FUNDAMENTAL CLASSES IN MOTIVIC HOMOTOPY THEORY

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ABSTRACT. We develop the theory of fundamental classes in the setting of motivic homotopy theory. Using this we construct, for any motivic spectrum, an associated twisted bivariant theory, extending the formalism of Fulton and MacPherson. We import the tools of Fulton's intersection theory into this setting: (refined) Gysin maps, specialization maps, and formulas for excess of intersection, self-intersections, and blow-ups. We also develop a theory of Euler classes of vector bundles in this setting. For the Milnor–Witt spectrum recently constructed by Déglise–Fasel, we get a bivariant theory extending the Chow–Witt groups of Barge–Morel, in the same way the higher Chow groups extend the classical Chow groups. As another application we prove a motivic Gauss–Bonnet formula, computing Euler characteristics in the motivic homotopy category.

CONTENTS

1. INTRODUCTION

Historical background. From Poincaré to Grothendieck, duality has been a central component in the study of (co)homology theories. It led Grothendieck, building on Serre's duality, to what can nowadays be considered as the summit of such a theory, the *formalism of the six operations.* This formalism appeared in two flavours in Grothendieck's works in algebraic geometry: that of coherent sheaves and that of étale sheaves.

Date: March 22, 2020.

In the coherent setting, abstract duality was realized through the adjunction of the exceptional functors $(f_!, f^!)$ (see [Har66, Chap. 5]). The concept of dualizing complex was pivotal: it was discovered soon after that Borel–Moore homology [BM60] can be described as homology with coefficients in the (topological) dualizing complex. The theory of ℓ-adic sheaves developed in SGA4 [SGA4] was the first complete incarnation of the six functors formalism, and for a long time the only one available in algebraic geometry. A key aspect of the six functor formalism that was highlighted in the seminar SGA5 [SGA5] is the *absolute purity* property. Stated in *op. cit.* as a conjecture, it was partially solved by Thomason [Tho84], and completely settled later by Gabber [Fuj02, ILO14].

More recently, Morel and Voevodsky introduced motivic homotopy theory [MV99, Voe98]. As in algebraic topology, the stable motivic homotopy category classifies cohomology theories which satisfy homotopy invariance with respect to the algebraic affine line $\mathbb{A}^1$. The stable motivic homotopy category also satisfies the six functors formalism (see [Ayo07]). Moreover, it satisfies a suitable universal property [Rob15] and contains the classical theories of algebraic geometry, such as Betti cohomology, étale ℓ-adic cohomology, algebraic de Rham cohomology in characteristic 0, and rigid cohomology in positive characteristic. It also incorporates newer theories such as motivic cohomology, algebraic K-theory and algebraic cobordism. The latter-mentioned theories share the common property of being *oriented*, like their respective topological analogues, singular cohomology, complex K-theory, and complex cobordism. However, a salient feature of the motivic homotopy category is that it also contains theories which are *not* oriented, such as Chow–Witt groups [BM00a, Fas07, Fas08] and Milnor–Witt motivic cohomology [DF17b], Balmer's higher Witt groups [Bal99], hermitian K-theory (also called higher Grothendieck groups, [Hor05, PW18b]), certain variants of algebraic cobordism [PW18a], and the stable cohomotopy groups, represented by the motivic sphere spectrum.

The formalism of six operations gives rise to a great deal of structure at the level of cohomology and Borel–Moore homology groups. Parts of this structure were axiomatized by Bloch and Ogus [BO74], via their notion of *Poincaré duality theory*, and later through the *bivariant theories* of Fulton and MacPherson [FM81]. The key element of these axiomatizations was the notion of the *fundamental class*, which was used to express duality isomorphisms.

Main problematic. Our goal in this paper is to incorporate Fulton and MacPherson's ideas into stable motivic homotopy theory, thereby obtaining a universal bivariant theory. In order to treat oriented and non-oriented spectra in a single theory, we have to replace Tate twists, as used for example in the Bloch–Ogus axiomatic, by "Thom twists", i.e., twists with respect to vector bundles (or more generally, with respect to virtual vector bundles). Let us explain the justification for this idea.

Our first inspiration is Morel and Voevodsky's homotopy purity theorem, which asserts that, for smooth closed pairs (X, Z), the homotopy type of X with support in Z is isomorphic to $\mathrm{Th}_Z(N_Z X)$, the Thom space of the normal bundle of Z in X. Here the homotopy type of $\mathrm{Th}_Z(N_Z X)$ should be understood as the homotopy type of Z twisted by the vector bundle $N_Z X$. Another motivation is Morel's work [Mor12] on computations of homotopy groups, in which a crucial role is played by the construction of good transfer maps for finite field extensions in the unstable homotopy category. In this work, twists are usually avoided but at the cost of choosing orientations. Similar constructions enter into play in Voevodsky's theory of framed correspondences [Voe01, GP18, EHK$^+$17], where the "framing" provides a chosen trivialization of the normal bundle. Finally, Calmès and Fasel have introduced the notion of GW-correspondences, based on Chow–Witt theory, where transfers do appear with a twist. These examples show the utmost importance of having good transfer or Gysin morphisms in $\mathbb{A}^1$-homotopy theory. The last indication which points out to our central construction is the extension obtained in [Jin16] of the finer operations of Fulton's intersection theory, such as refined Gysin morphisms, in the motivic Borel-Moore theory (see in particular [Jin16, Def. 3.1]). The translation becomes possible once one recognizes Borel-Moore motivic homology as a particular instance of bivariant theory.

Another fundamental model for our theory is that of Chow–Witt groups. In this theory, the necessity of considering twists appears most notably when Gysin morphisms are at stake (see [Fas07, Fas08]). Much of the interest in these groups comes from the fact that they are natural receptacles for *Euler classes* of vector bundles. The Euler class provides an obstruction for a vector bundle to split off a trivial summand of rank one (see [BM00b, Mor12, FS09]). In this paper we also develop a general theory of Euler classes in $\mathbb{A}^1$-homotopy theory, and show that they satisfy the expected obstruction-theoretic property (Corollary 3.1.8). Our motivation to introduce these Euler classes is to formulate excess intersection formulas in our bivariant theories, see the following theorem.

Main construction. The Thom space functor Th_X, associating to a vector bundle E over a scheme X its (stable) Thom space $\mathrm{Th}_X(E) \in \mathscr{SH}(X)$, canonically extends to the Picard groupoid of virtual vector bundles over X (see [Rio10, 4.1]). Given any motivic spectrum $\mathbb{E} \in \mathscr{SH}(S)$, we can define the *(twisted) bivariant theory* with coefficients in $\mathbb{E}$, graded by integers $n \in \mathbb{Z}$ and virtual vector bundles v on X, as the following group:

$$\mathbb{E}_n(X/S, v) := \mathrm{Hom}_{\mathscr{SH}(S)}(\mathrm{Th}_X(v)[n], p^!(\mathbb{E})),$$

for any morphism $p : X \to S$ that is separated of finite type. For $\mathbb{E} = \mathbb{S}_S$, we simply write $H_n(X/S, v)$ and call this *bivariant* $\mathbb{A}^1$*-theory*. The construction is functorial in $\mathbb{E}$ so that, given any ring spectrum $\mathbb{E}$ over S with unit $\nu : \mathbb{S}_S \to \mathbb{E}$, we get a canonical map:

$$H_*(X/S, v) \to \mathbb{E}_*(X/S, v),$$

expressing the universal role of bivariant $\mathbb{A}^1$-theory.[1] These bivariant theory groups satisfy a rich functoriality, including covariance for proper maps and contravariance for étale maps; see Paragraph 2.2.7 for details. There is also a composition product, which takes the form

$$\mathbb{E}_*(Y/X, w) \otimes \mathbb{E}_*(X/S, v) \to \mathbb{E}_*(Y/S, w + q^*v), (y, x) \mapsto y.x,$$

for schemes $Y/X/S$, and virtual vector bundles v/X, w/Y (see again Paragraph 2.2.7). That being given, here is the central construction of this paper.

Theorem (Theorem 3.3.2 and Proposition 3.3.4)**.** *For any* smoothable lci[2] *morphism* $f : X \to Y$*, there exists a canonical class* η_f*, called the* fundamental class *of* f*:*

$$\eta_f \in H_0(X/Y, \langle L_f \rangle),$$

where $\langle L_f \rangle$ *is the virtual tangent bundle of* f *(equivalently, the virtual bundle associated with the cotangent complex of* f*, which is perfect under our assumption).*

These classes satisfy the following properties:

(i) Associativity*. Consider morphisms* $f : X \to Y$ *and* $g : Y \to Z$ *such that* f*,* g*, and* $g \circ f$ *are smoothable and lci. Then one has:*

$$\eta_g.\eta_f \simeq \eta_{g\circ f}$$

in $H_0(X/Z, \langle L_f \rangle + f^*\langle L_g \rangle) \simeq H_0(X/Z, \langle L_{g\circ f} \rangle)$.

(ii) Excess intersection*. Consider a cartesian square*

(1.0.0.a)
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "u"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

such that f *and* g *are smoothable lci. Consider a factorization* $X \xrightarrow{i} P \xrightarrow{p} S$ *of* f *such that* p *is smooth and* i *is a closed immersion. Let* k *be the pullback of* i *along* u *and denote by* ξ

[1] We call this map the $\mathbb{A}^1$-regulator map, by extension of Beilinson's terminology; see Definition 4.1.2.

[2] A morphism of schemes is smoothable if it admits a (global) factorization into a closed immersion followed by a smooth morphism; such a morphism is lci (a local complete intersection) if the closed immersion is a regular immersion; see our conventions.

the quotient bundle of the natural monomorphism of vector bundles: $N_k \to u^{-1}N_i$. Then there exists an Euler class $e(\xi) \in H_0(Y/Y, \langle \xi \rangle)$ such that the following formula holds:

$$\Delta^*(\eta_f) \simeq e(\xi).\eta_g$$

in $H_0(Y/T, v^*\langle L_f \rangle) \simeq H_0(Y/T, \langle L_g \rangle - \langle \xi \rangle)$.

In particular, if the square Δ is tor-independent, we get $\Delta^*(\eta_f) \simeq \eta_g$.

This construction is universal in the stable motivic homotopy category. Indeed, given a motivic ring spectrum $\mathbb{E}$ the regulator map (Definition 4.1.2) gives rise to fundamental classes with coefficients in $\mathbb{E}$. This in turn yields Gysin homomorphisms

$$f^! : \mathbb{E}_n(Y/S, e) \to \mathbb{E}_n(X/S, f^*(e) + \langle L_f \rangle), \quad y \mapsto \eta_f.y,$$

for a smoothable lci morphism $f : X \to Y$, and the associativity (resp. excess intersection) property above corresponds to the compatibility with composition (resp. excess intersection formula) satisfied by these Gysin morphisms, as in Chow theory. Note that for oriented theories like Borel–Moore motivic homology, the orientation provides a Thom isomorphism that replaces the virtual twist $\langle L_f \rangle$ with a shift by the relative virtual dimension $\chi(L_f)$, so that the Gysin homomorphism takes a more familiar shape.[3]

From a categorical point of view, the universality of our construction is best stated in the language of the six operations: the fundamental class η_f as above corresponds to a natural transformation of functors

$$\mathfrak{p}_f : f^*(-) \otimes \mathrm{Th}_X(L_f) \to f^!$$

that we call the *purity transformation* associated to f (see Section 4.3). Then by adjunction, we get *trace* and *cotrace* maps, extending the classical construction of SGA4:

$$\mathrm{tr}_f : f_! \Sigma^{L_f} f^* \to \mathrm{Id}$$
$$\mathrm{cotr}_f : \mathrm{Id} \to f_* \Sigma^{-L_f} f^!.$$

These natural transformations can be considered as a natural extension, and in fact an important part, of the six functors formalism.[4] The Gysin morphisms in bivariant theory above are immediate consequences of these transformations, when applied to the ring spectrum $\mathbb{E}$. Moreover, one gets Gysin morphisms (wrong-way variance) for the traditional package of co/homological theories, with and without proper support, associated with a spectrum (even without a ring structure). We refer the interested reader to Paragraph 4.3.3 for details, and to Section 4.4 for a list of concrete examples.

When f is *smooth*, the map $\mathfrak{p}_f$ considered above is invertible and coincides with the classical purity isomorphism of the six functors, as defined by Ayoub (see Paragraph 2.1.7). In general, the purity transformation measures the failure of a given motivic spectrum $\mathbb{E} \in \mathcal{SH}(Y)$ to satisfy the *purity* property with respect to f (see Definition 4.3.7); when f is a closed immersion between regular schemes, this property corresponds to the notion of *absolute purity* (see Definition 4.3.11), axiomatizing the original conjecture of Grothendieck. Such an axiomatization is not new (see [Dég18b, 1.3], [CD16, A.2]). However, the formulation we obtain here is more flexible and has a number of advantages (see Example 4.3.12). The absolute purity is essential for arithmetic applications, and has already been obtained in several contexts (rational motives, étale motives, **KGL**-modules). We believe that new examples will be obtained in the future (and have work in progress in that direction). Finally, our construction has been applied in [FS17, Appendix A] to prove a new absolute purity result for motivic cohomology.

[3] Similar simplifications of twists occur for the so-called *Sp-oriented theories* of Panin and Walter. We leave the general formulation for future works. The reader can also consult Example 4.4.6.

[4] In fact, we show that our construction allows one to define these transformations in a greater generality, say for arbitrary motivic $(\infty, 1)$-category of coefficients in the sense of [Kha16, Chap. 2, Def. 3.5.2]. See Paragraph 4.3.4.

Further applications. We finally give several applications of our formalism. The first one is the possibility of extending Fulton's theory of *refined Gysin morphism* to an arbitrary ring spectrum (Definition 4.2.5). These new refined Gysin morphisms are used to define *specialization maps* in any representable theory, on the model of Fulton's definition of specialization for the Chow group. In fact, our specialization maps can be lifted to natural transformations of functors (see Paragraph 4.5.6 for details). Most interestingly, the theory can be applied to Chow–Witt groups and give specializations of quadratic cycles (see Example 4.5.5).

Note that the idea of refining classical formulas to the quadratic setting has been explored recently by many authors [Fas07, Fas09, Hoy15, KW16, Lev17b]. In this direction another application of the theory we develop is a motivic refinement of the classical Gauss-Bonnet formula [SGA5, VII 4.9]. Given a smooth proper S-scheme X, the *categorical Euler characteristic* $\chi^{cat}(X/S)$ is the endomorphism of the motivic sphere spectrum $\mathbb{S}_S$ given by the trace of the identity map of $\Sigma^\infty_+(X) \in S\mathscr{H}(S)$. A simple application of our excess intersection formula then computes this invariant as the degree of the Euler class of the tangent bundle $T_{X/S}$ (see Theorem 4.6.1). This result is a generalization of a theorem of Levine [Lev17b, Thm. 1], which applies when S is the spectrum of a field and $p : X \to S$ is smooth projective. It also recovers the SL-oriented variant in [LR18, Theorem 1.5], used in *op. cit.* to prove a certain explicit formula for the quadratic Euler characteristic conjectured by Serre.

Related work and further developments. Bivariant theories represented by *oriented* motivic spectra were studied in detail in [Dég18a]. In that setting, the fundamental class of a regular closed immersion is given by a construction of Navarro [Nav18], which itself is based on a construction of Gabber in the setting of étale cohomology [ILO14, Exp. XVI]. A similar construction to our fundamental class for closed immersions, in the context of equivariant stable $\mathbb{A}^1$-homotopy, has been developed independently in recent work of Levine [Lev17a].

