

# PERFECTION IN MOTIVIC HOMOTOPY THEORY

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ABSTRACT. We prove a topological invariance statement for the Morel–Voevodsky motivic homotopy category, up to inverting exponential characteristics of residue fields. This implies in particular that  $\mathbf{SH}[\frac{1}{p}]$  of characteristic  $p > 0$  schemes is invariant under passing to perfectness. Among other applications we prove Grothendieck–Verdier duality in this context.

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## 1. INTRODUCTION

Let  $S$  be a scheme and denote by  $S_{\text{ét}}$  its small étale topos. The starting point for this note is Grothendieck’s “équivalence remarquable de catégories” [EGA IV<sub>4</sub>, Théorème 18.1.2], which asserts that for any nil-immersion  $f : S_0 \hookrightarrow S$ , there is an induced equivalence

$$f^* : S_{\text{ét}} \rightarrow (S_0)_{\text{ét}}.$$

In fact, Grothendieck further generalized this to a *topological invariance* statement for the small étale topos: for any universal homeomorphism of schemes  $f : T \rightarrow S$ , the functor  $f^* : S_{\text{ét}} \rightarrow T_{\text{ét}}$  is an equivalence (see [SGA1, Exposé IX, Théorème 4.10], [SGA4, Exposé VIII, Théorème 1.1]).

The large étale topos fails to satisfy nil-invariance. An observation of Morel and Voevodsky [MV99] was that this failure can be repaired by working in the setting of  $\mathbf{A}^1$ -invariant sheaves. Indeed, it is a consequence of the Morel–Voevodsky localization theorem that the stable motivic homotopy category  $\mathbf{SH}$  satisfies nil-invariance (see e.g. [CD12, Proposition 2.3.6(1)]). However, the topological invariance property still fails, at least in positive characteristic (see Remark 2.1.12). Our goal in this paper is to show that topological invariance is in fact true for  $\mathbf{SH}$ , after inverting the exponential characteristic of the base field (Theorem 2.1.1). This also recovers the analogous statement in other related contexts, such as various variants of mixed motives [Ayo14, CD12, CD15, CD16] (see Remark 2.1.2).

A particularly useful consequence of topological invariance is that, for any scheme  $S$  of characteristic  $p$ ,  $\mathbf{SH}(S)[\frac{1}{p}]$  is invariant under passing to the perfection  $S_{\text{perf}}$  (Corollary 2.1.7). This allows us to remove perfectness hypotheses on the base field in many results, see §3.2. It

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also yields a Grothendieck–Verdier duality statement, following ideas of Cisinski–Déglise [CD15] (Theorem 3.1.1).

### 1.1. Conventions.

1.1.1. All schemes will implicitly be assumed to be quasi-compact quasi-separated.

Recall that a morphism of schemes  $f : X \rightarrow Y$  is a *universal homeomorphism* if it induces a homeomorphism on underlying topological spaces after any base change, or equivalently, if it is integral, universally injective, and surjective [EGA IV<sub>4</sub>, Corollary 18.12.11].

1.1.2. If  $S$  is a scheme of characteristic  $p > 0$ , i.e., an  $\mathbf{F}_p$ -scheme, we write  $F_S : S \rightarrow S$  for the Frobenius endomorphism [SGA5, Exposé XIV=XV]. Recall that  $S$  is *perfect* if the Frobenius  $F_S : S \rightarrow S$  is an isomorphism. Any  $\mathbf{F}_p$ -scheme  $S$  admits a *perfection*  $S_{\text{perf}}$ , defined as the limit of the tower

$$\dots \xrightarrow{F_S} S \xrightarrow{F_S} S,$$

see [BS17, Section 3].

1.1.3. Given a scheme  $S$ , we denote by  $\mathbf{SH}(S)$  the stable  $\infty$ -category of motivic spectra over  $S$ . We will use the language of six operations, see [Hoy14, Appendix C] or [Kha16] for the non-noetherian setting. Any motivic spectrum  $E \in \mathbf{SH}(S)$  represents a cohomology theory on  $S$ -schemes, given by the formula

$$E(X, \xi) = \text{Maps}_{\mathbf{SH}(S)}(\mathbf{1}_S, f_* \Sigma^\xi f^*(E))$$

for any morphism  $f : X \rightarrow S$  and any K-theory class  $\xi \in K(X)$ . Similarly, there is a Borel–Moore homology theory

$$E^{\text{BM}}(X/S, \xi) = \text{Maps}_{\mathbf{SH}(S)}(\mathbf{1}_S, f_* \Sigma^{-\xi} f^!(E)).$$

We refer to [DJK18] for details.

