism will then follow by applying $*$-inverse image along the quotient morphism $S \twoheadrightarrow BG$. We may simplify notation by replacing G by its base change $G \times_S X$. Since all functors involved commute with $*$-inverse image (f_* satisfies base change because f is universally of finite cohomological dimension, and f_+ satisfies base change by adjunction, see e.g. [HLP, Lem. 5.1.8]), we may use fpqc descent to reduce to the case where X is a noetherian scheme. Since $*$-inverse image to residue fields is jointly conservative (by noetherianness), we may then further assume that X is the spectrum of a field k. Since G is linearly reductive (and embeddable), BG is perfect (see e.g. [Kha4, Thm. 1.42]) so we may assume that $\mathscr{F}$ is a perfect complex (again, f_* preserves colimits because f is universally of finite cohomological dimension). Note that f^* is t-exact (since f is flat), f_* is t-exact (since G is linearly reductive), and f_+ is t-exact on perfect complexes (since $f_+(-) \simeq f_*(-^\vee)^\vee$). Thus we may also reduce to the case where $\mathscr{F}$ is a (discrete) coherent sheaf. In other words, we are reduced to showing that for every finite-dimensional vector space V over k with G-action, the canonical morphism $V^G \subseteq V \twoheadrightarrow V_G$ (from G-invariants to G-coinvariants) is invertible, which is well-known (for example, this follows easily from the characterization of linear reductivity in [Alp1, Prop. 12.6(vi)]). ☐

Definition A.34. Suppose G is linearly reductive. Let X be a locally noetherian Artin stack over S (with trivial G-action). Given a quasi-coherent sheaf $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X \times_S BG) \simeq \mathbf{D}^G_{\mathrm{qc}}(X)$, the *fixed part* of $\mathscr{F}$ is defined as $\mathscr{F}^{\mathrm{fix}} := f^* f_*(\mathscr{F})$ and the *moving part* of $\mathscr{F}$ is the cofibre of the canonical morphism $\mathscr{F}^{\mathrm{fix}} \to \mathscr{F}$. By Lemma A.33 the latter admits a (natural) retraction $\mathscr{F} \to f^* f_+(\mathscr{F}) \simeq f^* f_*(\mathscr{F}) = \mathscr{F}^{\mathrm{fix}}$. Thus the exact triangle

$$\mathscr{F}^{\mathrm{fix}} \to \mathscr{F} \to \mathscr{F}^{\mathrm{mov}}$$

is split, i.e., there are canonical isomorphisms $\mathscr{F} \simeq \mathscr{F}^{\mathrm{fix}} \oplus \mathscr{F}^{\mathrm{mov}}$, natural in $\mathscr{F}$. (These definitions are compatible with Definition 5.2 in case G is diagonalizable.)

Corollary A.35. *Suppose G is linearly reductive. Let X be a locally noetherian Artin stack over S with G-action. If L_X is perfect of Tor-amplitude $\leqslant n$, then so is $L_{X^{hG}}$. In particular, if X is smooth (resp. quasi-smooth) over S, then so is X^{hG}.*

Proof. Since G is linearly reductive, the functor of derived G-invariants is t-exact on quasi-coherent complexes. □

Corollary A.36. *Suppose G is linearly reductive. Let X be a locally noetherian Artin stack over S with G-action. There is a canonical identification of exact triangles in $\mathbf{D}^G_{\mathrm{qc}}(X^{hG}) \simeq \mathbf{D}_{\mathrm{qc}}(X^{hG} \times_S BG)$*

$$\begin{tikzcd}
\varepsilon^* L_X \arrow[r] \arrow[d, equal] & L_{X^{hG}} \arrow[r] \arrow[d, equal] & L_{X^{hG}/X} \arrow[d, equal] \\
\varepsilon^* L_X \arrow[r] & (\varepsilon^* L_X)^{\mathrm{fix}} \arrow[r] & (\varepsilon^* L_X)^{\mathrm{mov}}[1].
\end{tikzcd}$$