An immediate but important consequence of our work here is that the cohomology theory represented by any motivic spectrum $\mathbb{E}$ admits a canonical structure of *framed transfers*; that is, it extends to a presheaf on the category of framed correspondences (see [Voe01, EHK$^+$17]). It is proven in [EHK$^+$17] that this structure can be used to recognize infinite $\mathbb{P}^1$-loop spaces, in the same way that $\mathcal{E}_\infty$-structures can be used to recognize infinite loop spaces in topology (see also [GP18]). In [EHK$^+$18, DK18], framed transfers are applied to construct categories of finite $\mathbb{E}$-correspondences, for any motivic spectrum $\mathbb{E}$, together with canonical functors from the category of framed correspondences. As explained in *op. cit.*, such functors play an important role in the yoga of motivic categories. Another application of the existence of framed transfers is a topological invariance statement for the motivic homotopy category, up to inverting the exponential characteristic of the base field [EK20].

An application of our theory of Euler classes can be found in [JY18], where it is used to give a characterization of the characteristic class of a motive.

Finally, our constructions can be extended to the setting of *quasi-smooth* morphisms in derived algebraic geometry. This yields a formalism of motivic virtual fundamental classes, see [Kha19].

Contents. In Section 2 we construct the bivariant theory and cohomology theory associated to a motivic ring spectrum, and study their basic properties. Following Fulton–MacPherson [FM81], we also introduce the abstract notion of *orientations* of morphisms in this setting; fundamental classes will be examples of orientations. We then show how any choice of orientation gives rise to a purity transformation.

The heart of the paper is Section 3, where we construct fundamental classes and verify their basic properties. In the case where f is smooth, the fundamental class comes from the purity theorem (see Definition 2.3.5). For the case of a regular closed immersion we use the technique of deformation to the normal cone. We then explain how to glue these to obtain a fundamental class for any quasi-projective lci morphism. Throughout this section, we restrict our attention to the bivariant theory represented by the motivic sphere spectrum.

Finally in Section 4, we return to the setting of the bivariant theory represented by any motivic ring spectrum. We show how the fundamental class gives rise to Gysin homomorphisms. We prove the excess intersection formula in this setting (Proposition 4.2.2). We also discuss the purity transformation, the absolute purity property, and duality isomorphisms (identifying bivariant groups with certain cohomology groups). We then import some further constructions from Fulton's intersection theory, including refined Gysin maps and specialization maps. Finally, we conclude with a proof of the motivic Gauss-Bonnet formula mentioned above.

Conventions. The following conventions are in place throughout the paper:

(1) All schemes in this paper are assumed to be quasi-compact and quasi-separated.
(2) The term *s-morphism* is an abbreviation for "separated morphism of finite type". Similarly, an *s-scheme over S* is an S-scheme whose structural morphism is an s-morphism.
(3) We write $\mathbb{A}^1$ for the affine line over $\mathrm{Spec}(\mathbb{Z})$ and $\mathbb{G}_m$ for the complement of the origin. For a scheme X, we write $\mathbb{A}^1X$ and $\mathbb{G}_mX$ for $\mathbb{A}^1 \times X$ and $\mathbb{G}_m \times X$, respectively.
(4) We follow [SGA6, Exps. VII–VIII] for our conventions on regular closed immersions and lci morphisms. Recall that if X and Z are regular schemes, or are both smooth over some base S, then any closed immersion $Z \to X$ is regular. Given a regular closed immersion $i : Z \to X$, we write N_i, N_ZX, or occasionally $N(X,Z)$, for its normal bundle. Recall that a morphism of schemes $X \to S$ is *lci* (= a local complete intersection) if it admits, Zariski-locally on the source, a factorization $X \xrightarrow{i} Y \xrightarrow{p} S$, where p is smooth and i is a regular closed immersion. If it is also *smoothable*, then it admits such a factorization globally. This is for example the case if f is quasi-projective (in the sense that it factors through an immersion into some projective space $\mathbb{P}^n_S$).
(5) A cartesian square of schemes

(1.0.0.b)
$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

is *tor-independent* if the groups $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'})$ vanish for $i > 0$. In this case we also say that p is *transverse* to f. Recall that if p or f is flat, then this condition is automatic.
(6) We will make use of the language of stable ∞-categories [HA]. Given a stable ∞-category $\mathscr{C}$, we write $\mathrm{Maps}_{\mathscr{C}}(X,Y)$ for the mapping spectrum of any two objects X and Y. We write $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ or simply $[X,Y]$ for the abelian group of connected components $\pi_0 \mathrm{Maps}_{\mathscr{C}}(X,Y)$.
(7) Given a topological S^1-spectrum E, we write $x \in E$ to mean that x is a point in the infinite loop space $\Omega^\infty(X)$ (or an object in the corresponding ∞-groupoid).

Acknowledgments. We would like to thank Denis-Charles Cisinski for many helpful discussions. In particular, we learned the interpretation of the Gysin map in terms of the six operations from him. We also thank Martin Frankland for comments on a previous revision. F. Jin learned from Marc Levine his motivic Gauss-Bonnet formula during the lectures at the the 2017 Trimester Program *K-theory and related fields* at the Hausdorff Research Institute for Mathematics in Bonn. The third author would like to thank Elden Elmanto, Marc Hoyois, Vova Sosnilo and Maria Yakerson for their collaboration on the papers [EHK+17, EHK+18], which served as a strong motivation in writing this paper. We would also like to thank the referees for their comments which helped clarify some missing points. Finally, we thank the FRIAS for its hospitality while part of this work was completed.

F. Déglise received support from the French "Investissements d'Avenir" program, project ISITE-BFC (contract ANR-lS-IDEX-OOOB) and from a Marie Skłodowska-Curie FCFP Senior Fellowship hosted by the FRIAS. F. Jin is partially supported by the DFG Priority Programme SPP 1786.

2. Bivariant theories and cohomology theories

2.1. **The six operations.** Given any (quasi-compact quasi-separated) scheme S, we write $\mathcal{SH}(S)$ for the stable ∞-category of motivic spectra, as in [Hoy15, Appendix C] or [Kha16]. When S is noetherian and of finite dimension, then the homotopy category of $\mathcal{SH}(S)$ is equivalent, as a triangulated category, to the stable $\mathbb{A}^1$-homotopy category originally constructed by Voevodsky [Voe98]. As S varies, these categories are equipped with the formalism of Grothendieck's six operations [Ayo07, CD19]. In this subsection we briefly recall this formalism, and its ∞-categorical refinement as constructed in [Kha16, Chap. 2] (see also [Rob14] for another approach).

2.1.1. First, the stable presentable ∞-category $\mathcal{SH}(S)$ is symmetric monoidal, and we denote the monoidal product and monoidal unit by $\otimes$ and $\mathbb{S}_S$, respectively. It also admits internal hom objects $\underline{\mathrm{Hom}}(\mathbb{E},\mathbb{F}) \in \mathcal{SH}(S)$ for all $\mathbb{E},\mathbb{F} \in \mathcal{SH}(S)$. For any morphism of schemes $f : T \to S$, we have a pair of adjoint functors

$$f^* : \mathcal{SH}(S) \to \mathcal{SH}(T), \qquad f_* : \mathcal{SH}(T) \to \mathcal{SH}(S),$$

called the functors of *inverse* and *direct image* along f, respectively. If f is an *s-morphism*[5], i.e., a separated morphism of finite type, then there is another pair of adjoint functors

$$f_! : \mathcal{SH}(T) \to \mathcal{SH}(S), \qquad f^! : \mathcal{SH}(S) \to \mathcal{SH}(T),$$

called the functors of *exceptional* direct and inverse image along f, respectively. Each of these operations is 2-functorial.

2.1.2. The six operations $(\otimes, \underline{\mathrm{Hom}}, f^*, f_*, f_!, f^!)$ satisfy a variety of compatibilities. These include:

(1) For every morphism f, the functor f^* is symmetric monoidal.
(2) There is a natural transformation $f_! \to f_*$ which is invertible when f is proper.
(3) There is an invertible natural transformation $f^* \to f^!$ when f is an open immersion.
(4) The operation of exceptional direct image $f_!$ satisfies a projection formula against inverse image. That is, there is a canonical isomorphism

$$\mathbb{E} \otimes f_!(\mathbb{F}) \to f_!(f^*(\mathbb{E}) \otimes \mathbb{F})$$

for any s-morphism $f : T \to S$ and any $\mathbb{E} \in \mathcal{SH}(S)$, $\mathbb{F} \in \mathcal{SH}(T)$.
(5) The operation $f_!$ satisfies base change against inverse images g^*, and similarly $f^!$ satisfies base change against direct images g_*. That is, for any cartesian square

(2.1.2.a)
$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"'] & S' \arrow[d, "p"] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

where f and g are s-morphisms, there are canonical isomorphisms

$$p^* f_! \to g_! q^*, \qquad q_* g^! \to f^! p_*.$$

All the above data are subject to a homotopy coherent system of compatibilities (see [Kha16, Chap. 2, Sect. 5]).

2.1.3. The $\mathbb{A}^1$-homotopy invariance property of $\mathcal{SH}$ is encoded in terms of the six operations as follows. For a scheme S and any vector bundle $\pi : E \to S$, the functor $\pi^* : \mathcal{SH}(S) \to \mathcal{SH}(E)$ is *fully faithful*. In particular, the unit $\mathrm{Id} \to \pi_*\pi^*$ is invertible.

[5] Using Zariski descent, the operations $(f_!, f^!)$ can be extended to the case where f is locally of finite type; assuming this extension, the reader can globally redefine the term "s-morphism" as "locally of finite type morphism".

2.1.4. Given a locally free sheaf $\mathcal{E}$ of finite rank over S, let $E = \mathrm{Spec}_S(\mathcal{S}(\mathcal{E}))$ denote the associated vector bundle[6]. There is an auto-equivalence

$$\Sigma^{\mathcal{E}} : S\mathscr{H}(S) \to S\mathscr{H}(S),$$

called the *$\mathcal{E}$-suspension functor*, with inverse denoted $\Sigma^{-\mathcal{E}}$. This functor is compatible with the monoidal product $\otimes$ via a projection formula that provides canonical isomorphisms $\Sigma^{\mathcal{E}}(\mathbb{E}) \simeq \mathbb{E} \otimes \Sigma^{\mathcal{E}}(\mathbb{S}_S)$ for any $\mathbb{E} \in S\mathscr{H}(S)$. It is also compatible with the other operations in the sense that we have canonical isomorphisms

$$f^*\Sigma^{\mathcal{E}} \simeq \Sigma^{f^*(\mathcal{E})} f^*, \quad f_*\Sigma^{f^*(\mathcal{E})} \simeq \Sigma^{\mathcal{E}} f_*, \quad f_!\Sigma^{f^*(\mathcal{E})} \simeq \Sigma^{\mathcal{E}} f_!, \quad f^!\Sigma^{\mathcal{E}} \simeq \Sigma^{f^*(\mathcal{E})} f^!. \tag{2.1.4.a}$$

The motivic spectrum $\Sigma^{\mathcal{E}}(\mathbb{S}_S) \in S\mathscr{H}(S)$ is (the suspension spectrum of) the *Thom space* of $\mathcal{E}$, and is denoted $\mathrm{Th}_S(\mathcal{E})$. Its $\otimes$-inverse $\Sigma^{-\mathcal{E}}(\mathbb{S}_S)$ is denoted $\mathrm{Th}_S(-\mathcal{E})$.

Given a motivic spectrum $\mathbb{E} \in S\mathscr{H}(S)$, we denote by $\mathbb{E}(n) \in S\mathscr{H}(S)$ the motivic spectrum $\Sigma^{\mathcal{O}_S^n}(\mathbb{E})[-2n]$ for each $n \geq 0$. The assignment $\mathbb{E} \mapsto \mathbb{E}(n)$ defines then another auto-equivalence of $S\mathscr{H}(S)$, inverse to $\mathbb{E} \mapsto \mathbb{E}(-n) = \Sigma^{-\mathcal{O}_S^n}(\mathbb{E})[2n]$.

2.1.5. Let $\mathrm{Vect}(S)$ denote the groupoid of locally free sheaves on S of finite rank, and $\mathrm{Pic}(S\mathscr{H}(S))$ the ∞-groupoid of $\otimes$-invertible objects in $S\mathscr{H}(S)$. The assignment $\mathcal{E} \mapsto \mathrm{Th}_S(\mathcal{E})$ determines a map of presheaves of ∞-groupoids

$$\mathrm{Th} : \mathrm{Vect} \to \mathrm{Pic}(S\mathscr{H}).$$

Moreover, if K denotes the presheaf sending S to its Thomason–Trobaugh K-theory space $\mathrm{K}(S)$, this extends to a map of $\mathcal{E}_\infty$-groups

$$\mathrm{Th} : \mathrm{K} \to \mathrm{Pic}(S\mathscr{H}),$$

see [BH17, Subsect. 16.2]. In particular, any perfect complex $\mathcal{E}$ on S defines a K-theory class[7] $\langle\mathcal{E}\rangle \in \mathrm{K}(S)$ and thus an auto-equivalence $\Sigma^{\mathcal{E}} : S\mathscr{H}(S) \to S\mathscr{H}(S)$ and a Thom space $\mathrm{Th}_S(\mathcal{E}) \in S\mathscr{H}(S)$. The formulas (2.1.4.a) also extend. Moreover, any exact triangle $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}''$ of perfect complexes induces canonically a path $\langle\mathcal{E}\rangle \simeq \langle\mathcal{E}'\rangle + \langle\mathcal{E}''\rangle$ in the space $\mathrm{K}(S)$, hence also identifications

$$\Sigma^{\mathcal{E}} \simeq \Sigma^{\mathcal{E}'}\Sigma^{\mathcal{E}''} \simeq \Sigma^{\mathcal{E}''}\Sigma^{\mathcal{E}'} \tag{2.1.5.a}$$

and an isomorphism $\mathrm{Th}_S(\mathcal{E}) \simeq \mathrm{Th}_S(\mathcal{E}') \otimes \mathrm{Th}_S(\mathcal{E}'')$ in $S\mathscr{H}(S)$.

2.1.6. If $i : Z \to S$ is a closed immersion, then the direct image functor $i_* : S\mathscr{H}(Z) \to S\mathscr{H}(S)$ is fully faithful. Moreover, if the complementary open immersion $j : U \to S$ is quasi-compact, then by the Morel–Voevodsky localization theorem there is an exact triangle

$$i_*i^! \to \mathrm{Id} \to j_*j^*.$$

2.1.7. If f is a smooth s-morphism, then by Ayoub's purity theorem, there is a canonical isomorphism of functors

$$\mathfrak{p}_f : \Sigma^{T_f} f^* \to f^!,$$

where T_f is the relative tangent bundle. It follows in particular that f^* admits a left adjoint

$$f_\sharp = f_!\Sigma^{T_f}$$

which satisfies base change and projection formulas against inverse images g^*. If f is étale then we get an isomorphism $\mathfrak{p}_f : f^* \simeq f^!$, generalizing Paragraph 2.1.2(3).

[6]Throughout the text we will generally not distinguish between a locally free sheaf $\mathcal{E}$ and its associated vector bundle E. Thus for example we will also write Σ^E instead of $\Sigma^{\mathcal{E}}$, or similarly $\langle E\rangle \in \mathrm{K}(S)$ instead of $\langle\mathcal{E}\rangle \in \mathrm{K}(S)$ (see Paragraph 2.1.5 below). It should always be clear from the context what is intended.

[7]An abuse of language we will commit often is to say "K-theory class" when it would be more precise to say "point of the K-theory space".

2.1.8. Let $f : X \to Y$ be a closed immersion of s-schemes over S. Suppose that X and Y are *smooth* over S, with structural morphisms $p : X \to S$ and $q : Y \to S$. Then by the relative purity theorem of Morel–Voevodsky, there exist isomorphisms of functors

$$q_\sharp f_* \simeq p_\sharp \Sigma^{N_X Y}, \qquad \Sigma^{-N_X Y} p^* \simeq f^! q^*,$$

where $N_X Y$ denotes the normal bundle.