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## 2. TOPOLOGICAL INVARIANCE

2.1. **Main result and corollaries.** Let  $\mathcal{P}$  be a set of prime numbers. For a scheme  $S$ , we denote by  $\mathbf{SH}(S)[\mathcal{P}^{-1}]$  the localization of  $\mathbf{SH}(S)$  at the morphisms  $p : E \rightarrow E$ , for  $E \in \mathbf{SH}(S)$  and  $p \in \mathcal{P}$ . When  $\mathcal{P}$  contains a single prime  $p$ , we write simply  $\mathbf{SH}(S)[\frac{1}{p}]$ .

**Theorem 2.1.1.** *Let  $S$  be a scheme and  $\mathcal{P}$  a set of prime numbers. Suppose that every prime  $q \notin \mathcal{P}$  is invertible in  $\mathcal{O}_S$ . Then for any universal homeomorphism  $f : T \rightarrow S$ , the functor*

$$f^* : \mathbf{SH}(S)[\mathcal{P}^{-1}] \rightarrow \mathbf{SH}(T)[\mathcal{P}^{-1}]$$

*is an equivalence.*

*Remark 2.1.2.* The proof of Theorem 2.1.1 will in fact apply to any motivic  $\infty$ -category of coefficients  $\mathbf{D}$  as in [Kha16, Chap. 2, Definition 3.5.2]. See Remark 2.2.9 for details.

We now record some immediate consequences. Taking  $\mathcal{P}$  to be the set of all primes, we get:

**Corollary 2.1.3.** *For any universal homeomorphism  $f : T \rightarrow S$ , the functor*

$$f^* : \mathbf{SH}(S)_{\mathbf{Q}} \rightarrow \mathbf{SH}(T)_{\mathbf{Q}}$$

*is an equivalence.*

*Remark 2.1.4.* Recall that for every scheme  $S$ , the category  $\mathbf{SH}(S)_{\mathbf{Q}}$  admits natural splittings

$$\mathbf{SH}(S)_{\mathbf{Q}} \simeq \mathbf{SH}(S)_{\mathbf{Q},+} \times \mathbf{SH}(S)_{\mathbf{Q},-},$$

see [CD12, Sect. 16.2]. The analogue of Corollary 2.1.3 is known for the plus part  $\mathbf{SH}(S)_{\mathbf{Q},+}$ , via the identification with Beilinson motives (see Theorems 14.3.3 and 16.2.13 in *op. cit.*). For the minus part, the statement appears to be new.

Taking  $\mathcal{P}$  to be a single prime, we get:

**Corollary 2.1.5.** *Let  $S$  be a scheme of exponential characteristic  $p$ . Then for any universal homeomorphism  $f : T \rightarrow S$ , the functor*

$$f^* : \mathbf{SH}(S)_{[\frac{1}{p}]} \rightarrow \mathbf{SH}(T)_{[\frac{1}{p}]}$$

*is an equivalence.*

**Corollary 2.1.6.** *For every scheme  $S$  of characteristic  $p > 0$ , the absolute Frobenius induces an equivalence*

$$F_S^* : \mathbf{SH}(S)_{[\frac{1}{p}]} \rightarrow \mathbf{SH}(S)_{[\frac{1}{p}]}$$

**Corollary 2.1.7.** *For every scheme  $S$  of characteristic  $p > 0$ , the canonical morphism  $S_{\text{perf}} \rightarrow S$  induces an equivalence*

$$\mathbf{SH}(S)_{[\frac{1}{p}]} \rightarrow \mathbf{SH}(S_{\text{perf}})_{[\frac{1}{p}]}$$

*Proof.* Follows from Corollary 2.1.6 in view of continuity of  $\mathbf{SH}$  [Hoy14, Proposition C.12(4)].  $\square$

2.1.8. At the level of cohomology and Borel–Moore homology, we have the following reformulation (we consider the case  $\mathcal{P} = \{p\}$  for simplicity):

**Corollary 2.1.9.** *Let  $S$  be a scheme of exponential characteristic  $p$ . Let  $E \in \mathbf{SH}(S)$  be a motivic spectrum over  $S$ . Then we have:*

(i) *For any universal homeomorphism  $f : X \rightarrow Y$  of  $S$ -schemes, the induced maps*

$$\begin{aligned} f^* &: E(Y, \xi)_{[\frac{1}{p}]} \rightarrow E(X, f^*(\xi))_{[\frac{1}{p}]} \\ f_* &: E^{\text{BM}}(X, f^*(\xi))_{[\frac{1}{p}]} \rightarrow E^{\text{BM}}(Y, \xi)_{[\frac{1}{p}]} \end{aligned}$$