Corollary A.37. *Suppose G is linearly reductive. Let X be a locally noetherian Artin stack over S with G-action. Then the morphism $\varepsilon : X^{hG} \to X$ is formally unramified if and only if, for every geometric point x of X^{hG}, the dual Lie algebra $\mathfrak{a}(x)^\vee$ of $\underline{\mathrm{Aut}}_X(x)$ has vanishing moving part (with respect to the G-action defined in Notation A.10*[11]*).*

Proof. Recall that formal unramifiedness is the condition that $\mathrm{H}^0(L_{X^{hG}/X}) = \Omega^1_{X^{hG}_{\mathrm{cl}}/X_{\mathrm{cl}}}$ vanishes. We may therefore replace X by its classical truncation. By Remark A.9, the canonical monomorphism $B\underline{\mathrm{Aut}}_X(x) \hookrightarrow X$ is G-equivariant. The relative cotangent complex of the latter vanishes, so that there is a canonical isomorphism

$$x^* L_X \simeq x^* L_{B\underline{\mathrm{Aut}}_X(x)} \simeq \mathfrak{a}(x)^\vee[-1]$$

in $\mathbf{D}^G_{\mathrm{qc}}(\mathrm{Spec}(k(x)))$ (where x also denotes the morphism $\mathrm{Spec}(k(x)) \to B\underline{\mathrm{Aut}}_X(x)$ by abuse of notation). By Corollary A.36 we get a canonical isomorphism

$$x^* L_{X^{hG}/X} \simeq x^*(L_X)^{\mathrm{mov}}[1] \simeq (\mathfrak{a}(x)^\vee)^{\mathrm{mov}},$$

whence the claim. □

A.6. **Reparametrized homotopy fixed points for torus actions.**

[11]By abuse of notation, we identify x with its image $\varepsilon(x)$ in X. Since the latter belongs to the fixed locus X^G (Remark A.19), the $\mathrm{St}^G_X(x)$-action defined in Notation A.10 is a G-action.

Remark A.38. Let $\rho : G' \twoheadrightarrow G$ be a surjective homomorphism between group schemes over S. Given an S-scheme A and an S-morphism $x : A \to X^{hG'}$, consider the commutative square

$$\begin{tikzcd} \underline{\mathrm{Aut}}_{[X/G']}(x) \arrow[r, two heads] \arrow[d] & G'_A \arrow[d, two heads, "\rho"] \\ \underline{\mathrm{Aut}}_{[X/G]}(x) \arrow[r] & G_A \end{tikzcd} \tag{A.39}$$

of group schemes over A.

(i) Note that the square is cartesian. Indeed, the induced map on kernels of the horizontal maps may be identified with the identity of $\underline{\mathrm{Aut}}_X(x)$.

(ii) Since the upper horizontal and right-hand vertical arrows are surjective, the same holds for the lower arrow. This shows that x factors through the fixed locus X^G. Allowing x to vary, we see that the canonical morphism $\varepsilon_X^{G'} : X^{hG'} \to X$ factors through

$$X^{hG'} \to X^G. \tag{A.40}$$

Definition A.41. Let T be a split torus over S. A *reparametrization* of T is an isogeny $\rho : T' \twoheadrightarrow T$ where T' is a split torus.

Remark A.42. Note that the category of reparametrizations of T (where the morphisms are T-morphisms) is filtered. In fact, there is a bijection between morphisms of reparametrizations $T'' \twoheadrightarrow T'$ (say of rank r) and diagonalizable $r \times r$-matrices $(d_1, \ldots, d_r)$, $d_i \in \mathbf{Z}$, so the category is equivalent to a poset.