Many further compatibilities can be derived from the ones already listed. A few that will be especially useful in this paper are as follows:

2.1.9. Given a cartesian square as in (2.1.2.a) where f and g are s-morphisms, the base change formula (Paragraph 2.1.2(5)) induces a natural transformation

$$Ex^{*!} : q^* f^! \to g^! p^*. \tag{2.1.9.a}$$

It follows from the purity theorem (Paragraph 2.1.7) that if f or p is smooth, then $Ex^{*!}$ is invertible. For example if $i : Z \to S$ is a closed immersion, then the cartesian square

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, equal] & Z \arrow[d, "i"] \\ Z \arrow[r, "i"] & S \end{tikzcd}$$

gives rise to a canonical natural transformation

$$i^! \to i^*. \tag{2.1.9.b}$$

2.1.10. For any s-morphism $f : X \to S$ and any pair of motivic spectra $\mathbb{E}, \mathbb{F} \in S\mathscr{H}(S)$, there is a canonical morphism

$$Ex^{!*}_\otimes : f^!(\mathbb{E}) \otimes f^*(\mathbb{F}) \to f^!(\mathbb{E} \otimes \mathbb{F})$$

induced by adjunction from the projection formula (Paragraph 2.1.2(4)). If $\mathbb{F}$ is $\otimes$-*invertible*, then $Ex^{!*}_\otimes$ is invertible.

2.2. **Bivariant theories.** In this subsection we construct the *bivariant theory* represented by a motivic spectrum, and state its main properties. In fact, bivariant theory is only one of the "four theories" associated to a motivic spectrum (cf. [VSF00, Chap. 4, Sect. 9]). For sake of completeness we define them all now:

Definition 2.2.1. Let S be a scheme and $\mathbb{E} \in S\mathscr{H}(S)$ a motivic spectrum.

(i) *Bivariant theory.* For any s-morphism $p : X \to S$ and any K-theory class $v \in \mathrm{K}(X)$, we define the *v-twisted bivariant spectrum* of X over S as the mapping spectrum

$$\begin{aligned} \mathbb{E}(X/S, v) &= \mathrm{Maps}_{S\mathscr{H}(S)}(\mathbb{S}_S, p_*(p^!(\mathbb{E}) \otimes \mathrm{Th}_X(-v))) \\ &\simeq \mathrm{Maps}_{S\mathscr{H}(S)}(p_!(\mathrm{Th}_X(v)), \mathbb{E}). \end{aligned}$$

We also write

$$\mathbb{E}_n(X/S, v) = \pi_n \mathbb{E}(X/S, v) = \left[\mathbb{S}_S[n], p_*(p^!(\mathbb{E}) \otimes \mathrm{Th}_X(-v))\right]$$

for each integer $n \in \mathbb{Z}$.

(ii) *Cohomology theory.* For any morphism $p : X \to S$ and any $v \in \mathrm{K}(X)$, we define the *v-twisted cohomology spectrum* of X over S as the mapping spectrum

$$\begin{aligned} \mathbb{E}(X, v) &= \mathrm{Maps}_{S\mathscr{H}(S)}(\mathbb{S}_S, p_*(p^*(\mathbb{E}) \otimes \mathrm{Th}_X(v))) \\ &\simeq \mathrm{Maps}_{S\mathscr{H}(X)}(\mathbb{S}_X, p^*\mathbb{E} \otimes \mathrm{Th}_X(v)). \end{aligned}$$

We also write

$$\mathbb{E}^n(X, v) = \pi_{-n} \mathbb{E}(X, v) = \left[\mathbb{S}_S, p_*(p^*(\mathbb{E}) \otimes \mathrm{Th}_X(v))[n]\right].$$

for each integer $n \in \mathbb{Z}$.

(iii) *Bivariant theory with proper support (or homology).* For any s-morphism $p : X \to S$ and any K-theory class $v \in \mathrm{K}(X)$, we define the spectrum of *v-twisted bivariant theory with proper support* of X over S as the mapping spectrum

$$\mathbb{E}^c(X/S, v) = \mathrm{Maps}_{S\mathscr{H}(S)}\left(\mathbb{S}_S, f_!(f^!(\mathbb{E}_S) \otimes \mathrm{Th}_X(-v))\right)$$

(iv) *Cohomology with proper support.* For any s-morphism $p : X \to S$ and any K-theory class $v \in \mathrm{K}(X)$, we define the spectrum of *v-twisted cohomology with proper support* of X over S as the mapping spectrum

$$\mathbb{E}_c(X/S, v) = \mathrm{Maps}_{S\mathscr{H}(S)}\left(\mathbb{S}_S, f_!(f^*(\mathbb{E}) \otimes \mathrm{Th}_X(v))\right).$$

Remark 2.2.2. Note that we have canonical identifications

$$\mathbb{E}(X, v) \simeq \mathbb{E}(X/X, -v)$$

for any s-scheme X over S and $v \in \mathrm{K}(X)$.

Remark 2.2.3. The bivariant groups $\mathbb{E}_*(X/S, *)$ were previously called *Borel–Moore homology* groups in [Dég18a]. This terminology is justified when S is the spectrum of a field, and coincides with that of [VSF00, Chap. 4, Sect. 9]. However, in the case where S is an arbitrary scheme, and especially singular, the homology groups $\mathbb{E}_*(X/S, *)$ are no longer given by the cohomology with coefficients in a dualizing object, which is a characteristic property of the original theory of Borel and Moore. For that reason we find the terminology "bivariant" more suitable.

Remark 2.2.4. Given a morphism $f : T \to S$, we can consider the inverse image $\mathbb{E}_T = f^*(\mathbb{E}) \in S\mathscr{H}(T)$ and the associated bivariant theory $\mathbb{E}_T(-/T, *)$ over T. When there is no risk of confusion, we will usually abuse notation by writing $\mathbb{E}(-/T, *) = \mathbb{E}_T(-/T, *)$.

Remark 2.2.5. Note that any isomorphism $v \simeq w$ in $\mathrm{K}(X)$ induces an isomorphism of bivariant spectra $\mathbb{E}(X/S, v) \simeq \mathbb{E}(X/S, w)$ and of cohomology spectra $\mathbb{E}(X, v) \simeq \mathbb{E}(X, w)$. More precisely, the assignments $v \mapsto \mathbb{E}(X/S, v)$ and $v \mapsto \mathbb{E}(X, v)$ are functors on $\mathrm{K}(X)$ (viewed as an ∞-groupoid).

Notation 2.2.6. In the notation $\mathbb{E}(X/S, v)$ and $\mathbb{E}(X, v)$, we will sometimes implicitly regard v as a class in $\mathrm{K}(X)$ even if it is actually defined over some deeper base. For example we will write $\mathbb{E}(X/S, v) = \mathbb{E}(X/S, f^*(v))$ where $f : X \to S$ and $v \in \mathrm{K}(S)$.

2.2.7. The bivariant theory represented by a motivic spectrum $\mathbb{E} \in S\mathscr{H}(S)$ satisfies the following axioms, which are K-graded and spectrum-level refinements of the axioms of Fulton and MacPherson [FM81]:

(1) *Base change.* For any cartesian square

$$\begin{tikzcd} X_T \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "p"] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

there is a canonical base change map

$$\Delta^* : \mathbb{E}(X/S, v) \to \mathbb{E}(X_T/T, g^*v).$$

This is induced by the natural transformation

$$p_*\Sigma^{-v}p^! \xrightarrow{\text{unit}} p_*\Sigma^{-v}p^!f_*f^* \simeq p_*\Sigma^{-v}g_*q^!f^* \simeq p_*g_*\Sigma^{-g^*v}q^!f^* \simeq f_*q_*\Sigma^{-g^*v}q^!f^*,$$

where we have used the base change formula (Paragraph 2.1.2(5)) and the formula (2.1.4.a).

(2) *Proper covariance.* For any proper morphism $f : X \to Y$ of s-schemes over S, there is a direct image map

$$f_* : \mathbb{E}(X/S, f^*v) \to \mathbb{E}(Y/S, v).$$

This covariance is induced by the unit map $f_!f^! \to \mathrm{Id}$ and the identification $f_! \simeq f_*$ as f is proper (Paragraph 2.1.2(2)).

(3) *Étale contravariance.* For any étale s-morphism $f : X \to Y$ of s-schemes over S, there is an inverse image map
$$f^! : \mathbb{E}(Y/S, v) \to \mathbb{E}(X/S, f^*v).$$
This contravariance is induced by the purity isomorphism $\mathfrak{p}_f : f^! \simeq f^*$ (Paragraph 2.1.7).

(4) *Product.* If $\mathbb{E}$ is equipped with a multiplication map $\mu_{\mathbb{E}} : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$, then for s-morphisms $p : X \to S$ and $q : Y \to X$, and any K-theory classes $v \in \mathrm{K}(X)$ and $w \in \mathrm{K}(Y)$, there is a map
$$\mathbb{E}(Y/X, w) \otimes \mathbb{E}(X/S, v) \to \mathbb{E}(Y/S, w + q^*v).$$
Given maps $y : \mathrm{Th}_Y(w)[m] \to q^!\mathbb{E}_X$ and $x : \mathrm{Th}_X(v)[n] \to p^!\mathbb{E}_S$, the product $y.x$ is defined as follows:
$$\mathrm{Th}_Y(w + q^*v)[m + n] \xrightarrow{y \otimes \mathrm{Id}} q^!\mathbb{E}_X \otimes \mathrm{Th}_Y(q^*v)[n] \xrightarrow{Ex^{!*}_{\otimes}} q^!(\mathbb{E}_X \otimes \mathrm{Th}_Y(v)[n])$$
$$\xrightarrow{q^!(\mathrm{Id} \otimes x)} q^!(\mathbb{E}_X \otimes p^!\mathbb{E}_S) \xrightarrow{Ex^{!*}_{\otimes}} q^!p^!(\mathbb{E}_S \otimes \mathbb{E}_S) \xrightarrow{\mu_{\mathbb{E}}} q^!p^!(\mathbb{E}_S) = (pq)^!(\mathbb{E}_S).$$

These structures satisfy the usual properties stated by Fulton and MacPherson (functoriality, base change formula both with respect to base change and étale contravariance, compatibility with pullbacks and projection formulas; see [Dég18a, 1.2.8] for the precise formulation).

Remark 2.2.8. One of the main objectives of this paper concerns the extension of contravariant functoriality from *étale* morphisms to *smoothable lci* morphisms. This will be achieved in Theorem 4.2.1.

Remark 2.2.9. Note that a particular case of the product of Paragraph 2.2.7(4) is the cap-product:
$$\cap : \mathbb{E}(X, v) \otimes \mathbb{E}(X/S, w) \to \mathbb{E}(X/S, w - v).$$

The localization theorem (Paragraph 2.1.6) gives the following direct corollary:

Proposition 2.2.10. *Let $i : Z \to X$ be a closed immersion of s-schemes over S, with quasi-compact complementary open immersion $j : U \to X$. Then there exists, for any $e \in \mathrm{K}(X)$, a canonical exact triangle of spectra*
$$\mathbb{E}(Z/S, e) \xrightarrow{i_*} \mathbb{E}(X/S, e) \xrightarrow{j^*} \mathbb{E}(U/S, e)$$
called the localization triangle. *Moreover, this triangle is natural with respect to the contravariance in S, the contravariance in X/S for étale S-morphisms, and the covariance in X/S for proper S-morphisms (see parts (1)–(3) of Paragraph 2.2.7).*

A special case of the naturality in Proposition 2.2.10 is the following:

Corollary 2.2.11. *Suppose given a commutative square*
$$\begin{tikzcd} T \arrow[r, "l"] \arrow[d, "k"'] & Z' \arrow[d, "j"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$
of closed immersions of s-schemes over S. Assume that the respective complementary open immersions i', j', k', and l' are quasi-compact. Then the localization triangles (Proposition 2.2.10) assemble into a commutative diagram of spectra
$$\begin{tikzcd} \mathbb{E}(T/S, e) \arrow[r, "l_*"] \arrow[d, "k_*"] & \mathbb{E}(Z'/S, e) \arrow[r, "l'^*"] \arrow[d, "j_*"] & \mathbb{E}(Z' - T/S, e) \arrow[d, "\tilde{j}_*"] \\ \mathbb{E}(Z/S, e) \arrow[r, "i_*"] \arrow[d, "k'^*"] & \mathbb{E}(X/S, e) \arrow[r, "i'^*"] \arrow[d, "j'^*"] & \mathbb{E}(X - Z/S, e) \arrow[d, "\tilde{j}'^*"] \\ \mathbb{E}(Z - T/S, e) \arrow[r, "\tilde{i}_*"] & \mathbb{E}(X - Z'/S, e) \arrow[r, "\tilde{i}'^*"] & \mathbb{E}(X - Z \cup Z'/S, e) \end{tikzcd}$$
for any $e \in \mathrm{K}(X)$. Here $\tilde{j}$, $\tilde{i}$ denote the obvious closed immersions obtained by restriction, and $\tilde{i}'$, $\tilde{j}'$ the complementary open immersions.

Proposition 2.2.12. *Suppose $\mathbb{E} \in \mathscr{SH}(S)$ is equipped with a multiplication map $\mu_{\mathbb{E}} : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$. Consider cartesian squares of s-schemes over S:*

$$\begin{tikzcd} T \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Delta_Z"] & Y \arrow[d] \arrow[dr, phantom, "\Delta_U"] & V \arrow[l, "k'"'] \arrow[d] \\ Z \arrow[r, "i"'] & X & U \arrow[l, "i'"] \end{tikzcd}$$

such that i and k are closed immersions with quasi-compact complementary open immersions i' and k', respectively. For any $\pi \in \mathbb{E}(Y/X, e')[r]$ with $e' \in \mathrm{K}(Y)$, $r \in \mathbb{Z}$, set:

$$\pi_Z = \Delta_Z^*(\pi) \in \mathbb{E}(T/Z, e')[r], \quad \pi_U = \Delta_U^*(\pi) \in \mathbb{E}(V/U, e')[r].$$

Then the following diagram of localization triangles is commutative:

$$\begin{tikzcd} \mathbb{E}(Z/S, e) \arrow[r, "i_*"] \arrow[d, "\gamma_{\pi_Z}"] & \mathbb{E}(X/S, e) \arrow[r, "i'^*"] \arrow[d, "\gamma_{\pi}"] & \mathbb{E}(U/S, e) \arrow[d, "\gamma_{\pi_U}"] \\ \mathbb{E}(T/S, e+e')[r] \arrow[r, "k_*"] & \mathbb{E}(Y/S, e+e')[r] \arrow[r, "k'^*"] & \mathbb{E}(V/S, e+e')[r] \end{tikzcd}$$

where γ_x denotes multiplication by $x \in \{\pi, \pi_Z, \pi_U\}$.

Proof. The left-hand square commutes by the projection formula. The right-hand square commutes since products are compatible with base change. □

2.2.13. Suppose that $\mathbb{E}$ is equipped with a multiplication $\mu_{\mathbb{E}} : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$ as in Paragraph 2.2.7(4). If this multiplication is unital, resp. associative, resp. commutative, then the bivariant theory represented by $\mathbb{E}$ inherits the same property. This is in particular the case when $\mathbb{E}$ is equipped with an $\mathcal{E}_\infty$-ring structure.

For example, assume that the multiplication is commutative in the sense that it is further equipped with a commutative diagram in $\mathscr{SH}(S)$:

$$\begin{tikzcd} \mathbb{E} \otimes \mathbb{E} \arrow[r, "\sim", "\tau"'] \arrow[d, "\mu_{\mathbb{E}}"'] & \mathbb{E} \otimes \mathbb{E} \arrow[d, "\mu_{\mathbb{E}}"] \\ \mathbb{E} \arrow[r, equal] & \mathbb{E}, \end{tikzcd}$$

where τ is the isomorphism swapping the two factors. Given s-schemes $p : X \to S$ and $q : Y \to S$ and $x \in \mathbb{E}(X/S, v)[-m]$ and $y \in \mathbb{E}(Y/S, w)[-n]$ (where $m, n \in \mathbb{Z}$ and $v \in \mathrm{K}(X)$, $w \in \mathrm{K}(Y)$), consider the cartesian square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "q"] \\ X \arrow[r, "p"'] & S. \end{tikzcd}$$

Then there is an identification

$$\Delta^*(x).y \simeq (-1)^{m+n}.\big(\Delta^*(y).x\big)$$

in $\mathbb{E}(X \times_S Y/S, \pi_1^* v + \pi_2^* w)[-m-n] \simeq \mathbb{E}(X \times_S Y/S, \pi_2^* w + \pi_1^* v)[-m-n]$ using the permutation isomorphism $\pi_2^* w + \pi_1^* v \simeq \pi_1^* v + \pi_2^* w$.