*are equivalences for every  $\xi \in K(Y)$ .*

(ii) *The canonical morphism  $f : S_{\text{perf}} \rightarrow S$  induces equivalences*

$$\begin{aligned} f^* &: E(S, \xi)_{[\frac{1}{p}]} \rightarrow E(S_{\text{perf}}, f^*(\xi))_{[\frac{1}{p}]} \\ f_* &: E^{\text{BM}}(S_{\text{perf}}, f^*(\xi))_{[\frac{1}{p}]} \rightarrow E^{\text{BM}}(S, \xi)_{[\frac{1}{p}]} \end{aligned}$$

*for every  $\xi \in K(S)$ .*

*Proof.* Note that we have canonical identifications

$$E(T, \psi)_{[\frac{1}{p}]} \simeq \text{Maps}_{\mathbf{SH}(S)}(\mathbf{1}_S, (\pi_T)_* \Sigma^\psi \pi_T^*(E))_{[\frac{1}{p}]} \simeq \text{Maps}_{\mathbf{SH}(S)_{[\frac{1}{p}]}}(\mathbf{1}_S, (\pi_T)_* \Sigma^\psi \pi_T^*(E))$$

for every  $\pi_T : T \rightarrow S$  and  $\psi \in K(T)$ . Therefore the map on cohomology spaces is induced from the natural transformation

$$(\pi_Y)_* \Sigma^\xi \pi_Y^* \rightarrow (\pi_Y)_* \Sigma^\xi f_* f^* \pi_Y^* \simeq (\pi_Y)_* f_* \Sigma^{f^*(\xi)} f^* \pi_Y^* \simeq (\pi_X)_* \Sigma^{f^*(\xi)} \pi_X^*$$

For  $f$  as in (i) (resp. (ii)), the unit map  $\text{id} \rightarrow f_* f^*$  is invertible after inverting  $p$  by Theorem 2.1.1 (resp. by Corollary 2.1.7), whence the claim. The proof for Borel–Moore homology is similar, using the fact that the co-unit map  $f_* f^! \rightarrow \text{id}$  is also invertible in both cases (after inverting  $p$ ).  $\square$

*Remark 2.1.10.* Corollary 2.1.9 also holds for the compactly supported variants (cohomology with compact support and relative homology), with the same proofs.

*Example 2.1.11.* Let  $\text{KGL} \in \mathbf{SH}(\mathbf{F}_p)$  denote the homotopy invariant K-theory spectrum over  $\text{Spec}(\mathbf{F}_p)$ . For every (possibly singular and non-noetherian)  $\mathbf{F}_p$ -scheme  $S$ , we have functorial equivalences

$$\text{KGL}(S, 0) \left[ \frac{1}{p} \right] \simeq \text{K}(S) \left[ \frac{1}{p} \right]$$

by [Cis13, Theorem 2.20] and [TT90, Exercise 9.11(h)]. Under these identifications, Corollary 2.1.9 recovers in particular the recent observation of Kelly and Morrow [KM18, Lemma 4.1] that the canonical map

$$\text{K}(S) \left[ \frac{1}{p} \right] \rightarrow \text{K}(S_{\text{perf}}) \left[ \frac{1}{p} \right]$$

is an equivalence.

*Remark 2.1.12.* Corollary 2.1.5 is *false* before inverting  $p$ . Indeed, let  $k$  be a field such that the Frobenius induces an equivalence  $F^* : \mathbf{SH}(k) \rightarrow \mathbf{SH}(k)$ . Then as in Corollary 2.1.9, the induced map on algebraic K-theory spectra

$$F^* : \text{K}(k) \rightarrow \text{K}(k)$$

is also an equivalence. But under the canonical identification  $\text{K}_1(k) \simeq k^\times$ , the induced endomorphism of  $k^\times$  is  $x \mapsto x^p$ , which is an isomorphism if and only if  $k$  is perfect.

*Remark 2.1.13.* Under the assumptions of Theorem 2.1.1, suppose further that  $f$  is of finite type (hence a finite radicial surjection). Then the equivalence  $f^* : \mathbf{SH}(S)[\mathcal{P}^{-1}] \rightarrow \mathbf{SH}(T)[\mathcal{P}^{-1}]$  is quasi-inverse to  $f_* \simeq f_!$ . Hence the left adjoint and right adjoints of the latter functor are equivalent. That is, we have an equivalence of functors

$$f^* \simeq f^! : \mathbf{SH}(S)[\mathcal{P}^{-1}] \rightarrow \mathbf{SH}(T)[\mathcal{P}^{-1}].$$