Definition A.43. Let $G = T$ be a split torus over S. Let X be an Artin stack over S with T-action. The *reparametrized homotopy fixed point stack* $X^{rhT} \to X$ is defined as the filtered colimit

$$X^{rhT} := \varinjlim_{\rho} X^{hT'}$$

over all reparametrizations $\rho : T' \twoheadrightarrow T$.

Proposition A.44. *Let $G = T$ be a split torus of rank r over S acting on a quasi-separated Deligne–Mumford stack X locally of finite presentation over S.*

(i) *For any reparametrization $T' \twoheadrightarrow T$, the induced map*

$$X^{hT} \to X^{hT'} \tag{A.45}$$

is an open and closed immersion.

(ii) *There is a canonical decomposition*

$$X^{rhT} = \bigsqcup_{\rho} X^{rhT}_{\rho}$$

over reparametrizations $\rho : T' \twoheadrightarrow T$, where X^{rhT}_{ρ} is the open and closed substack

$$X^{rhT}_{\rho} = X^{hT'} \smallsetminus \bigcup_{\rho' : T'' \twoheadrightarrow T} X^{rhT''}$$

where ρ' varies over reparametrizations that factor ρ via some non-identical reparametrization $T' \twoheadrightarrow T''$.

(iii) *If X is quasi-compact, then X^{rhT} is quasi-compact. In particular, we have $X^{rhT} = X^{hT'}$ for some reparametrization $\rho : T' \twoheadrightarrow T$.*

Proof. (i): For any reparametrization $T' \twoheadrightarrow T$, the induced map

$$X^{hT} \to X^{hT'}$$

is formally étale, since the relative cotangent complex vanishes by Corollary A.36. Moreover, (A.45) is locally of finite presentation, hence étale, at least if X has affine and finitely presented diagonal, see [Rom3]. If X is quasi-separated, Deligne–Mumford and locally of finite presentation over S, then (A.45) is also a closed immersion (Proposition A.27). Thus in that case it is an open and closed immersion.

(ii): follows from (i).

(iii): Suppose X is quasi-compact. The closed substacks $X^{hT'}$ stabilize as $\rho : T' \twoheadrightarrow T$ varies among reparametrizations. Indeed, recall that each $X^{hT'}$ is a closed substack of X (Proposition A.27) and the colimit $\varinjlim_\rho X^{hT'}$ over reparametrizations $\rho : T' \twoheadrightarrow T$ is also closed in X by Theorem A.48 and Proposition A.15. In particular, it is quasi-compact because X is quasi-compact. □

Remark A.46. We have the following more explicit description of the open and closed substack X^{rhT}_ρ, for a given reparametrization $\rho : T' \twoheadrightarrow T$. Let x be a geometric point of X^{rhT}. Then we have $x \in X^{rhT}_\rho$ if and only if ρ is identified with the reparametrization $\alpha_A : \underline{\mathrm{Aut}}_{\mathcal{X}}(x)^0_{\mathrm{red}} \to T_A$, where the source is the reduced identity component of $\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$. Indeed, the group-theoretic section of $\underline{\mathrm{Aut}}_{[X/T']}(x) \to T'$ gives rise to a group homomorphism $T' \to \underline{\mathrm{Aut}}_{[X/T]}(x)$ from a reduced and connected algebraic group, which thus factors through $\underline{\mathrm{Aut}}_{[X/T]}(x)^0_{\mathrm{red}}$.

Remark A.47. For T a rank one torus over a field acting on a Deligne–Mumford stack, the reparametrized homotopy fixed stack X^{rhT} is used as the definition of T-fixed locus in [AHR, Def. 5.25] (see also [Kre, Prop. 5.3.4]).