2.2.14. For any $\mathbb{E} \in \mathscr{SH}(S)$ and $v \in \mathrm{K}(S)$, the functor $X \mapsto \mathbb{E}(X/S, v)$ satisfies descent with respect to Nisnevich squares and abstract blow-up squares (hence satisfies cdh descent), on the category of s-schemes over S.

2.3. **Orientations and systems of fundamental classes.** Following Fulton–MacPherson, we now introduce the notion of *orientation* of a morphism f. As we recall in the next subsection, any choice of orientation gives rise to a Gysin map in bivariant theory (Paragraph 2.4.1). The fundamental classes we construct in Section 3 will be examples of orientations.

For simplicity, throughout this discussion we will restrict our attention to the bivariant theory represented by the sphere spectrum $\mathbb{S}$:

Notation 2.3.1. We set

$$H(X/S,v) := \mathbb{S}(X/S,v) = \mathrm{Maps}_{S\mathscr{H}(X)}(\mathrm{Th}_X(v), p^!(\mathbb{S}_S))$$

for any s-morphism $p : X \to S$ and any $v \in \mathrm{K}(X)$. We will refer to this simply as *bivariant* $\mathbb{A}^1$*-theory*. Similarly, we set $H(X,v) := \mathbb{S}(X,v)$ and more generally $H_Z(X,v) := \mathbb{S}_Z(X,v) = \mathbb{S}(Z/X,-v)$, where Z is a closed subscheme of X.

Definition 2.3.2. Let S be a scheme and $f : X \to S$ an s-morphism. An *orientation* of f is a pair (η_f, e_f), where

$$\eta_f \in H(X/S, e_f)$$

and $e_f \in \mathrm{K}(X)$. When there is no risk of confusion, we write simply η_f instead of (η_f, e_f).

Remark 2.3.3. The above use of the term "orientation" is taken from [FM81]. We warn the reader however that it is *unrelated* to the notion of "oriented motivic spectrum" (see Definition 4.4.1).

Example 2.3.4. Let $f : X \to S$ be a smooth s-morphism with tangent bundle T_f. The purity isomorphism $\mathfrak{p}_f : \Sigma^{T_f} f^* \xrightarrow{\sim} f^!$ (Paragraph 2.1.7) induces a canonical isomorphism

$$\eta_f : \mathrm{Th}_X(T_f) \xrightarrow{\sim} f^!(\mathbb{S}_S).$$

This defines a canonical orientation $\eta_f \in H(X/S, \langle T_f \rangle)$.

It will be useful to have the following description of the purity isomorphism $\mathfrak{p}_f$ (cf. [CD19, Def. 2.4.25, Cor. 2.4.37]). We begin by considering the cartesian square Δ:

$$\begin{tikzcd} X \times_S X \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

Write $\delta : X \to X \times_S X$ for the diagonal embedding. Then $\Sigma^{-T_f}\mathfrak{p}_f$ is inverse to the composite:

(2.3.4.a) $$\Sigma^{-T_f} f^! \xrightarrow{\sim} \delta^! f_1^* f^! \xrightarrow{Ex^{*!}} \delta^! f_2^! f^* = f^*$$

Here the first isomorphism is induced by the relative purity isomorphism $\Sigma^{-T_f} \simeq \delta^! f_1^*$ (Paragraph 2.1.8), modulo the tautological identification between T_f and the normal bundle N_δ. The exchange transformation $Ex^{*!}$ is invertible because f is smooth (see Paragraph 2.1.9).

Definition 2.3.5. Let $f : X \to S$ be a smooth s-morphism. The *fundamental class* of f is the orientation $\eta_f \in H(X/S, \langle T_f \rangle)$ defined in Example 2.3.4.

Definition 2.3.6. Let S be a scheme and let $\mathscr{C}$ be a class of morphisms between s-schemes over S. A *system of fundamental classes* for $\mathscr{C}$ consists of the following data:

(i) *Fundamental classes.* For each morphism $f : X \to Y$ in $\mathscr{C}$, there is an orientation $(\eta_f^{\mathscr{C}}, e_f)$.

(ii) *Normalisation.* For $f = \mathrm{Id}_S$ the identity morphism, there is an isomorphism $e_f \simeq 0$ in $\mathrm{K}(S)$, and an isomorphism $\eta_f^{\mathscr{C}} \simeq 1$ in $H(S/S, e_f) \simeq H(S/S, 0)$.

(iii) *Associativity formula.* Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathscr{C}$ such that the composite $g \circ f$ is also in $\mathscr{C}$. Then there are identifications

(2.3.6.a) $$e_{g\circ f} \simeq e_f + f^*(e_g)$$

in $\mathrm{K}(X)$ and

$$\eta_g^{\mathscr{C}}.\eta_f^{\mathscr{C}} \simeq \eta_{g\circ f}^{\mathscr{C}}$$

in $H(X/Z, e_{g\circ f})$.

We say that a system of fundamental classes $(\eta_f^{\mathscr{C}})_f$ is *stable under transverse base change* if it is equipped with the following further data:

(iv) *Transverse base change formula.* For any *tor-independent* cartesian square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "p"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

such that f and g are in $\mathscr{C}$, there are identifications $e_g \simeq q^*(e_f)$ in $\mathrm{K}(Y)$ and $\Delta^*(\eta_f^{\mathscr{C}}) \simeq \eta_g^{\mathscr{C}}$ in $H(X/S, e_g)$.

Remark 2.3.7. The previous definition admits an obvious extension to general bivariant theories (i.e., the contexts of Definition 2.2.1 and Paragraph 4.3.4), and we will freely use this extension. Then our definition is both a generalization of [FM81, I, 2.6.2] and of [Dég18a, 2.1.9].

Remark 2.3.8. Let S be a scheme and let $\mathscr{C}$ be a class of morphisms between s-schemes over S. Suppose $(\eta_f)_f$ is a system of fundamental classes for $\mathscr{C}$ as in Definition 2.3.6. For any morphisms $f: X \to Y$ and $g: Y \to Z$ in $\mathscr{C}$ such that the composite $g \circ f$ is also in $\mathscr{C}$, the isomorphisms (2.3.6.a) induce canonical isomorphisms of functors $S\mathscr{H}(Z) \to S\mathscr{H}(X)$

$$(g\circ f)^*\Sigma^{e_{gf}} \simeq \Sigma^{e_f} f^*\Sigma^{e_g} g^*,$$

and isomorphisms of Thom spaces

$$\begin{aligned} \mathrm{Th}(e_{gf}) &\simeq \mathrm{Th}(f^*e_g) \otimes \mathrm{Th}(e_f) \\ &\simeq f^*\,\mathrm{Th}(e_g) \otimes \mathrm{Th}(e_f) \end{aligned}$$

in $S\mathscr{H}(X)$.

Example 2.3.9. It follows from [Ayo07, 1.7.3] that the family of orientations η_f for f smooth (Definition 2.3.5) forms a system of fundamental classes for the class of smooth s-morphisms. Moreover, this system is stable under (arbitrary) base change: explicit homotopies $\Delta^*(\eta_f) \simeq \eta_g$ as in Definition 2.3.6(iv) are provided by the deformation to the normal cone space, as in the proof of [Dég18a, Lem. 2.3.13] (where the right-hand square (3) can be ignored).

Example 2.3.10. In Section 3, we will extend Example 2.3.9 to the class of *smoothable lci* s-morphisms. Recall from [Ill06] that an lci morphism $f: X \to S$ admits a perfect cotangent complex $L_f = L_{X/S}$, which induces a point $\langle L_f\rangle \in \mathrm{K}(X)$ (which represents the "virtual tangent bundle" of f in $\mathrm{K}_0(X)$ in the sense of [SGA6, Exp. VIII]). For example, if f is smooth then $\langle L_f\rangle = \langle T_f\rangle$ is the class of the relative tangent bundle; if f is a regular closed immersion then $\langle L_f\rangle = -\langle N_f\rangle$, where $\langle N_f\rangle$ is the class of the normal bundle. Every *smoothable* lci morphism f factors through a regular closed immersion i followed by a smooth morphism p, and such a factorization induces an identification $\langle L_f\rangle \simeq i^*\langle T_p\rangle - \langle N_i\rangle$ in K-theory. More generally, given lci morphisms $f: X \to Y$ and $g: Y \to Z$, the composite $g\circ f$ is lci with cotangent complex canonically identified in $\mathrm{K}(Y)$ with

(2.3.10.a) $$\langle L_{g\circ f}\rangle \simeq \langle L_f\rangle + f^*\langle L_g\rangle.$$

The fundamental class of a smoothable lci s-morphism $f: X \to S$ will then be an orientation in $H(X/S, \langle L_f\rangle)$ (see Theorem 3.3.2).

We finish with a discussion of *strong* orientations and duality isomorphisms.

Definition 2.3.11. Let $f: X \to S$ be an s-morphism, and (η_f, e_f) an orientation of f.

(1) We say that (η_f, e_f) is *strong* if for any $v \in \mathrm{K}(X)$, cap-product with η_f induces an isomorphism
$$\gamma_{\eta_f} : H(X, v) \to H(X/S, e_f - v), \quad x \mapsto x.\eta_f.$$
In that case, we refer to γ_{η_f} as the *duality isomorphism* associated with the strong orientation η_f.

(2) We say that (η_f, e_f) is *universally strong* if the morphism $\eta_f : \mathrm{Th}_X(e_f) \to f^!(\mathbb{S}_S)$ is an isomorphism.

Remark 2.3.12. It follows immediately from the construction of the cap product that a universally strong orientation is strong.

Example 2.3.13. If f is *smooth*, then the orientation η_f of Definition 2.3.5 is universally strong by the purity theorem (Paragraph 2.1.7).

The following lemma explains the terminology "universally strong".

Lemma 2.3.14. *Let $f : X \to S$ be an s-morphism. Let (η_f, e_f) be a* universally strong *orientation for f. Then for any cartesian square*

$$\begin{tikzcd} X_T \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] & S \end{tikzcd}$$

with p smooth, the orientation $\Delta^*(\eta_f) \in H(X_T/T, q^*(e_f))$ of g is universally strong.

Proof. Since p is smooth, the exchange transformation $Ex^{*!} : p^*f^! \to g^!q^*$ (Paragraph 2.1.9) is invertible. It follows then from the construction of the change of base map Δ^* that $\Delta^*(\eta_f) : \mathrm{Th}_{X_T}(q^*(e_f)) \to g^!p^*(\mathbb{S}_S)$ is an isomorphism, as claimed. □

2.4. **Gysin maps.**

Definition 2.4.1. Let $f : X \to Y$ be a morphism of s-schemes over S. Then any orientation $\eta_f \in H(X/Y, e_f)$ gives rise to a *Gysin map*:
$$\eta_f^! : H(Y/S, e) \to H(X/S, e_f + f^*(e)), \quad x \mapsto \eta_f.x$$
using the product in bivariant $\mathbb{A}^1$-theory, for all $e \in \mathrm{K}(Y)$. When the orientation η_f is clear, we simply put: $f^! = \eta_f^!$.

Proposition 2.4.2. *Let S be a scheme and let $\mathscr{C}$ be a class of morphisms between s-schemes over S. Suppose $(\eta_f)_f$ is a system of fundamental classes for $\mathscr{C}$ as in Definition 2.3.6.*

(i) Functoriality. *Let f and g be morphisms in $\mathscr{C}$ such that the composite $g \circ f$ is also in $\mathscr{C}$. Then for every $e \in \mathrm{K}(Z)$ there is an induced identification*
$$(g \circ f)^! \simeq f^! \circ g^!$$
of Gysin maps $H(Z/S, e) \to H(X/S, e_{g\circ f} + (g \circ f)^*(e))$, modulo the identification of Remark 2.3.8.

(ii) Transverse base change. *Assume that the system $(\eta_f)_f$ is stable under transverse base change. Suppose given a* tor-independent *cartesian square of s-schemes over S of the form*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"'] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where f and g are in $\mathscr{C}$, and u and v are proper. Then for every $e \in \mathrm{K}(Y)$ there is an induced identification
$$f^! \circ v_* \simeq u_* \circ g^!$$

of maps $H(Y'/S, v^*(e)) \to H(X/S, e_f + f^*(e))$, *where we use the identification* $e_g + g^*v^*(e) \simeq u^*(e_f + f^*(e))$ *in* $\mathrm{K}(X')$.

Example 2.4.3. If f is a smooth s-morphism, then the fundamental class η_f (Definition 2.3.5) gives rise to canonical Gysin maps

$$f^! : H(X/S, e) \to H(Y/S, \langle T_f \rangle + f^*(e)).$$

This extends the contravariant functoriality from étale morphisms to smooth morphisms.

Lemma 2.4.4. *Let* X *be an s-scheme over* S *and let* $p : E \to X$ *be a vector bundle. Then the tangent bundle* T_p *is identified with* $p^{-1}(E)$ *and the Gysin map*

$$p^! : H(X/S, e) \to H(E/S, p^*\langle E \rangle + p^*e)$$

is invertible.

Proof. In view of the construction of the Gysin map, the claim follows directly from the facts that the morphism $\eta_p : \mathrm{Th}_X(p^*E) \to p^!(\mathbb{S}_X)$ is invertible, and that the functor $p^* : \mathscr{SH}(X) \to \mathscr{SH}(E)$ is fully faithful (Paragraph 2.1.3). □

Definition 2.4.5. In the context of Lemma 2.4.4, we define the *Thom isomorphism*[8]

$$\phi_{E/X} : H(E/S, e) \to H(X/S, e - \langle E \rangle),$$

associated to E/X, as the inverse of the Gysin map $p^! : H(X/S, e - \langle E \rangle) \to H(E/S, e)$.

Remark 2.4.6. The Thom isomorphism satisfies the properties of compatibility with base change and with direct sums (that is, $\phi_{E\oplus F/X} \simeq \phi_{E\oplus F/F} \circ \phi_{F/X}$ for vector bundles E and F over X). These follow respectively from the compatibility of the Gysin morphisms $p^!$ with base change and with composition.

We conclude this subsection by recording the naturality of the localization sequences (Proposition 2.2.10) with respect to Gysin maps of smooth morphisms.

Proposition 2.4.7. *Consider a commutative diagram of cartesian squares of* S*-schemes*

$$\begin{tikzcd} T \arrow[r, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] & V \arrow[l, "l"'] \arrow[d, "h"] \\ Z \arrow[r, "i"'] & X & U \arrow[l, "j"] \end{tikzcd}$$

such that f *is smooth,* i *is a closed immersion and* j *is the complementary open immersion. Then for any* $e \in \mathrm{K}(X)$, *the diagram of spectra*

$$\begin{tikzcd} H(Z/S, e) \arrow[r, "i_*"] \arrow[d, "g^!"] & H(X/S, e) \arrow[r, "j^!"] \arrow[d, "f^!"] & H(U/S, e) \arrow[d, "h^!"] \\ H(T/S, \langle T_g \rangle + e) \arrow[r, "k_*"] & H(Y/S, \langle T_f \rangle + e) \arrow[r, "l^!"] & H(V/S, \langle T_h \rangle + e) \end{tikzcd}$$

commutes.