**2.2. Proof of Theorem 2.1.1.** In order to prove Theorem 2.1.1, we have to show that the unit and co-unit maps

$$\text{id} \rightarrow f_* f^*, \quad f^* f_* \rightarrow \text{id}$$

are both invertible. For the latter, this turns out to be the case before inverting  $p$ :

**Proposition 2.2.1.** *For any universal homeomorphism  $f : T \rightarrow S$ , the co-unit transformation*

$$f^* f_* \rightarrow \text{id}_{\mathbf{SH}(T)}$$

*is invertible. In other words, the functor  $f_* : \mathbf{SH}(T) \rightarrow \mathbf{SH}(S)$  is fully faithful.*

*Proof.* We argue as in the proof of [CD12, Proposition 2.1.9]. By the proper base change formula, this co-unit is identified with the natural transformation  $(\pi_2)_*(\pi_1)^* \rightarrow \text{id}$ , where  $\pi_1$  and  $\pi_2$  are the respective projections  $T \times_S T \rightarrow T$ . Since  $f$  is a universal homeomorphism, its diagonal  $\Delta : T \rightarrow T \times_S T$  is a nilpotent closed immersion. Then by the localization theorem (cf. [CD12, Proposition 2.3.6(1)]),  $\Delta^*$  is an equivalence. Since  $\pi_1$  and  $\pi_2$  are retractions of  $\Delta$  it follows that we have canonical identifications  $\Delta^* \simeq (\pi_\varepsilon)_*$  and  $\Delta_* \simeq (\pi_\varepsilon)^*$  for each  $\varepsilon \in \{1, 2\}$ . In particular, the natural transformation  $(\pi_2)_*(\pi_1)^* \rightarrow \text{id}$  is identified with the co-unit  $\Delta^* \Delta_* \rightarrow \text{id}$ , which is invertible.  $\square$

For the unit map, we will require a more involved argument. We begin by introducing some notation.

*Notation 2.2.2.* Given a unit  $a \in \Gamma(S, \mathcal{O}_S)^\times$ , we write  $\langle a \rangle$  for the induced point of  $\Omega\mathbf{K}(S)$ . For an integer  $n \geq 0$ , we write  $n_\epsilon$  for the formal sum

$$n_\epsilon = 1 + \langle -1 \rangle + 1 + \cdots$$

which consists of  $n$  terms.

*Remark 2.2.3.* Recall that there is a canonical map of spaces  $\mathbf{K}(S) \rightarrow \text{Aut}(\mathbf{SH}(S))$  which sends the class of a perfect complex  $\xi$  to the auto-equivalence  $\Sigma^\xi = \text{Th}_S(\xi) \otimes -$  of  $\mathbf{SH}(S)$ . It sends 0 to the identity  $\text{id}_{\mathbf{SH}(S)}$  and thus induces a canonical map

$$(2.2.4) \quad \Omega\mathbf{K}(S) \rightarrow \text{Aut}(\text{id}_{\mathbf{SH}(S)}),$$

via which we may also regard  $n_\epsilon$  (Notation 2.2.2) as an automorphism of  $\text{id}_{\mathbf{SH}(S)}$ .

We are grateful to Marc Hoyois for suggesting the following re-interpretation of [EHK<sup>+</sup>18, Proposition B.1.4].

**Proposition 2.2.5.** *Let  $S$  be a scheme,  $P \in \Gamma(S, \mathcal{O}_S)[x]$  a monic polynomial of degree  $d$ , and  $T \subset \mathbf{A}_S^1$  the closed subscheme cut out by  $P$ . If  $f : T \rightarrow S$  denotes the canonical morphism, then there exists a canonical natural transformation*

$$\text{tr}_f : f_* f^* \rightarrow \text{id}_{\mathbf{SH}(S)}$$

such that the composites

$$\text{id} \xrightarrow{\text{unit}} f_* f^* \xrightarrow{\text{tr}_f} \text{id}, \quad f_* f^* \xrightarrow{\text{tr}_f} \text{id} \xrightarrow{\text{unit}} f_* f^*$$

are homotopic to  $d_\epsilon$  and  $f_* * d_\epsilon * f^*$ , respectively.