A.7. **Fixed vs. reparametrized homotopy fixed.** In this subsection our goal is to compare the fixed locus X^G with the homotopy fixed point stack X^{hG}. The two constructions are typically different, as $\varepsilon : X^{hG} \to X$ may not be a monomorphism (Example A.29). But even if we just consider the set-theoretic image of X^{hG} in $|X|$, it will typically not coincide with $|X^G|$. In the case where $G = T$ is a torus, we can somewhat bridge the gap by replacing X^{hT} by its reparametrized version X^{rhT}. In this case, we will prove:

Theorem A.48. *Suppose $G = T$ is a split torus acting on a 1-Artin stack X over S with affine stabilizers. Then the morphisms (A.40) induce a canonical surjection*

$$X^{rhT} \twoheadrightarrow X^T$$

over X.

Proof. As $\rho : T' \twoheadrightarrow T$ varies, the canonical morphisms $X^{hT'} \to X^T$ (A.40) are compatible by construction. Thus there is a canonical morphism

$$X^{rhT} \to X^T.$$

For surjectivity, let x be a geometric point of X^T. Then we have the canonical surjection $\underline{\mathrm{Aut}}_{[X/T]}(x) \twoheadrightarrow T_{k(x)}$ (A.5). By [Bor2, Cor. 1 of Prop. 11.14], there exists a (split) subtorus of $\underline{\mathrm{Aut}}_{[X/T]}(x)$ on which this restricts to an isogeny. By [SGA3, Exp. VIII, Cor. 1.6], we may lift this to an isogeny $\rho : T' \to T$ over S.

Using the cartesian square (A.39) (taking now $A = \mathrm{Spec}(k(x))$), we see that in order to lift x to $X^{hT'}$ it is enough to show that $\underline{\mathrm{Aut}}_{[X/T]}(x) \to T_{k(x)}$ becomes surjective after base change along ρ. But ρ factors through $\underline{\mathrm{Aut}}_{[X/T]}(x) \to T_{k(x)}$ by construction, so this is clear. □

Corollary A.49. *Suppose $G = T$ is a split torus acting on a quasi-compact 1-Artin stack X over S with affine stabilizers. Then there exists a reparametrization $\rho : T' \twoheadrightarrow T$ such that the morphism (A.40) induces a surjection*

$$X^{hT'}_{\mathrm{red}} \twoheadrightarrow X^T_{\mathrm{red}}$$

of reduced 1-Artin stacks (where the target is the reduced T-fixed locus, see Definition A.16). In particular, the set-theoretic image of $\varepsilon : X^{hT'} \to X$ coincides with $|X^T| \subseteq |X|$.

Remark A.50. Let X be a derived 1-Artin stack over S with T-action. Let $f : Z \to X$ be a T-equivariant finite unramified morphism satisfying the following properties:

(i) There exists some reparametrization $T' \twoheadrightarrow T$ such that the induced T'-action on Z is trivial.

(ii) The conormal sheaf $\mathscr{N}_{Z/X} \in \mathbf{D}^{T'}_{\mathrm{qc}}(Z) \simeq \mathbf{D}_{\mathrm{qc}}(Z \times BT')$ has no T'-fixed part.

(For example, this applies to $\varepsilon : X^{hT'} \to X$ in the situation of Corollary A.49.) Then the canonical morphism $f^* L_X \to L_Z$ in $\mathbf{D}^{T'}_{\mathrm{qc}}(Z) \simeq \mathbf{D}_{\mathrm{qc}}(Z \times BT')$ induces an isomorphism $L_Z \simeq (L_Z)^{\mathrm{fix}} \simeq (f^* L_X)^{\mathrm{fix}}$ (where the first isomorphism is because L_Z has no moving part by (i), and the second is because the cofibre $L_{Z/X}$ has no fixed part by (ii)). In particular, if X is smooth (resp. quasi-smooth) then so is Z.

A.8. **Edidin–Rydh fixed vs. reparametrized homotopy fixed.**

Definition A.51. Let $G = T$ be a split torus of rank r over S acting on a 1-Artin stack X over S with finite stabilizers. We define X^{sT} as the stack over X whose A-valued points, for an X-scheme $x : A \to X$, are closed subgroups of $\underline{\mathrm{Aut}}_{[X/T]}(x)$ which are affine and smooth over A with connected fibres of dimension r.