Proof. The right-hand square commutes by the associativity property of fundamental classes of smooth morphisms (Example 2.3.9). The left-hand square commutes by the naturality of the relative purity isomorphism of Morel–Voevodsky [Hoy15, Prop. A.4], in view of the construction of the fundamental classes η_f and η_g (Example 2.3.4). □

[8] Not to be confused with the Thom isomorphism of (4.4.1.a).

2.5. **Purity transformations.** The notion of orientation seen in the preceding subsection is part of our twisted version of the bivariant formalism of Fulton and MacPherson. We state in this subsection a variant, or a companion, of this notion in the spirit of Grothendieck's six functors formalism.

2.5.1. Let us fix an s-morphism $f : X \to S$, and an orientation (η_f, e_f). According to our definitions, the class $\eta_f \in H(X/S, e_f)$ can be seen as a morphism in $\mathscr{SH}(X)$:

$$\eta_f : \mathrm{Th}_X(e_f) \to f^!(\mathbb{S}_S).$$

This gives rise to a natural transformation

$$\mathfrak{p}(\eta_f) : \Sigma^{e_f} f^* \to f^! \tag{2.5.1.a}$$

associated to the orientation η_f, defined as the following composite:

$$f^*(-) \otimes \mathrm{Th}_X(e_f) \xrightarrow{\mathrm{Id} \otimes \eta_f} f^*(-) \otimes f^!(\mathbb{S}_S) \xrightarrow{Ex^{!*}_{\otimes}} f^!(- \otimes \mathbb{S}_S) \simeq f^!,$$

where the exchange transformation $Ex^{!*}_{\otimes}$ is as in Paragraph 2.1.10.

Remark 2.5.2.

(i) Suppose f is smooth and consider the canonical orientation η_f of Definition 2.3.5. It follows by construction that in this case the associated purity transformation $\mathfrak{p}(\eta_f)$ is nothing else than the purity isomorphism $\mathfrak{p}_f$ (2.3.4.a). In particular, $\mathfrak{p}(\eta_f)$ is an isomorphism in this case.

(ii) Note that the datum of an orientation (η_f, e_f) and that of the associated purity transformation $\mathfrak{p}(\eta_f)$ are essentially interchangeable. Indeed we recover η_f by evaluating $\mathfrak{p}(\eta_f)$ at the unit object $\mathbb{S}_S$.

2.5.3. Consider the notation of the previous definition. Then one associates to $\mathfrak{p}(\eta_f)$, using the adjunction properties, two natural transformations:

$$\mathrm{tr}_f : f_! \Sigma^{e_f} f^* \to \mathrm{Id}$$
$$\mathrm{cotr}_f : \mathrm{Id} \to f_* \Sigma^{-e_f} f^!.$$

The first (resp. second) natural transformation will be called the *trace map* (resp. *co-trace map*) associated with the orientation η_f, following the classical usage in the literature. These two maps are functorial incarnations of the Gysin map defined earlier (Paragraph 2.4.1), as we will see later (see Paragraph 4.3.3).

The notion of a system of fundamental classes (Definition 2.3.6) was introduced to reflect the functoriality of Gysin morphisms. For completeness, we now formulate the analogous functoriality property for the associated purity transformations.

Proposition 2.5.4. *Let S be a scheme and let $\mathscr{C}$ be a class of morphisms between s-schemes over S. Suppose $(\eta_f)_f$ is a system of fundamental classes for $\mathscr{C}$ as in Definition 2.3.6.*

(i) Functoriality. *Let f and g be morphisms in $\mathscr{C}$ such that the composite $g \circ f$ is also in $\mathscr{C}$. Then there is a commutative square*

$$\begin{tikzcd}
\Sigma^{e_{g\circ f}}(g \circ f)^* \arrow[r, "\mathfrak{p}(\eta_{g\circ f})"] \arrow[d, equal] & (g \circ f)^! \arrow[d, equal] \\
\Sigma^{e_f} f^* \Sigma^{e_g} g^* \arrow[r, "\mathfrak{p}(\eta_f)*\mathfrak{p}(\eta_g)"] & f^! g^!
\end{tikzcd}$$

where the left-hand vertical isomorphism is from Remark 2.3.8, and the lower horizontal arrow is the horizontal composition of the 2-morphisms $\mathfrak{p}(\eta_f)$ and $\mathfrak{p}(\eta_g)$.

(ii) Transverse base change. *Assume that the system* $(\eta_f)_f$ *is stable under transverse base change. Suppose given a* tor-independent *cartesian square of the form*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where f and g are in $\mathscr{C}$. Then there are commutative squares of natural transformations:

$$\begin{tikzcd} u^*\Sigma^{e_f} f^* \arrow[r, equal] \arrow[d, "u^* * \mathfrak{p}_f"] & \Sigma^{e_g} u^* f^* \arrow[r, equal] \arrow[d, phantom, "(a)"] & \Sigma^{e_g} g^* v^* \arrow[d, "\mathfrak{p}_g * v^*"] \\ u^* f^! \arrow[rr, "Ex^{*!}"'] & & g^! v^* \end{tikzcd} \qquad \begin{tikzcd} \Sigma^{e_f} f^* v_* \arrow[r, "Ex^*_*"] \arrow[d, "\mathfrak{p}_f * v_*"] & \Sigma^{e_f} u_* g^* \arrow[r, equal] \arrow[d, phantom, "(b)"] & u_* \Sigma^{e_g} g^* \arrow[d, "u_* * \mathfrak{p}_g"] \\ f^! v_* \arrow[rr, "\sim"] & & u_* g^! \end{tikzcd}$$

If u and v are s-morphisms, then there are also commutative squares

$$\begin{tikzcd} u^!\Sigma^{e_f} f^* \arrow[r, equal] \arrow[d, "u^! * \mathfrak{p}_f"] & \Sigma^{e_g} u^! f^* \arrow[r, equal] \arrow[d, phantom, "(c)"] & \Sigma^{e_g} g^* v^! \arrow[d, "\mathfrak{p}_g * v^!"] \\ u^! f^! \arrow[rr, equal] & & g^! v^! \end{tikzcd} \qquad \begin{tikzcd} u_!\Sigma^{e_g} g^* \arrow[r, equal] \arrow[d, "u_! * \mathfrak{p}_g"] & \Sigma^{e_f} u_! g^* \arrow[r, "\sim"] \arrow[d, phantom, "(d)"] & \Sigma^{e_f} f^* v_! \arrow[d, "\mathfrak{p}_f * v_!"] \\ u_! g^! \arrow[rr, "Ex^!_!"'] & & f^! v_! \end{tikzcd}$$

Proof. Claim (i) follows from axioms (ii) and (iii) of Definition 2.3.6.

In claim (ii), commutativity of (a) follows directly from axiom (iv) of Definition 2.3.6. The square (b) can be derived from (a) by applying u_* on the left and v_* on the right, and using the naturality of the unit and counit of the adjunctions (u^*, u_*) and (v^*, v_*), respectively. Similarly, commutativity of square (c) will follow similarly from (d). For (d), we may unravel the definition of the purity transformation (Paragraph 2.5.1) to write the square as follows:

$$\begin{tikzcd} u_!\Sigma^{e_g} g^* \arrow[r, equal] \arrow[d, equal] & \Sigma^{e_f} u_! g^* \arrow[r, "\sim"] & \Sigma^{e_f} f^* v_! \arrow[d, equal] \\ u_!(g^*(-) \otimes \Sigma^{e_g} g^*(\mathbb{S})) \arrow[d, "u_!(g^*(-)\otimes \mathfrak{p}_g(\mathbb{S}))"] & & f^* v_!(-) \otimes \Sigma^{e_f} f^*(\mathbb{S}) \arrow[d, "f^* v_!(-)\otimes \mathfrak{p}_f(\mathbb{S})"] \\ u_!(g^*(-) \otimes g^!(\mathbb{S})) \arrow[d, "Ex^{*!}_{\otimes}"] & & f^* v_!(-) \otimes f^!(\mathbb{S}) \arrow[d, "Ex^{*!}_{\otimes}"] \\ u_! g^! \arrow[rr, "Ex^!_!"'] & & f^! v_! \end{tikzcd}$$

Observing that $\mathfrak{p}_g(\mathbb{S}) = \mathfrak{p}_g(v^*(\mathbb{S}))$, we can use square (a) to decompose the left-hand middle arrow into a natural transformation induced by $\mathfrak{p}_f(\mathbb{S})$ and an exchange transformation. The commutativity of the resulting square is then a formal exercise. □

Remark 2.5.5. Suppose that the class $\mathscr{C}$ contains all identity morphisms and is closed under composition, and let $\mathscr{S}^{\mathscr{C}}$ denote the subcategory of the category of schemes $\mathscr{S}$ whose morphisms all belong to $\mathscr{C}$. At the level of homotopy categories, Proposition 2.5.4(i) implies that the assignment $f \mapsto \mathfrak{p}(\eta_f)$ defines a natural transformation of contravariant pseudofunctors

$$\mathfrak{p} : \mathrm{Ho}(S\mathscr{H})^{e*} \to \mathrm{Ho}(S\mathscr{H})^!$$

on the category $\mathscr{S}^{\mathscr{C}}$, where the notation is as follows:

- $\mathscr{T}ri$ denotes the $(2,1)$-category of large triangulated categories, triangulated functors, and invertible triangulated natural transformations.

- $\mathrm{Ho}(\mathscr{SH})^!$ is the pseudofunctor $(\mathscr{S}^{\mathscr{C}})^{\mathrm{op}} \to \mathscr{T}ri$, given by the assignments
$$S \mapsto \mathrm{Ho}(\mathscr{SH}(S)), f \mapsto f^!.$$
- Similarly $\mathrm{Ho}(\mathscr{SH})^{e*}$ is the pseudofunctor $(\mathscr{S}^{\mathscr{C}})^{\mathrm{op}} \to \mathscr{T}ri$ given by the assignments
$$S \mapsto \mathrm{Ho}(\mathscr{SH}(S)), f \mapsto \Sigma^{e_f} f^*.$$

The expected enhancement to a natural transformation at the level of ∞-categories requires further work that we do not undertake in this paper.

We can also reformulate the transverse base change property (Proposition 2.5.4(ii)) in terms of the (co)trace maps.

Corollary 2.5.6. *Under the assumptions of Proposition 2.5.4(ii), the following diagrams commute:*

$$\begin{tikzcd}
f_!\Sigma^{e_f} f^* v_! \arrow[r, "\sim"] \arrow[d, "\mathrm{tr}_f * v_!"'] & f_!\Sigma^{e_f} u_! g^* \arrow[r, equal] & f_! u_! \Sigma^{e_g} g^* \arrow[r, equal] & v_! g_! \Sigma^{e_g} g^* \arrow[d, "v_! * \mathrm{tr}_g"] \\
v_! \arrow[rrr, equal] & & & v_!
\end{tikzcd}$$

$$\begin{tikzcd}
v^! \arrow[rrr, equal] \arrow[d, "\mathrm{cotr}_g * v^!"'] & & & v^! \arrow[d, "v^! * \mathrm{cotr}_f"] \\
g_*\Sigma^{-e_g} g^! v^! \arrow[r, equal] & g_*\Sigma^{-e_g} u^! f^! \arrow[r, equal] & g_* u^! \Sigma^{-e_f} f^! \arrow[r, "\sim"] & v^! f_* \Sigma^{-e_f} f^!
\end{tikzcd}$$

3. Construction of fundamental classes

3.1. **Euler classes.** Before proceeding to our construction of fundamental classes, we begin with a discussion of Euler classes in the setting of bivariant theories. Our basic definition is very simple and can be formulated unstably.

3.1.1. Let X be a scheme and E be a vector bundle over X. Recall that the Thom space $\mathrm{Th}_X(E) \in \mathscr{SH}(X)$ is in fact the suspension spectrum of a pointed motivic space in $\mathscr{H}_\bullet(X)$. Moreover, the latter can be modelled by the pointed Nisnevich sheaf of sets
$$\mathrm{Th}_X(E) := \mathrm{coKer}(E^\times \to E),$$
where $E^\times$ is complement of the zero section.

Note that Thom spaces are functorial with respect to monomorphisms of vector bundles. That is, if $\nu : F \to E$ is a monomorphism of vector bundles over X, one gets a canonical morphism of pointed sheaves:
$$\nu_* : \mathrm{Th}_X(F) \to \mathrm{Th}_X(E).$$

Definition 3.1.2. Let E be a vector bundle over a scheme X, and s be its zero section. We can regard s as a monomorphism of vector bundles $s : X \to E$. We define the *Euler class* $e(E)$ of E/X as the induced map in $\mathscr{H}_\bullet(X)$:
$$s_* : X_+ = \mathrm{Th}_X(X) \to \mathrm{Th}_X(E).$$

Remark 3.1.3. We will often view the Euler class as a class
$$e(E) \in H(Y, \langle E \rangle) \simeq H(Y/Y, -\langle E \rangle),$$
via the canonical map
$$\mathrm{Maps}_{\mathscr{H}_\bullet(X)}(X_+, \mathrm{Th}_X(E)) \to \mathrm{Maps}_{\mathscr{SH}(X)}(\mathbb{S}_X, \mathrm{Th}_X(E)).$$
It can then also be realized as a class in the (twisted) cohomology spectrum of any motivic ring spectrum $\mathbb{E}$ (Definition 4.1.5). When $\mathbb{E}$ is oriented, our Euler class coincides with the top Chern class (see Paragraph 4.4.3). When $\mathbb{E}$ is the Milnor–Witt spectrum (Example 4.4.6) it recovers the classical Euler class in the Chow–Witt group.

It is easy to see that Euler classes commute with base change:

Lemma 3.1.4. *For any morphism $f : Y \to X$ and any vector bundle E over X, the following diagram is commutative:*

$$\begin{tikzcd} Y_+ \arrow[r, "e(f^{-1}(E))"] \arrow[d, equal] & \mathrm{Th}_Y(f^{-1}(E)) \arrow[d, equal] \\ f^*(X_+) \arrow[r, "f^*(e(E))"] & f^*(\mathrm{Th}_X(E)) \end{tikzcd}$$

Proof. This follows from the fact that the functor f^* commutes with cokernels, and the base change of the zero section of E is the zero section of $f^{-1}(E)$. □

3.1.5. By construction, the Thom space fits into a cofiber sequence in $\mathscr{H}_\bullet(X)$:

$$(E^\times)_+ \to E_+ \to \mathrm{Th}_X(E).$$

By $\mathbb{A}^1$-homotopy invariance, the projection map $E \to X$ induces an isomorphism in $\mathscr{H}_\bullet(X)$, whose inverse is induced by the zero section $s : X \to E$. It follows from our construction that the following diagram commutes:

$$\begin{tikzcd} (E^\times)_+ \arrow[r] \arrow[d, equal] & E_+ \arrow[r] & \mathrm{Th}_X(E) \arrow[d, equal] \\ (E^\times)_+ \arrow[r] & X_+ \arrow[u, "s_*"', "\wr"] \arrow[r, "e(E)"] & \mathrm{Th}_X(E). \end{tikzcd}$$

Definition 3.1.6. For any vector bundle E over X, the *Euler cofiber sequence* is the cofiber sequence

$$(E^\times)_+ \to X_+ \xrightarrow{e(E)} \mathrm{Th}_X(E)$$

in $\mathscr{H}_\bullet(X)$.