*Proof.* Note that  $f : T \rightarrow S$  is finite and syntomic. The conormal sheaf  $\mathcal{N}_{T/\mathbf{A}_S^1} \simeq (P)/(P^2)$  is free of rank 1, and the generator  $P$  induces a canonical trivialization  $\tau : \mathbf{L}_f \simeq 0$  in  $\mathbf{K}(T)$ . Therefore, the trace transformation  $\text{tr}_f$  of [DJK18, 4.3.1] induces a canonical natural transformation

$$f_* f^* \simeq f_* \Sigma^{\mathbf{L}_f} f^* \xrightarrow{\text{tr}_f} \text{id}$$

which we denote again by  $\text{tr}_f$ . The claim can be reformulated in cohomological terms as the assertion that, for every  $E \in \mathbf{SH}(S)$ , the composites

$$(2.2.6) \quad E(S, 0) \xrightarrow{f^*} E(T, 0) \simeq E(T, \langle \mathbf{L}_f \rangle) \xrightarrow{f_!} E(S, 0),$$

$$(2.2.7) \quad E(T, 0) \simeq E(T, \langle \mathbf{L}_f \rangle) \xrightarrow{f_!} E(S, 0) \xrightarrow{f^*} E(T, 0)$$

are homotopic to multiplication by  $d_\epsilon$ .

Regarding the assignment  $X \mapsto E(X, 0)$  as a presheaf with framed transfers, the first composite is induced by the framed correspondence

$$\begin{array}{ccc} & T & \\ f, \tau \swarrow & & \searrow f \\ S & & S. \end{array}$$

Therefore the claim follows from [EHK<sup>+</sup>18, Proposition B.1.4]. The second composite is identified, by the transverse base change property of the trace transformation  $\text{tr}_f$  [DJK18, Proposition 2.5.6], with

$$E(T, 0) \xrightarrow{\pi_2^*} E(T \times_S T, 0) \simeq E(T \times_S T, \langle \mathbf{L}_f \rangle) \xrightarrow{(\pi_1)_!} E(T, 0),$$

where  $\pi_1$  and  $\pi_2$  are the first and second projections of  $T \times_S T$ , respectively. As above, this is induced by the framed correspondence

$$\begin{array}{ccc} & T \times_S T & \\ \pi_1, \pi_2^*(\tau) \swarrow & & \searrow \pi_2 \\ T & & T \end{array}$$

so the claim follows by another application of [EHK<sup>+</sup>18, Proposition B.1.4].  $\square$

**Lemma 2.2.8.** *Let  $S$  be the spectrum of a field  $k$  of exponential characteristic  $p$ . Then for any power  $q$  of  $p$ , the canonical map*

$$\mathrm{End}_{\mathbf{SH}(S)}(\mathbf{1}_S) \rightarrow \mathrm{End}_{\mathbf{SH}(S)}(\mathbf{1}_S)_{[p]} \simeq \mathrm{End}_{\mathbf{SH}(S)_{[p]}}(\mathbf{1}_S)$$

sends  $q_\epsilon \in \mathrm{End}_{\mathbf{SH}(S)}(\mathbf{1}_S)$  to a unit.

*Proof.* We only need to consider the case  $p > 1$ . Using Morel's identification  $\mathrm{End}(\mathbf{1}_S) \simeq \mathrm{GW}(k)$  [Mor04], which has been extended in [BH18, Lemma 10.12], it will suffice to show that the induced element  $q_\epsilon \in \mathrm{GW}(k)_{[p]}$  is invertible. In view of the cartesian square

$$\begin{array}{ccc} \mathrm{GW}(k) & \xrightarrow{\dim} & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathrm{W}(k) & \longrightarrow & \mathbf{Z}/2 \end{array}$$

as in [Mor12, (3.1)] (cf. [Bac18, Lemma 17], [KK82, Lemma 1.16]), it will in fact suffice to only check invertibility in  $\mathbf{Z}_{[p]}$  and in  $\mathrm{W}(k)_{[p]}$ . The former is obvious. For the latter, we first assume that  $p$  is odd. In this case, we note that  $d_\epsilon = \frac{d-1}{2}h + 1$  and thus  $q_\epsilon$  is invertible in  $\mathrm{W}(k)$  (without inverting  $p$ ). When  $p = 2$ ,  $-1$  is trivially a sum of squares in  $k$ , so the Witt ring is 2-torsion [MH73, Theorem III.3.6] and the claim follows.  $\square$

We are now ready to complete the proof of Theorem 2.1.1:

*Proof of Theorem 2.1.1.* After Proposition 2.2.1 it remains to show that unit map  $\mathrm{id}_{\mathbf{SH}(S)} \rightarrow f_* f^*$  becomes invertible after inverting the primes in  $\mathcal{P}$ . By continuity [Hoy14, Proposition C.12(4)] and proper base change, we may use a noetherian approximation argument [TT90, Theorem C.9] to assume that  $S$  is noetherian and of finite dimension. Then using [BH18, Proposition A.3] (and proper base change again), we may assume that  $S$  is a henselian local scheme (which is still noetherian and finite-dimensional); we denote its closed point by  $i : \{s\} \rightarrow S$  and the complement by  $j : U \rightarrow S$ . By the localization theorem, the pair of functors  $(i^*, j^*)$  is jointly conservative (see [CD12, Section 2.3]). Since  $U$  has dimension strictly lower than that of  $S$ , we can argue by induction on the dimension of  $S$  to reduce to the case where  $S = \{s\}$ , i.e., where  $S$  is the spectrum of a field  $k$ . Since  $f$  is radicial, it is then induced by a purely inseparable field extension  $k \subset K$ . In characteristic zero ( $p = 1$ ), we are already done. Otherwise, by using continuity again, we may assume that the extension  $k \subset K$  is finite, i.e., that  $K = k(\alpha)$  with  $\alpha^q \in k$  for  $q$  some power of the prime  $p$ . Now it follows from Proposition 2.2.5 and Lemma 2.2.8 that the unit map  $\mathrm{id}_{\mathbf{SH}(S)} \rightarrow f_* f^*$  is invertible after inverting  $p$ . But the assumption implies that  $p \in \mathcal{P}$ , so the conclusion follows.  $\square$

**Remark 2.2.9.** We now explain how the above proof can be generalized to an arbitrary motivic  $\infty$ -category of coefficients  $\mathbf{D}$  as in Remark 2.1.2. First, we recall that the theory of fundamental classes developed in [DJK18] applies in this more general setting (see 4.3.4 in *op. cit.*). Therefore, for any object  $E \in \mathbf{D}(S)$ , the cohomology theory  $X \mapsto E(X, 0)$  defines a presheaf with framed transfers on the category of  $S$ -schemes. The proof of Proposition 2.2.5 then applies *mutatis mutandis*. To show that Lemma 2.2.8 holds for  $\mathbf{D}$ , i.e., that  $q_\epsilon$  is a unit for every power  $q$  of

$p$ , we argue as follows. Since  $\mathbf{D}$  satisfies Nisnevich descent [CD12, Prop. 2.3.8],  $\mathbf{A}^1$ -invariance, and Thom stability, it follows from the universal property of  $\mathbf{SH}$  [Rob15, Corollary 1.2] that there is a canonical monoidal realization functor  $\mathbf{SH}(S) \rightarrow \mathbf{D}(S)$ . Moreover, the canonical map  $\Omega K(S) \rightarrow \text{Aut}(\text{id}_{\mathbf{D}(S)})$  factors through the induced map  $\text{Aut}(\text{id}_{\mathbf{SH}(S)}) \rightarrow \text{Aut}(\text{id}_{\mathbf{D}(S)})$ , so the claim follows from the universal case of  $\mathbf{SH}$ . The rest of the proof of Theorem 2.1.1 only relies on the formalism of six operations, which is available for  $\mathbf{D}$  [Kha16, Cor. 4.2.3].

### 3. APPLICATIONS

**3.1. Duality.** Let  $S$  be a scheme that is locally of finite type over a field  $k$  of exponential characteristic  $p$ . The structural morphism  $\pi : S \rightarrow \text{Spec}(k)$  determines a *duality functor* defined by

$$D_S(E) = \underline{\text{Hom}}(E, \pi^!(\mathbf{1}_k)).$$

To justify this name, we must show that the object  $\pi^!(\mathbf{1}_k)$  is *dualizing*. That is:

**Theorem 3.1.1.** *For any compact object  $E \in \mathbf{SH}(S)$ , the canonical map*

$$E \rightarrow D_S(D_S(E))$$

*is an equivalence in  $\mathbf{SH}(S)[\frac{1}{p}]$ .*

*Remark 3.1.2.* As remarked in [CD15, Remark 7.4], Theorem 3.1.1 implies the formalism of Grothendieck–Verdier duality for  $\mathbf{SH}[\frac{1}{p}]$ , for locally of finite type  $k$ -schemes. In particular, this gives an improvement of [BD15, Theorem 2.4.8].

Theorem 3.1.1 follows from the following statement, analogous to [CD15, Proposition 7.2].

**Proposition 3.1.3.** *The full subcategory of compact objects in  $\mathbf{SH}(S)[\frac{1}{p}]$  is generated as a thick subcategory by objects of the form  $f_!(\mathbf{1})(n)$ , where  $f : X \rightarrow S$  proper,  $X$  is smooth over a purely inseparable extension of  $k$ , and  $n \in \mathbf{Z}$  is an integer.*

*Proof.* If  $k$  is perfect, the statement is [BD15, Corollary 2.4.7]. In general, the morphism  $\varphi : S_{\text{perf}} \simeq S \times_{\text{Spec}(k)} \text{Spec}(k_{\text{perf}}) \rightarrow S$  induces an equivalence  $\varphi^* : \mathbf{SH}(S) \rightarrow \mathbf{SH}(S_{\text{perf}})$  by Corollary 2.1.7. If  $f : X \rightarrow S_{\text{perf}}$  is a proper morphism with  $X$  smooth over  $k_{\text{perf}}$ , then the composite  $X \rightarrow S_{\text{perf}} \rightarrow S$  is as in the statement, so we conclude.  $\square$