Remark A.52. The definition of X^{sT} is a variant of the construction in [ER, Prop. C.5] of, for a 1-Artin stack $\mathcal{X}$, a stack $\mathcal{X}^{\max} \to \mathcal{X}$ that can be thought of as the locus of points with maximal-dimensional stabilizer. Whenever $X^{sT} \neq \varnothing$, then we have

$$X^{sT} = \mathcal{X}^{\max} \underset{\mathcal{X}}{\times} X$$

where $\mathcal{X} = [X/T]$.

In this subsection we will prove:

Theorem A.53. *Let $G = T$ be a split torus over S. Let X be a tame Deligne–Mumford stack which is quasi-separated and locally of finite presentation over S with T-action. Then there is a canonical isomorphism*

$$X^{rhT} \simeq X^{sT}$$

over X.

Since X^{rhT} is a closed substack of X (see Proposition A.44), Theorem A.53 shows that X^{sT} is also a closed substack of X and in particular is Deligne–Mumford. When X is noetherian and $[X/T]$ admits a good moduli space in the sense of [Alp1], this follows from [ER, Prop. C.5]. We begin with the following generalization to our situation:

Theorem A.54. *Let X be a 1-Artin stack over S with finite diagonal and tame stabilizers and T-action. Then the morphism $X^{sT} \to X$ is a closed immersion. In particular, X^{sT} is 1-Artin with finite diagonal.*

Proof. Note that X^{sT} is stable under base change by étale representable T-equivariant morphisms $p : Y \to X$, i.e., the induced morphism

$$Y^{sT} \to X^{sT} \underset{X}{\times} Y$$

is an isomorphism. Equivalently, let us show that for every Y-scheme $y : A \to Y$ the map of sets

$$Y^{sT}(A) \to X^{sT} \underset{X}{\times} Y(A) \tag{A.55}$$

is bijective. Since p is étale and representable, the morphism $I_{\mathcal{Y}} \to I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{Y}$ (where $\mathcal{Y} = [Y/T]$) is an open immersion, so in particular $\underline{\mathrm{Aut}}_{\mathcal{Y}}(y) \to \underline{\mathrm{Aut}}_{\mathcal{X}}(p(y)) \times_{\mathcal{X}} \mathcal{Y}$ is an open immersion. This shows that (A.55) is injective, so it remains to show that surjectivity. Let H be a closed subgroup of $\underline{\mathrm{Aut}}_{\mathcal{X}}(x) \times_{\mathcal{X}} \mathcal{Y}$ which is smooth and affine over A with connected r-dimensional fibres. We claim that the open immersion of group schemes over A

$$H \underset{\underline{\mathrm{Aut}}_{\mathcal{X}}(x) \times_{\mathcal{X}} \mathcal{Y}}{\times} \underline{\mathrm{Aut}}_{\mathcal{Y}}(y) \to H$$

is invertible (and hence H lifts to a closed subgroup of $\underline{\mathrm{Aut}}_{\mathcal{Y}}(y)$ as desired). This can be checked over points of A, so we may assume that A is a field. Now by [SP, Tag 047T], this morphism is also a closed immersion, hence an inclusion of connected components. But H is connected, so we are done.

Since X has finite tame (hence linearly reductive) stabilizers, it follows from the short exact sequence (A.5) that $\mathcal{X} = [X/T]$ has linearly stabilizers.