The Euler cofiber sequence immediately yields the following characteristic property of Euler classes:

Proposition 3.1.7. *Let E be a vector bundle over X. Any nowhere vanishing section s of $E \to X$ induces a null-homotopy of the morphism $X_+ \to \mathrm{Th}_X(E)$ corresponding to the Euler class $e(E)$. In particular, s induces an identification $e(E) \simeq 0$ in $H(X, \langle E \rangle)$.*

Proof. Such a section s induces a section of the morphism $(E^\times)_+ \to X_+$ in the Euler cofiber sequence. □

Corollary 3.1.8. *Let E be a vector bundle over X. If E contains the trivial line bundle $\mathbb{A}^1_X$ as a direct summand, then there is an identification $e(E) \simeq 0$ in $H(X, \langle E \rangle)$.*

3.1.9. Suppose we have an exact sequence of vector bundles

$$0 \to E' \xrightarrow{\nu} E \to E'' \to 0 \tag{3.1.9.a}$$

over a scheme X. Then the isomorphism $\mathrm{Th}_X(E) \simeq \mathrm{Th}_X(E') \otimes \mathrm{Th}_X(E'')$ in $S\mathscr{H}(X)$ (Paragraph 2.1.5) in fact exists already unstably:

$$\mathrm{Th}_X(E) \simeq \mathrm{Th}_X(E') \wedge \mathrm{Th}_X(E'')$$

in $\mathscr{H}_\bullet(X)$. This is clear when the sequence (3.1.9.a) is split, and one reduces to this case by pulling back to the $\underline{\mathrm{Hom}}_X(E'', E')$-torsor of splittings of the sequence (which is fully faithful by $\mathbb{A}^1$-invariance).

Lemma 3.1.10. *Given an exact sequence of vector bundles as in* (3.1.9.a)*, the following diagram is commutative:*

$$\begin{tikzcd} \mathrm{Th}_X(E') \arrow[r, "\mathrm{Id} \wedge e(E'')"] \arrow[d, equal] & \mathrm{Th}_X(E') \wedge \mathrm{Th}_X(E'') \arrow[d, equal] \\ \mathrm{Th}_X(E') \arrow[r, "\nu_*"] & \mathrm{Th}_X(E) \end{tikzcd}$$

Proof. Argue as above to reduce to the case where (3.1.9.a) is split. □

The additivity property of Euler classes is then a direct corollary:

Proposition 3.1.11. *Given an exact sequence of vector bundles as in* (3.1.9.a)*, the following diagram is commutative:*

$$\begin{tikzcd} X_+ \arrow[r, "e(E')\wedge e(E'')"] \arrow[d, equal] & \mathrm{Th}_X(E') \wedge \mathrm{Th}_X(E'') \arrow[d, equal] \\ X_+ \arrow[r, "e(E)"] & \mathrm{Th}_X(E) \end{tikzcd}$$

Proof. Compose the diagram of Lemma 3.1.10 with the map $e(E') : X_+ \to \mathrm{Th}_X(E')$ (on the left). □

3.2. **Fundamental classes: regular closed immersions.** In this section we construct the fundamental class of a regular closed immersion and demonstrate its expected properties. Before proceeding, we make a brief digression to consider a certain preliminary construction.

3.2.1. Let X be a scheme and consider the diagram

$$\begin{tikzcd} \mathbb{G}_m X \arrow[rr, "j"] \arrow[dr, "\pi"'] & & \mathbb{A}^1 X \arrow[dl, "\bar\pi"] \\ & X & \end{tikzcd}$$

For any $e \in \mathrm{K}(X)$, we have a commutative diagram

$$\begin{tikzcd} H(X, 1-e) \arrow[r, equal] \arrow[d, "\pi^*"'] & H(X/X, e-1) \arrow[r, "\bar\pi^!"] \arrow[d, "\pi^!"'] & H(\mathbb{A}^1 X/X, e) \arrow[d, "j^!"'] \\ H(\mathbb{G}_m X, 1-e) \arrow[r, "\gamma_{\eta_\pi}"] & H(\mathbb{G}_m X/X, e) \arrow[r, equal] & H(\mathbb{G}_m X/X, e) \end{tikzcd}$$

using the identifications $\langle T_\pi \rangle \simeq 1$ in $\mathrm{K}(\mathbb{G}_m X)$, $\langle T_{\bar\pi} \rangle \simeq 1$ in $\mathrm{K}(\mathbb{A}^1 X)$. (Here, as usual, $1 \in \mathrm{K}(X)$ denotes the unit, i.e., the class $\langle \mathbb{A}^1 X \rangle$, for any scheme X.) The right-hand square consists of Gysin maps and commutes by Example 2.3.9; moreover, the morphism $\bar\pi^!$ is invertible (Lemma 2.4.4). In the left-hand square, the morphism γ_{η_π} is the duality isomorphism (Definition 2.3.11) associated to the strong orientation (Example 2.3.13) of π, and the square evidently commutes by construction of the morphisms involved. Since the left vertical arrow π^* admits a retraction s_1^*, given by the inverse image by the unit section $s_1 : X \to \mathbb{G}_m X$ in cohomology, we also get a canonical retraction ν_t of the right vertical arrow $j^!$.

3.2.2. Consider now the localization triangle

$$H(\mathbb{A}^1 X/X, e)[-1] \xrightarrow{j^!} H(\mathbb{G}_m X/X, e)[-1] \xrightarrow{\partial_{s_0}} H(X/X, e)$$

associated with the zero section $s_0 : X \to \mathbb{A}^1_X$. By Paragraph 3.2.1 it is canonically split, and we get an $H(X/X, e)$-linear section

$$\gamma_t : H(X/X, e) \to H(\mathbb{G}_m X/X, e)[-1]$$

of ∂_{s_0}. By linearity, this map is determined uniquely by a morphism

$$\{t\} : \mathbb{S}_{\mathbb{G}_m X} \to \pi^!(\mathbb{S}_X)[-1]$$

in $\mathscr{SH}(\mathbb{G}_m X)$; that is, γ_t is multiplication by $\{t\} \in H(\mathbb{G}_m X/X, 0)[-1]$. By construction, $\{t\}$ is stable under arbitrary base changes. If X is an s-scheme over some base S, we will abuse notation and write γ_t also for the map

$$\gamma_t : H(X/S, e) \to H(\mathbb{G}_m X/S, e)[-1],$$

given again by the assignment $x \mapsto \{t\}.x$.

We now proceed to the construction of the fundamental class.

3.2.3. Let X be an S-scheme and $i : Z \to X$ a regular closed immersion. We write $D_Z X$ or $D(X, Z)$ for the (affine) deformation space $B_{Z\times 0}(X \times \mathbb{A}^1) - B_{Z\times 0}(X \times 0)$, as defined by Verdier (denoted $M(Z/X)$ in [Ver76, §2]); here $B_Z X$ denotes the blow-up of X along Z. This fits into a diagram of tor-independent cartesian squares

$$\begin{tikzcd} N_Z X \arrow[r, "k"] \arrow[d] & D_Z X \arrow[d, "r"] & \mathbb{G}_m X \arrow[l, "h"'] \arrow[d, equal] \\ X \arrow[r, "\{0\}"] & \mathbb{A}^1 X & \mathbb{G}_m X \arrow[l] \end{tikzcd}$$

where k and h are complementary closed/open immersions, and the left-hand arrow is the composite of the projection $p : N_Z X \to Z$ and $i : Z \to X$. Consider the associated localization triangle (Proposition 2.2.10):

$$H(N_Z X/S, e) \xrightarrow{k_*} H(D_Z X/S, e) \xrightarrow{h^!} H(\mathbb{G}_m X/S, e) \xrightarrow{\partial_{N_Z X/D_Z X}} H(N_Z X/S, e)[1]$$

for any $e \in \mathrm{K}(X)$.

Definition 3.2.4. With notation as above, the *specialization to the normal cone* map associated to i is the composite

$$\sigma_{Z/X} : H(X/S, e) \xrightarrow{\gamma_t} H(\mathbb{G}_m X/S, e)[-1] \xrightarrow{\partial_{N_Z X/D_Z X}} H(N_Z X/S, e),$$

where γ_t is the map constructed in Paragraph 3.2.2.

Definition 3.2.5. The *fundamental class* $\eta_i \in H(Z/X, -\langle N_Z X\rangle)$ associated to the regular closed immersion i is the image of $1 \in H(X/X, 0)$ by the composite

$$H(X/X, 0) \xrightarrow{\sigma_{Z/X}} H(N_Z X/X, 0) \xrightarrow{\phi_{N_Z X/Z}} H(Z/X, -\langle N_Z X\rangle),$$

where $\phi_{N_Z X/Z}$ is the Thom isomorphism of $p : N_Z X \to Z$ (Definition 2.4.5). In other words, $\eta_i = \phi_{N_Z X/Z}\big(\sigma_{Z/X}(\{t\})\big)$.

Remark 3.2.6.

(i) By definition of the Thom isomorphism, the fundamental class η_i is determined uniquely by the property that there is a canonical identification

$$p^!(\eta_i) \simeq \sigma_{Z/X}(\{t\})$$

in $H(N_Z X/X, 0)$, where $p : N_Z X \to Z$ is the projection of the normal bundle and we have used the notation of Lemma 2.4.4.

(ii) Our definition of the specialization map is formally very close to the corresponding map in Rost's theory of cycle modules, denoted by $J(X, Z)$ in [Ros96, §11].

(iii) For each regular closed immersion i, the fundamental class defines an orientation $(\eta_i, -\langle N_i\rangle)$ in the sense of Definition 2.3.2. It can be viewed equivalently as a morphism

$$\eta_i : \mathrm{Th}_Z(-N_i) \to i^!(\mathbb{S}_X)$$

in $\mathscr{SH}(Z)$.

(iv) Let us assume that X is an s-scheme over a base S. Then the orientation $(\eta_i, -\langle N_i\rangle)$ gives rise to a Gysin map (Definition 2.4.1):

$$i^! : H(X/S, e) \to H(Z/S, -\langle N_i\rangle + e), \quad x \mapsto \eta_i.x.$$

It follows from the definitions that this map can also be described as the composite:

$$H(X/S, e) \xrightarrow{\sigma_{Z/X}} H(N_Z X/S, e) \xrightarrow{(p^!)^{-1}} H(Z/S, -\langle N_Z X\rangle + e),$$

therefore comparing our construction with that of Verdier [Ver76]. Note also that $\sigma_{Z/X} \simeq p^! i^!$ so that the Gysin map and the specialization map uniquely determine each other.

(v) One can describe the map η_i more concretely as follows. Let us recall the deformation diagram:

$$\begin{tikzcd} \mathbb{G}_m X \arrow[r, hook, "h"] \arrow[rd, "\pi"'] & D_Z X \arrow[d, "r"] & N_Z X \arrow[l, hook', "k"'] \arrow[d, "p"] \\ & X & Z \arrow[l, hook', "i"'] \end{tikzcd}$$

First the map $\{t\} : \mathbb{S}_{\mathbb{G}_m X}[1] \to \pi^!(\mathbb{S}_X)$ corresponds by adjunction and after one desuspension to a map:

$$\sigma_t : \mathbb{S}_{D_Z X} \to h_* \pi^!(\mathbb{S}_X[-1]) = h_* h^* r^!(\mathbb{S}_X[-1]).$$

Then one gets the following composite map:

$$\mathbb{S}_{D_Z X} \xrightarrow{\sigma_t} h_* h^* r^!(\mathbb{S}_X[-1]) \xrightarrow{\text{boundary}} k_! k^! r^!(\mathbb{S}_X) = k_! p^! i^!(\mathbb{S}_X) \simeq k_* p^* \big(\mathrm{Th}_Z(N_Z X) \otimes i^!(\mathbb{S}_X)\big)$$

where the last isomorphism uses the purity isomorphism $\mathfrak{p}_p$. Using the identification $\mathbb{S}_D \simeq r^*(\mathbb{S}_X)$ and the adjunction (r^*, r_*) we deduce a map:

$$\mathbb{S}_Z \to p_* p^* \big(\mathrm{Th}_Z(N_Z X) \otimes i^!(\mathbb{S}_X)\big) \simeq \mathrm{Th}_Z(N_Z X) \otimes i^!(\mathbb{S}_X)$$

where the last isomorphism follows from the $\mathbb{A}^1$-homotopy invariance of $\mathscr{SH}$ (Paragraph 2.1.3). The latter composite is nothing else than the morphism $\mathrm{Th}_Z(N_Z X) \otimes \eta_i$.

3.2.7. Consider a cartesian square where i and k are regular closed immersions

$$\begin{tikzcd} T \arrow[r, "k"] \arrow[d, "q"'] \arrow[rd, phantom, "\Delta"] & Y \arrow[d, "p"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

Then we get a morphism of deformation spaces $D_T Y \to D_Z X$ and similarly a morphism of vector bundles:

$$N_T Y \xrightarrow{\nu} q^{-1} N_Z X \to N_Z X$$

where ν is in general a monomorphism of vector bundles (*i.e.* the codimension of T in Y can be strictly smaller than that of Z in X: there is excess of intersection). We put $\xi = q^{-1} N_Z X / N_T Y$, the excess intersection bundle.

Proposition 3.2.8 (Excess intersection formula)**.** *With notation as above, we have a canonical isomorphism*

$$\Delta^*(\eta_i) \simeq e(\xi).\eta_k$$

in $H(T/Y, -\langle q^{-1} N_Z X \rangle)$, modulo the identification $\langle \xi \rangle + \langle N_T Y \rangle \simeq \langle q^{-1} N_Z X \rangle$, where $e(\xi) \in H(T/T, \langle \xi \rangle)$ is the Euler class of ξ (Definition 3.1.2 and Remark 3.1.3).

Proof. Let us put $D'_T Y = D_Z X \times_X Y$ and $N'_T Y = q^{-1} N_Z X$. Then we get the following commutative diagram of schemes, in which each square is cartesian:

$$\begin{tikzcd} N_T Y \arrow[r, hook] \arrow[d] & D_T Y \arrow[d] & \mathbb{G}_m Y \arrow[l, hook'] \arrow[d, equal] \\ N'_T Y \arrow[r, hook] \arrow[d] & D'_T Y \arrow[d] & \mathbb{G}_m Y \arrow[l, hook'] \arrow[d] \\ N_Z X \arrow[r, hook] & D_Z X & \mathbb{G}_m X \arrow[l, hook'] \end{tikzcd}$$

Therefore, one gets the following commutative diagram:

$$\begin{tikzcd} H(\mathbb{G}_m Y/Y,0)[-1] \arrow[r, "\partial_{T/Y}"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & H(N_T Y/Y,0) \arrow[r, "\phi_{N_T Y/T}"] \arrow[d] \arrow[dr, phantom, "(3)"] & H(T/Y,-\langle N_T Y\rangle) \arrow[d, "\nu_*"] \\ H(\mathbb{G}_m Y/Y,0)[-1] \arrow[r] \arrow[dr, phantom, "(2)"] & H(N'_T Y/Y,0) \arrow[r, "\phi_{N'_T Y/T}"] \arrow[dr, phantom, "(4)"] & H(T/Y,-\langle N'_T Y\rangle) \\ H(\mathbb{G}_m X/X,0)[-1] \arrow[r, "\partial_{Z/X}"'] \arrow[u, "\Delta^*"] & H(N_Z X/Z,0) \arrow[r, "\phi_{N_T Y/T}"'] \arrow[u, "\Delta^*"'] & H(Z/X,-\langle N_Z X\rangle). \arrow[u, "\Delta^*"'] \end{tikzcd}$$

Here the right-hand arrow labelled Δ^* is the change of base map (Paragraph 2.2.7(1)); we have abused notation by also writing Δ^* for the two analogous maps on the left and middle (induced by the obvious cartesian squares). Square (1) (resp. (2)) is commutative because of the naturality of localization triangles with respect to the proper covariance (resp. base change). Square (3) is commutative by definition of ν_*, and square (4) by compatibility of Thom isomorphisms with respect to base change.

Now observe that the image of $\{t\} \in H(\mathbb{G}_m X/X,0)[-1]$ by the counter-clockwise composite in the above diagram is nothing else than the class $\Delta^*(\eta_i) \in H(T/Y,-\langle N'_T Y\rangle)$. Similarly the image by the clockwise composite is the class $\nu_*(\eta_k) \in H(T/Y,-\langle N'_T Y\rangle)$. We conclude using Lemma 3.1.10. □

Example 3.2.9. We get the following usual applications of the preceding formula.