*Proof of Theorem 3.1.1.* As in the proof of [CD15, Theorem 7.3], this follows immediately from Proposition 3.1.3, and Ayoub’s purity theorem for smooth morphisms [Ayo08, Section 1.6], [Hoy14, Appendix C].  $\square$

**3.2. Removal of perfectness hypotheses.** Corollary 2.1.7 allows us to immediately drop perfectness hypotheses in many known results, at least after inverting the exponential characteristic. Some examples are listed below.

**Theorem 3.2.1.** *Let  $S$  be the spectrum of a field  $k$  of exponential characteristic  $p$ . For any smooth  $S$ -scheme  $X$ , the suspension spectrum  $\Sigma_+^\infty(X)$  is strongly dualizable in  $\mathbf{SH}(S)[\frac{1}{p}]$ . In particular,  $\mathbf{SH}(S)[\frac{1}{p}]$  is generated under colimits by the strongly dualizable objects.*

Indeed, we can use Corollary 2.1.7 to reduce the case where  $k$  is perfect, which is due to Riou, see [LYZ13, Corollary B.2].

*Remark 3.2.2.* Proposition 2.2.5 gives the following refinement of [LYZ13, Lemma B.3]. Suppose  $f : Y \rightarrow X$  is a finite étale morphism of degree  $d$  between smooth connected  $k$ -schemes. Then,

up to replacing  $X$  by a dense open subset  $U \subseteq X$  and  $f$  by its base change  $f_U : Y_U \rightarrow U$ , there are isomorphisms of natural transformations

$$\begin{aligned} (\mathrm{id} \xrightarrow{\mathrm{unit}} f_* f^* \simeq f_{\sharp} f^* \xrightarrow{\mathrm{counit}} \mathrm{id}) &\simeq d_\epsilon, \\ (f_* f^* \simeq f_{\sharp} f^* \xrightarrow{\mathrm{counit}} \mathrm{id} \xrightarrow{\mathrm{unit}} f_* f^* \mathrm{id}) &\simeq f_* * d_\epsilon * f^*, \end{aligned}$$

where  $d_\epsilon$  is as in Notation 2.2.2. To prove this, note there are canonical identifications  $\Sigma^{L_f} \simeq \mathrm{id}$  and  $f_* \simeq f_! \simeq f_{\sharp}$  since  $f$  is finite and étale, and the composite

$$f_* f^* \simeq f_{\sharp} f^* \xrightarrow{\mathrm{counit}} \mathrm{id}$$

is canonically homotopic to the trace transformation  $\mathrm{tr}_f : f_* f^* \simeq f_* \Sigma^{L_f} f^* \rightarrow \mathrm{id}$ . Replacing  $X$  by its generic point, we may assume that  $X = \mathrm{Spec}(k)$ . Then  $Y = \mathrm{Spec}(K)$  with  $K/k$  a finite separable field extension, so by the primitive element theorem we are now in the situation of Proposition 2.2.5.

3.2.3. We also have the following variant of Bachmann's conservativity theorem [Bac18].

**Theorem 3.2.4.** *Let  $k$  be a field with finite 2-étale cohomological dimension and exponential characteristic  $p$ . Then the canonical functor*

$$\mathbf{SH}(k)[\tfrac{1}{p}] \rightarrow \mathbf{DM}(k; \mathbf{Z}[\tfrac{1}{p}])$$

*is conservative on compact objects.*

*Proof.* Using Corollary 2.1.7 and the analogous result for mixed motives [CD15, Lemma 3.15], we may replace  $k$  by its perfection. Then the result is proven in [Bac18, Theorem 16].  $\square$

*Remark 3.2.5.* Using Theorem 3.2.4 we can deduce the Pic-injectivity result of [Bac18, Theorem 18]. This extends Bachmann's results on Po Hu's conjecture on invertibility of the the suspension spectra of affine quadrics to imperfect fields; see *loc. cit* for details.

3.2.6. We can similarly extend the recognition principle for infinite loop spaces [EHK<sup>+</sup>18, Theorem 3.5.13] to non-perfect fields.