Therefore we may apply the local structure theorem of [AHHR, Thm. 1.11] to find for every point x of X an affine étale neighbourhood $\mathcal{Y} \to \mathcal{X}$ of x where $\mathcal{Y}$ is a quotient $[V/\mathrm{GL}_n]$ with V affine. By base change this gives a T-equivariant affine étale neighbourhood $Y \to X$. Since the question is étale-local on X and because X^{sT} is stable under the base change $Y \to X$, we may therefore replace X by Y, so that the quotient $\mathcal{X} = [X/T]$ is now a global quotient of an affine scheme by GL_n. In this case, either $X^{sT} = \varnothing$ or the claim follows by combining Remark A.52 and [ER, Prop. C.5]. $\square$

Proof of Theorem A.53. Recall that X^{rhT} and X^{sT} are closed substacks of X (see Proposition A.44 and Theorem A.54). Let us first show that there is an inclusion $X^{sT} \subseteq X^{rhT}$. Let $x : A \to X$ be an A-valued point, where A is an S-scheme, and let $G \subseteq \underline{\mathrm{Aut}}_{\mathcal{X}}(x)$ be a closed subgroup which is smooth affine over A with r-dimensional connected fibres. The composite homomorphism

$$G \hookrightarrow \underline{\mathrm{Aut}}_{\mathcal{X}}(x) \xrightarrow{\alpha_A} T_A$$

is surjective over geometric points of A, since G has r-dimensional fibres and the kernel is contained in $\underline{\mathrm{Aut}}_X(x)$ which is quasi-finite. It follows that the geometric fibres of G are tori of rank r (for every geometric point a of A, by [Bor2, Cor. 1 of Prop. 11.14] the homomorphism $G_a \to T_a$ restricts to a finite surjection on a maximal subtorus H, but then $H = G_a$ because they are smooth and connected of the same dimension).

Since G is smooth and affine with geometric fibres of constant reductive rank, it follows from [SGA3, Exp. XII, Thm. 1.7(b)] that it admits a maximal subtorus $H \subseteq G$ in the sense of [SGA3, Exp. XII, Def. 1.3]. But then $H = G$ since we have $H_a = G_a$ for every geometric point a of A and G and H are flat over A. In particular G is a torus, which we may assume is split, since this holds étale-locally on A by [Con, B.3.4] (and X^{sT} and X^{rhT} are subsheaves of the étale sheaf X). Now $T' := G \to T_A$ is a reparametrization. Using the cartesian square (A.39), we get a group-theoretic section of the homomorphism $\underline{\mathrm{Aut}}_{[X/T']}(x) \to T'_A$, whence the desired lift of $x : A \to X^{sT}$ to X^{rhT}.

It remains to show that the inclusion $X^{sT} \subseteq X^{rhT}$ is an effective epimorphism. Take a scheme A and an A-valued point $x : A \to X^{rhT}$ which belongs to the open and closed substack X^{rhT}_ρ for some reparametrization $\rho : T' \twoheadrightarrow T$ (Proposition A.44). This point corresponds to a group-theoretic section s of $\alpha_A : \underline{\mathrm{Aut}}_{[X/T']}(x) \to T'_A$. This is a closed immersion since α_A is separated (as a morphism between quasi-affine schemes). We will show that the composite homomorphism

$$s' : T'_A \xrightarrow{s} \underline{\mathrm{Aut}}_{[X/T']}(x) \to \underline{\mathrm{Aut}}_{[X/T]}(x)$$

is a closed immersion, and hence defines an A-point of X^{sT}. Since the second morphism is finite, the composite s' is proper, so it is enough to show that it is a monomorphism. This can be checked over geometric points a of A. The base change s'_a yields a reparametrization $T'_a \twoheadrightarrow T_a$ which by Remark A.46 is isomorphic to the reparametrization $\underline{\mathrm{Aut}}_{[X/T]}(x_a)^0_{\mathrm{red}} \twoheadrightarrow T_a$.

Then $T'_a \to \underline{\mathrm{Aut}}_{[X/T]}(x_a)^0_{\mathrm{red}}$, as a morphism between abstractly isomorphic reparametrizations of T_a, must itself be an isomorphism (by Remark A.42). In particular, s'_a is a closed immersion. □

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