(i) *Transverse base change formula.* If we assume that p is transverse to i, then ν is an isomorphism and the excess bundle vanishes. Thus we get a canonical identification $\Delta^*(\eta_i) \simeq \eta_k$.

(ii) *Self-intersection formula.* If we apply the formula to the self-intersection square

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Delta"] & Z \arrow[d, "i"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

where the excess bundle equals $N_Z X$, we get a canonical identification

$$\Delta^*(\eta_i) \simeq e(N_Z X) \tag{3.2.9.a}$$

in $H(Z/Z,-\langle N_Z X\rangle) = H(Z,\langle N_Z X\rangle)$.

(iii) *Blow-up formula.* In the case where $p : Y \to X$ is the blow-up along Z, we obtain a generalization of the "key formula" for blow-ups in [Ful98, 6.7].

Remark 3.2.10. If X is a scheme, E is a vector bundle over X and $s_0 : X \to E$ is the zero section, the self-intersection formula (3.2.9.a) applied to s_0 says that we can recover the Euler class of E from the fundamental class of s_0 by base change along the self-intersection square. That is,

$$e(E) \simeq \Delta^*(\eta_{s_0}),$$

where Δ denotes the self-intersection square associated to s_0. More generally, if $s : X \to E$ is an arbitrary section, consider the cartesian square

$$\begin{tikzcd} Z_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta_s"] & X \arrow[d, "s_0"] \\ X \arrow[r, "s"'] & E \end{tikzcd}$$

We define the *Euler class with support* as

$$e(E;s) := \Delta_s^*(\eta_s) \in H(Z_s/X,-\langle E\rangle).$$

This notion corresponds to [Lev17b, Definition 5.1]. In particular the usual Euler class is the case $s = s_0$.

We will now state good properties of our constructions of orientations for regular closed immersions, culminating in the associativity formula. Note that all these formulas will be subsumed once we will get our final construction.

3.2.11. First consider a cartesian square of S-schemes:

$$\begin{tikzcd} T \arrow[r, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

such that i is a regular closed immersion and f is smooth. The isomorphisms of vector bundles $T_g \simeq T_f|_T$ and $N_T Y \simeq N_Z X|_T$ induce an identification

$$\langle T_g \rangle - \langle N_Z X|_T \rangle \simeq \langle L_{T/X} \rangle \simeq \langle T_f|_T \rangle - \langle N_T Y \rangle \tag{3.2.11.a}$$

in $\mathrm{K}(T)$, where $L_{T/X}$ is the cotangent complex of T over X.

Lemma 3.2.12. *With notation as above, one has the commutative square*

$$\begin{tikzcd} H(X/S,*) \arrow[r, "\sigma_{Z/X}"] \arrow[d, "f^!"] & H(N_Z X/S,*) \arrow[d, "N_g(f)^!"] \\ H(Y/S,\langle T_f \rangle + *) \arrow[r, "\sigma_{T/Y}"] & H(N_T Y/S, \langle T_f \rangle + *) \end{tikzcd}$$

modulo the canonical isomorphism $k^*(T_f)|_{N_T Y} \simeq N_{N_T Y/N_Z X}$ of vector bundles on $N_T Y$.

Proof. It suffices to show that both squares in the following diagram commute:

$$\begin{tikzcd} H(X/S,*) \arrow[r, "\gamma_t"] \arrow[d, "f^!"'] \arrow[dr, phantom, "(1)"] & H(\mathbb{G}_m X/S,*)[-1] \arrow[r, "\partial_{N_Z X/D_Z X}"] \arrow[d, dashed, "(1\times f)^!"] \arrow[dr, phantom, "(2)"] & H(N_Z X/S,*) \arrow[d, dashed, "N_g(f)^!"] \\ H(Y/S,\langle T_f \rangle + *) \arrow[r, "\gamma_t"'] & H(\mathbb{G}_m Y/S,\langle T_f \rangle + *)[-1] \arrow[r, "\partial_{N_T Y/D_T Y}"'] & H(N_T Y/S,\langle T_f \rangle + *) \end{tikzcd}$$

In fact, the commutativity of (1) (where we have denoted the canonical functions of $D_Z X$ and $D_T Y$ by the same letter t) is obvious, and (2) follows from Proposition 2.4.7. $\square$

Lemma 3.2.13. *With notation as above, one has a canonical identification*

$$\eta_g . \eta_i \simeq \eta_k . \eta_f$$

in $H(T/X, \langle L_{T/X} \rangle)$.

Proof. Consider the following diagram:

$$\begin{tikzcd} H(X/X,*) \arrow[r, "i^!"] \arrow[d, "f^!"] & H(Z/X, -\langle N_Z X \rangle + e) \arrow[r, "p^!_{N_Z X/Z}"] \arrow[d, "g^!"] & H(N_Z X/X,*) \arrow[d, "N_g(f)^!"] \\ H(Y/X,\langle T_f \rangle + *) \arrow[r, "k^!"] & H(T/X,\langle L_{T/X} \rangle + *) \arrow[r, "p^!_{N_T Y/T}"] & H(N_T Y/X,\langle T_f \rangle) \end{tikzcd}$$

The right-hand square commutes by the associativity formula for Gysin morphisms associated with smooth morphisms (Example 2.3.9). Furthermore, the horizontal arrows $p^!_{N_Z X/Z}$ and $p^!_{N_T Y/T}$ are invertible (Lemma 2.4.4), so it suffices to show that the composite square commutes. But the upper and lower composites are the respective specialization maps $\sigma_{Z/X}$ and $\sigma_{T/Y}$, so we conclude by Lemma 3.2.12. $\square$

3.2.14. Next we consider a commutative diagram of schemes:

$$\begin{tikzcd} Z \arrow[rr, hook, "i"] \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

such that i is a closed immersion and p, q are smooth s-morphisms. In this situation, the canonical exact sequence of vector bundles over Z

$$0 \to T_q \to T_p|_Z \to N_Z X \to 0$$

gives rise to an identification

$$\langle T_p|_Z \rangle - \langle N_Z X \rangle \simeq \langle T_q \rangle. \tag{3.2.14.a}$$

in $\mathrm{K}(Z)$.

Lemma 3.2.15. *With notation as above, one has a canonical identification*

$$\eta_p.\eta_i \simeq \eta_q$$

in $H(Z/S, \langle T_q \rangle)$.

Proof. Consider the cartesian square

$$\begin{tikzcd} D_Z X \arrow[d] \arrow[dr, phantom, "\Delta"] & N_Z X \arrow[l, hook', "k"'] \arrow[d, "\pi"] \\ \mathbb{A}^1_S & S \arrow[l, hook', "s"] \end{tikzcd}$$

where s is the zero section and π is the composite map $N_Z X \xrightarrow{p_N} Z \xrightarrow{q} S$. The claim will follow from the commutativity of the diagram

$$\begin{tikzcd} H(S/S, 0) \arrow[r, "\gamma_t"] \arrow[d, "p^!"'] \arrow[dr, phantom, "(1)"] & H(\mathbb{G}_m S/S, 0)[-1] \arrow[r, "\partial_s"] \arrow[d, dashed, "(1\times p)^!"] \arrow[dr, phantom, "(2)"] & H(S/S, 0) \arrow[r, equal] \arrow[d, dashed, "\pi^!"] \arrow[dr, phantom, "(3)"] & H(S/S, 0) \arrow[d, "q^!"] \\ H(X/S, T_p) \arrow[r, "\gamma_t"'] & H(\mathbb{G}_m X/S, T_p)[-1] \arrow[r, "\partial_{N_T Y/D_T Y}"'] & H(N_Z X/s, T_p) & H(Z/s, e), \arrow[l, "p_N^!"] \end{tikzcd}$$

by considering the image of $1 \in H(S/S, 0)$ (recall that we have $\partial_s \circ \gamma_t \simeq 1$ by construction, see Paragraph 3.2.2).

The commutativity of square (1) is obvious, that of (2) follows from Proposition 2.4.7 applied to the cartesian square Δ, and that of (3) follows from the associativity of Gysin morphisms associated with smooth morphisms (Example 2.3.9). □

Before proceeding, we draw out some corollaries of the previous lemma.

Corollary 3.2.16. *Consider the assumptions of Paragraph 3.2.14. Then the orientation η_i is universally strong (Definition 2.3.11). In other words, the morphism $\eta_i : \mathrm{Th}_Z(-N_i) \to i^!(\mathbb{S}_X)$ is invertible.*

Proof. This follows from the previous lemma and the fact that the maps η_p and η_q are isomorphisms (Definition 2.3.5). □

Corollary 3.2.17. *Let $p : X \to S$ be a smooth s-morphism and $i : S \to X$ a section of p. Then there is a canonical identification $\eta_i.\eta_p \simeq 1$ in $H(S/S, 0)$, modulo the identification $i^{-1}(T_p) \simeq N_i$.*

Example 3.2.18. Let X/S be an s-scheme. Consider a vector bundle $p : E \to X$, $s_0 : X \to E$ its zero section and $e \in \mathrm{K}(X)$. Then the associated Gysin map

$$s_0^! : H(E/S, e) \to H(X/S, -\langle E \rangle + e)$$

is precisely the Thom isomorphism (Definition 2.4.5). This follows from Corollary 3.2.17. In cohomological terms, we also get the Thom isomorphism:

$$H(X,e) \xrightarrow{p^*} H(E,e) \simeq H(E/E,-e) \xrightarrow{s_0^!} H(X/E,-\langle E\rangle - e) = H_X(E, e+\langle E\rangle).$$

3.2.19. We now proceed towards the formulation of the associativity formula for the fundamental classes of two composable regular closed immersions

$$Z \xrightarrow{k} Y \xrightarrow{i} X.$$

Recall that there is a short exact sequence

$$0 \to N_Z Y \to N_Z X \to N_Y X|_Z \to 0$$

of vector bundles over Z, whence an identification $\langle N_Z X\rangle \simeq \langle N_Z Y\rangle + \langle N_Y X|_Z\rangle$ in $\mathrm{K}(Z)$ (Paragraph 2.1.5). There is also a canonical isomorphism of vector bundles

$$N(N_Z X, N_Z Y) \simeq N(N_Y X, N_Y X|_Z)$$

over Z; we will abuse notation and write N for both.

We will make use of the double deformation space (cf. [Ros96, §10])

$$D = D(D_Z X, D_Z X|_Y).$$

That is, D is the deformation space associated to $D_Z X|_Y \to D_Z X$, which is a regular closed immersion because the cartesian square

$$\begin{tikzcd} D_Z X|_Y \arrow[r] \arrow[d] & D_Z X \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

is tor-independent. Indeed, this can be checked locally on X, so we can assume that $Z \to Y \to X$ is a (transverse) base change of $\{0\} \to \mathbb{A}^m \to \mathbb{A}^n$ for some $0 \le m \le n$. Since the deformation space is stable under transverse base change (as is the question of tor-independence), we may reduce to the latter situation, in which case $D_Z X \to X$ is just the projection $\mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$, which is transverse to any morphism.

Note that D is a scheme over $X \times \mathbb{A}^2$; we write s and t for the first and second projections to $\mathbb{A}^1$, respectively. Set

$$D_1 = D|_{\{0\}\times\mathbb{A}^1}, \quad D_2 = D|_{\mathbb{A}^1\times\{0\}}, \quad D_0 = D|_{\{0\}\times\{0\}}.$$

The fibres of D over various subschemes of $\mathbb{A}^2$ are summarized in the following table.

$\{0\}\times\mathbb{A}^1$	$D(N_Z X, N_Z Y)$	$\mathbb{G}_m\times\mathbb{G}_m$	$X\times\mathbb{G}_m\times\mathbb{G}_m$
$\mathbb{A}^1\times\{0\}$	$D(N_Y X, N_Y X\|_Z)$	$\mathbb{G}_m\times\{0\}$	$D_2 - D_0 = \mathbb{G}_m\times N_Z X$
$\mathbb{G}_m\times\mathbb{A}^1$	$D - D_1 = \mathbb{G}_m\times D_Z X$	$\{0\}\times\mathbb{G}_m$	$D_1 - D_0 = N_Y X\times\mathbb{G}_m$
$\mathbb{A}^1\times\mathbb{G}_m$	$D - D_2 = D_Y X\times\mathbb{G}_m$	$\{0\}\times\{0\}$	N

Lemma 3.2.20. *Under the assumptions and notation of Paragraph 3.2.19, the diagram*

$$\begin{tikzcd} H(X/X,e) \arrow[r, "\sigma_{Y/X}"] \arrow[d, "\sigma_{Z/X}"'] & H(N_Y X/X,e) \arrow[d, "\sigma_{N_Y X|_Z/N_Y X}"] \\ H(N_Z X/X,e) \arrow[r, "\sigma_{N_Z Y/N_Z X}"] & H(N/X,e) \end{tikzcd}$$

commutes for every $e \in \mathrm{K}(X)$.

Proof. By construction of the specialization maps, the square in question factors as in the following diagram:

$$\begin{tikzcd}
H(X/X,e) \arrow[r, "\gamma_s"] \arrow[d, "\gamma_t"'] \arrow[dr, phantom, "(1)"] & H(X\mathbb{G}_m^s/X,e)[-1] \arrow[r, "\partial_{N_Y X/D_Y X}"] \arrow[d, "\partial_{N_Z X/D_Z X}"] \arrow[dr, phantom, "(2)"] & H(N_Y X/X,e) \arrow[d, "\gamma_t"] \\
H(X\mathbb{G}_m^t/X,e) \arrow[r, "\gamma_s"] \arrow[d, "\partial_{N_X Z/D_Z X}"'] \arrow[dr, phantom, "(3)"] & H(X\mathbb{G}_m^s\mathbb{G}_m^t/X,e)[-2] \arrow[r, "\partial_{N_Y X\mathbb{G}_m^t/D_Y X\mathbb{G}_m^t}"] \arrow[d, "\partial_{\mathbb{G}_m^s N_Z X/\mathbb{G}_m^s D_Z X}"] \arrow[dr, phantom, "(4)"] & H(N_Y X\mathbb{G}_m^t/X,e)[-1] \arrow[d, "\partial_{N/D_1}"] \\
H(N_Z X/X,e) \arrow[r, "\gamma_s"'] & H(\mathbb{G}_m^s N_Z X/X,e)[-1] \arrow[r, "\partial_{N/D_2}"'] & H(N/X,e)
\end{tikzcd}$$

Some remarks on the notation are in order. First of all we have omitted the symbol $\times$ in the diagram. We have also used exponents s and t to indicate that $\mathbb{G}_m^s$, resp. $\mathbb{G}_m^t$, is viewed as a subset of the s-axis, resp. t-axis, in $\mathbb{A}^2$. Finally, we have written γ_u for multiplication with the class $\sigma_\pi \in H(\mathbb{G}_m^u/\mathbb{Z},0)[-1]$ with $u \in \{s,t\}$.

Now observe that squares (2) and (3) commute by Proposition 2.2.12. Square (1) is *anti*-commutative by Paragraph 2.2.13. Applying Corollary 2.2.11 to the commutative square

$$\begin{tikzcd}
D_0 \arrow[r] \arrow[d] & D_1 \arrow[d] \\
D_2 \arrow[r] & D
\end{tikzcd}$$

we deduce that square (4) is also anti-commutative, whence the claim. $\square$

Theorem 3.2.21. *Let S be a scheme. Then there exists a system of fundamental classes $(\eta_i)_i$ (Definition 2.3.6) on the class of regular closed immersions between s-schemes over S, satisfying the following properties:*

(i) *For every regular closed immersion $i : Z \to X$, the orientation $\eta_i \in H(Z/X, -\langle N_Z X\rangle)$ is the fundamental class defined in Definition 3.2.5.*

(ii) *The system is stable under transverse base change (Definition 2.3.6(iv)).*

Proof. According to Definition 2.3.6 we must give the following data:

(i) *Fundamental classes.* For any regular closed immersion $i : Z \to X$, we take the orientation (η_i, e_i) with $e_i = -\langle N_Z X\rangle \in \mathrm{K}(Z)$ and $\eta_i \in H(Z/X, -\langle N_Z X\rangle)$ as in Definition 3.2.5.