**Theorem 3.2.7.** *Let  $k$  be a field of exponential characteristic  $p$ . Then there are canonical equivalences of symmetric monoidal  $\infty$ -categories*

$$\begin{aligned} \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\tfrac{1}{p}] &\simeq \mathbf{SH}^{\mathrm{veff}}(k)[\tfrac{1}{p}], \\ \mathbf{SH}^{\mathrm{S}^1, \mathrm{fr}}(k)[\tfrac{1}{p}] &\simeq \mathbf{SH}^{\mathrm{eff}}(k)[\tfrac{1}{p}]. \end{aligned}$$

*Remark 3.2.8.* Theorem 3.2.7 also implies cancellation in the sense of [EHK<sup>+</sup>18, Theorem 3.5.8] for non-perfect fields, after inverting the exponential characteristic.

We thank Marc Hoyois for pointing out a gap in the original proof of the following lemma.

**Lemma 3.2.9.** *Let  $k$  be a field of exponential characteristic  $p$ . Then the morphism  $f : \mathrm{Spec}(k_{\mathrm{perf}}) \rightarrow \mathrm{Spec}(k)$  induces an equivalence*

$$f^* : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\tfrac{1}{p}] \rightarrow \mathbf{H}^{\mathrm{fr}}(k_{\mathrm{perf}})^{\mathrm{gp}}[\tfrac{1}{p}]$$

*of symmetric monoidal  $\infty$ -categories.*

*Proof.* We first show fully faithfulness. By continuity, it suffices to show that the Frobenius induces fully faithful functors  $F^* : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\tfrac{1}{p}] \rightarrow \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}[\tfrac{1}{p}]$ , i.e., that the counit map  $\mathrm{id} \rightarrow F_* F^*$  is an equivalence after inverting  $p$ . For this it suffices to show that, for every grouplike framed motivic space  $\mathcal{F}$  and every smooth  $k$ -scheme  $X$ , the induced map of spaces

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(F^{-1}(X), F^*(\mathcal{F}))$$



is an equivalence after inverting  $p$ . Let  $\mathcal{F}^+$  denote the motivic localization of the left Kan extension of  $\mathcal{F}$  along the inclusion from framed correspondences of smooth  $k$ -schemes to framed correspondences of all  $k$ -schemes of finite type. Since the functor  $\mathcal{F} \mapsto \mathcal{F}^+$  is fully faithful and commutes with  $F^*$  (compare [CD12, Prop. 11.1.19]), the map above is identified with the canonical map

$$F^* : \Gamma(X, \mathcal{F}^+) \rightarrow \Gamma(F^{-1}(X), F^*(\mathcal{F}^+)) \simeq \Gamma(F^{-1}(X), \mathcal{F}^+).$$

Arguing just as in the proof of Proposition 2.2.5, we see that the structure of framed transfers on  $\mathcal{F}^+$  gives rise to the two composites (2.2.6) and (2.2.7), so that [EHK<sup>+</sup>18, Proposition B.1.4] shows that the map in question is an equivalence after inverting  $p_\epsilon$ . We conclude by using the analogue of Lemma 2.2.8 for  $\mathrm{End}_{\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}}(\mathbf{1}_k)$ , which holds because there is a canonical equivalence

$$\mathrm{End}_{\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}}(\mathbf{1}_k) \rightarrow \mathrm{End}_{\mathbf{SH}(k)}(\mathbf{1}_k)$$

by [EHK<sup>+</sup>18, Theorem 3.5.17].

It remains now to show that  $f^*$  is essentially surjective. Since any smooth irreducible  $k_{\mathrm{perf}}$ -scheme is, up to a universal homeomorphism, the base change of a smooth irreducible  $k$ -scheme [Sus17, Lemma 1.12], it will suffice to show the following claim: for any universal homeomorphism of smooth schemes over  $k_{\mathrm{perf}}$ , the induced map in  $\mathbf{H}^{\mathrm{fr}}(k_{\mathrm{perf}})^{\mathrm{gp}}[\frac{1}{p}]$  is invertible. By [EHK<sup>+</sup>18, Theorem 3.5.13(i)] it suffices to show that the induced map in  $\mathbf{SH}(k_{\mathrm{perf}})[\frac{1}{p}]$  is invertible. This follows directly from Theorem 2.1.1.  $\square$

*Proof of Theorem 3.2.7.* Note that we need only prove the claim when  $p > 1$ , and that the second claim follows from the first by stabilization. The equivalence of Corollary 2.1.7 restricts to an equivalence

$$\mathbf{SH}^{\mathrm{veff}}(k)[\frac{1}{p}] \rightarrow \mathbf{SH}^{\mathrm{veff}}(k_{\mathrm{perf}})[\frac{1}{p}]$$

by construction. Combining this with Lemma 3.2.9, we see that we may replace  $k$  by its perfection. In that case, the statement is [EHK<sup>+</sup>18, Theorem 3.5.13(i)].  $\square$

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