(ii) *Normalisation.* If $i = \mathrm{Id}_S$ for a scheme S, then $N_S S = S$ and the specialization map and Thom isomorphism are both the identity maps on $H(S/S, 0)$, so we have a canonical identification $\eta_i \simeq 1$.

(iii) *Associativity formula.* Given regular closed immersions $i : Y \to X$ and $k : Z \to Y$, the composite $k \circ i$ is again a regular closed immersion and we have an identification $N_Z X \simeq k^*\langle N_Y X\rangle + \langle N_Z Y\rangle$ in $\mathrm{K}(Z)$. We obtain a canonical identification $\eta_k.\eta_i \simeq \eta_{i\circ k}$ in $H(Z/X, -\langle N_Z X\rangle)$ from the following commutative diagram

$$\begin{tikzcd}
H(X/X,0) \arrow[r, "\sigma_{Y/X}"'] \arrow[rr, bend left, "i^!"] \arrow[d, "\sigma_{Z/X}"] \arrow[dd, bend right=60, "(ik)^!"'] \arrow[dr, phantom, "(1)"] & H(N_YX/X,0) \arrow[d, "\sigma_{N_YX|_Z/N_YX}"] \arrow[dr, phantom, "(2)"] & H(Y/X,-\langle N_YX\rangle) \arrow[l, "p^!_{N_YX/Y}"] \arrow[d, "\sigma_{Z/Y}"] \arrow[dd, bend left=60, "k^!"] \\
H(N_ZX/X,0) \arrow[r, "\sigma_{N_ZY/N_ZX}"] \arrow[dr, phantom, "(3)"] & H(N/X,0) \arrow[dr, phantom, "(4)"] & H(N_YX|_Z/X,-\langle N\rangle) \arrow[l, "p^!_{N/N_YX|_Z}"'] \\
H(Z/X,-\langle N_ZX\rangle) \arrow[u, "p^!_{N_ZX/Z}"] \arrow[r, "p^!_{N_ZY/Z}"'] & H(N_ZY/X,-\langle N\rangle) \arrow[u, "p^!_{N/N_ZY}"] & H(Z/X,-\langle N_ZY-N_YX\rangle|_Z). \arrow[l, "p^!_{N_ZY/Z}"] \arrow[u, "p^!_{N_YX|_Z/Z}"']
\end{tikzcd}$$

by evaluating at $1 \in H(X/X,0)$, since the maps $p^!_{N_ZY/Z}$ and $p^!_{N/N_ZY}$ are invertible (Lemma 2.4.4). Note that each square is indeed commutative:

(1) Apply Lemma 3.2.20.

(2) Apply Lemma 3.2.12 to the cartesian square

$$\begin{tikzcd}
N_YX|_Z \arrow[r, hook] \arrow[d, "p_{N_YX|_Z/Z}"'] & N_YX \arrow[d, "p_{N_YX/Y}"] \\
Z \arrow[r, hook, "k"] & Y
\end{tikzcd}$$

(3) This square factors into two triangles:

$$\begin{tikzcd}
H(N_ZX/X,0) \arrow[r, "\sigma_{N_ZY/N_ZX}"] \arrow[dr, "N_Z(i)^!"] & H(N/X,0) \\
H(Z/X,-\langle N_ZX\rangle) \arrow[u, "p^!_{N_ZX/Z}"] \arrow[r, "p^!_{N_ZY/Z}"'] & H(N_ZY/X,-\langle N\rangle) \arrow[u, "p^!_{N/N_ZY}"']
\end{tikzcd}$$

The upper-right triangle commutes by construction of $N_Z(i)^!$, the Gysin map associated to $N_Z(i) : N_ZY \to N_ZX$. The lower-left triangle commutes by Lemma 3.2.15 applied to the commutative diagram:

$$\begin{tikzcd}
N_ZY \arrow[rr, hook] \arrow[dr, "p_{N_ZY/Z}"'] & & N_ZX \arrow[dl, "p_{N_ZX/Z}"] \\
& Z &
\end{tikzcd}$$

(4) Apply the associativity of Gysin morphisms associated with smooth morphisms (Example 2.3.9).

(iv) *Transverse base change formula.* Suppose given a tor-independent cartesian square

$$\begin{tikzcd}
T \arrow[r, "k"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "p"] \\
Z \arrow[r, "i"'] & X
\end{tikzcd}$$

where i is a regular closed immersion. Then k is also a regular closed immersion and there is a canonical identification of vector bundles $N_TY \simeq q^{-1}(N_ZX)$. The canonical identification $\Delta^*(\eta_i) \simeq \eta_k$ comes then from Example 3.2.9(i).

□

3.3. **Fundamental classes: general case.** In this subsection we conclude our main construction by gluing the system of fundamental classes defined on the class of smooth morphisms (Definition 2.3.5) together with the system defined on the class of regular closed immersions in the previous subsection (Theorem 3.2.21).

Lemma 3.3.1. *Suppose given a commutative diagram*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, hook, "i'"'] & Y \arrow[d, "f"] \\ & S \end{tikzcd}$$

where i and i' are regular closed immersions and f is a smooth s-morphism. Then there is a canonical identification

$$\eta_i.\eta_f \simeq \eta_{i'}$$

in $H(X/S, -\langle N_X S\rangle)$, modulo the identification $-\langle N_X S\rangle \simeq -\langle N_X Y\rangle + i^*\langle T_{Y/S}\rangle$ in $\mathrm{K}(X)$.

Proof. The diagram in question factors as follows:

$$\begin{tikzcd} X \arrow[r, hook, "\Gamma_i"] \arrow[dr, equal] & X \times_S Y \arrow[r, hook, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\ & X \arrow[r, hook, "i'"'] & S, \end{tikzcd}$$

where $\Gamma_i : X \to X \times_S Y$ is the graph of i, p_1 and p_2 are the respective projections, and the square is cartesian. By Lemma 3.2.13, Corollary 3.2.17 and Theorem 3.2.21 we get canonical identifications

$$\eta_i.\eta_f \simeq \eta_{\Gamma_i}.\eta_{p_2}.\eta_f \simeq \eta_{\Gamma_i}.\eta_{p_1}.\eta_{i'} \simeq \eta_{i'},$$

as claimed. □

We are now ready to state the main theorem, defining a system of fundamental classes on the class of smoothable lci morphisms:

Theorem 3.3.2. *Let S be a scheme. Then there exists a system of fundamental classes $(\eta_f)_f$ (Definition 2.3.6) on the class of smoothable lci s-morphisms between s-schemes over S, satisfying the following properties:*

(i) *The restriction of the system $(\eta_f)_f$ to the class of smooth s-morphisms coincides with the system of Example 2.3.9.*

(ii) *The restriction of the system $(\eta_f)_f$ to the class of regular closed immersions coincides with the system of Theorem 3.2.21.*

(iii) *The system is stable under transverse base change (Definition 2.3.6(iv)).*

Proof. To define the system $(\eta_f)_f$, we must give the following data (see Definition 2.3.6):

(i) *Fundamental classes.* Given a smoothable lci s-morphism $f : X \to S$, we may choose a factorization through a regular closed immersion $i : X \to Y$ and a smooth s-morphism $p : Y \to S$. We define the fundamental class $\eta_f = \eta_i.\eta_p \in H(X/S, \langle L_f\rangle)$. Note that, given another factorization through some $i' : X \to Y'$ and $p' : Y \to S$, we obtain a canonical identification $\eta_i.\eta_p \simeq \eta_{i'}.\eta_{p'}$ by applying Lemma 3.3.1 to the diagram

$$\begin{tikzcd} & Y \arrow[rr, "p"] & & S \\ X \arrow[ur, "i"] \arrow[r, "{(i,i')}"] \arrow[rrr, bend right, "i'"'] & Y \times_S Y' \arrow[u, "p_1"] \arrow[rr, "p_2"] & & Y'. \arrow[u, "p'"'] \end{tikzcd}$$

(ii) *Normalisation.* If $f = \mathrm{Id}_S$ for a scheme S, then we choose the trivial factorization $f = \mathrm{Id}_S \circ \mathrm{Id}_S$ and the normalization properties of Example 2.3.9 and Theorem 3.2.21 give a canonical identification $\eta_f \simeq 1.1 \simeq 1$.

(iii) *Associativity formula.* If $f : X \to Y$ and $g : Y \to Z$ are two smoothable lci s-morphisms, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i_1"] \arrow[dr, "f"'] & P \arrow[r, hook, "i_3"] \arrow[d, "p_1"] & R \arrow[d, "p_3"] \\ & Y \arrow[r, hook, "i_2"] \arrow[dr, "g"'] & Q \arrow[d, "p_2"] \\ & & Z, \end{tikzcd}$$

where the i_k's are closed immersions and the p_k's are smooth morphisms, and the square is cartesian. By Example 2.3.9, Lemma 3.2.13 and Theorem 3.2.21 we have identifications

$$\eta_f.\eta_g \simeq \eta_{i_1}.\eta_{p_1}.\eta_{i_2}.\eta_{p_2} \simeq \eta_{i_1}.\eta_{i_3}.\eta_{p_3}.\eta_{p_2} \simeq \eta_{i_3 \circ i_1}.\eta_{p_2 \circ p_3} \simeq \eta_{g \circ f}.$$

(iv) *Transverse base change formula.* Suppose given a tor-independent cartesian square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "u"] \\ X \arrow[r, "f"'] & S, \end{tikzcd}$$

where f and g are smoothable lci s-morphisms. Choosing a factorization $f = p \circ i$, where i is a regular closed immersion and p is a smooth s-morphism, there is an induced factorization of the square Δ:

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "v"'] \arrow[dr, phantom, "\Delta_i"] & Y' \arrow[r, "q"] \arrow[d, "v'"] \arrow[dr, phantom, "\Delta_p"] & T \arrow[d, "u"] \\ X \arrow[r, "i"'] & X' \arrow[r, "p"'] & S, \end{tikzcd}$$

where k is a regular closed immersion and q is a smooth s-morphism. Now by above and by the transverse base change properties of Example 2.3.9 and Theorem 3.2.21, we have identifications

$$\Delta^*(\eta_f) \simeq \Delta^*(\eta_i.\eta_p) \simeq \Delta_i^*(\eta_i).\Delta_p^*(\eta_p) \simeq \eta_k.\eta_q \simeq \eta_g$$

as claimed.

□

3.3.3. In fact, the transverse base change property of Theorem 3.3.2(iii) is a special case of an excess intersection formula generalizing Proposition 3.2.8. Consider a cartesian square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "v"] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "u"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

where f and g are smootable lci morphisms. Factor $f = p \circ i$ as a closed immersion followed by a smooth morphism and consider the diagram of cartesian squares

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "v"'] & Q \arrow[r, "q"] \arrow[d, "r"'] & T \arrow[d, "u"] \\ X \arrow[r, "i"'] & P \arrow[r, "p"'] & S \end{tikzcd}$$

where k and i are regular closed immersions and q and p are smooth. By 3.2.7 there is a canonical monomorphism of Y-vector bundles $N_Y Q \xrightarrow{\nu} v^{-1} N_X P$. We let ξ be the quotient bundle.

Proposition 3.3.4. *With notation and assumptions as above, there is an identification*

$$\Delta^*(\eta_f) \simeq e(\xi).\eta_g$$

in $H(Y/T, v^*\langle L_f \rangle)$, modulo the identification $v^*\langle L_f \rangle \simeq -\langle \xi \rangle + \langle L_g \rangle$ in $\mathrm{K}(Y)$, where $e(\xi) \in H(Y/Y, \langle \xi \rangle)$ is the Euler class of ξ (Definition 3.1.2 and Remark 3.1.3).

This follows from Proposition 3.2.8 and the fact that fundamental classes for smooth morphisms are compatible with any base change (Example 2.3.9).

4. Main results and applications

4.1. **Fundamental classes and Euler classes with coefficients.**

4.1.1. Let S be a scheme and $\mathbb{E} \in \mathscr{SH}(S)$ a motivic spectrum. Observe that the bivariant spectra $\mathbb{E}(X/S, v)$ of Definition 2.2.1 are natural in $\mathbb{E}$. That is, given any morphism $\varphi : \mathbb{E} \to \mathbb{F}$ in $\mathscr{SH}(S)$, there is an induced map of spectra

$$\varphi_* : \mathbb{E}(X/S, v) \to \mathbb{F}(X/S, v)$$

for every s-scheme X over S and every $v \in \mathrm{K}(X)$. Note that φ_* is compatible with the various functorialities of bivariant theory (Paragraph 2.2.7). Also, if $\mathbb{E}$ and $\mathbb{F}$ are equipped with multiplications which commute with φ, then the induced map φ_* preserves products (as defined in Paragraph 2.2.7(4)).

Definition 4.1.2. Let $\mathbb{E} \in \mathscr{SH}(S)$ be a motivic spectrum equipped with a unit map $\eta : \mathbb{S}_S \to \mathbb{E}$. By Paragraph 4.1.1 there is a canonical natural transformation of bivariant theories

$$\rho_{X/S} : H(X/S, v) \to \mathbb{E}(X/S, v) \tag{4.1.2.a}$$

that we call the $\mathbb{A}^1$-*regulator map.*

Definition 4.1.3. Let $\mathbb{E} \in \mathscr{SH}(S)$ be a unital motivic spectrum. Given a smoothable lci s-morphism $f : X \to S$, let $\eta_f \in H(X/S, \langle L_f \rangle)$ denote the fundamental class of f as in Theorem 3.3.2. We define the *fundamental class of f with coefficients in* $\mathbb{E}$, denoted

$$\eta_f^{\mathbb{E}} \in \mathbb{E}(X/S, \langle L_f \rangle),$$

as the image of η_f by the $\mathbb{A}^1$-regulator map $\rho_{X/S}$ (4.1.2.a).

If $\mathbb{E}$ is unital, associative and commutative, then because $\rho_{X/S}$ is compatible with products and "change of base" maps, the associativity and base change formulas provided by Theorem 3.3.2 are immediately inherited by the fundamental classes of Definition 4.1.3. If we extend Definition 2.3.6 as indicated in Remark 2.3.7, then we can state more precisely:

Theorem 4.1.4. *Let $\mathbb{E} \in \mathscr{SH}(S)$ be a motivic spectrum equipped with a unital associative commutative multiplication. Then there exists a system of fundamental classes $(\eta_f^{\mathbb{E}})_f$ on the class of smoothable lci s-morphisms between s-schemes over S. This system is stable under transverse base change, and recovers the system of Theorem 3.3.2 in the case $\mathbb{E} = \mathbb{S}_S$.*

Definition 4.1.5. Let $\mathbb{E} \in \mathscr{SH}(S)$ be a unital motivic spectrum. Then for any scheme X and any vector bundle E/X, one defines the *Euler class of E/X with coefficients in* $\mathbb{E}$, denoted

$$e(E, \mathbb{E}) \in \mathbb{E}(X, \langle E \rangle) \simeq \mathbb{E}(X/X, -\langle E \rangle)$$

as the image of the class $e(E) \in H(X, \langle E \rangle) \simeq H(X/X, -\langle E \rangle)$ of Definition 3.1.2 by the $\mathbb{A}^1$-regulator map $\rho_{X/S}$ (4.1.2.a).

Remark 4.1.6. It is possible to define fundamental classes with coefficients in arbitrary motivic spectra $\mathbb{E} \in \mathscr{SH}(S)$, without using any multiplicative or even unital structure. Indeed the constructions of Section 3 (where we only considered the case $\mathbb{E} = \mathbb{S}_S$ for simplicity) extend immediately to general $\mathbb{E}$ without any difficulty. Alternatively we can replace the use of the $\mathbb{A}^1$-regulator map above by using instead the *module* structure, i.e., the canonical action[9]

$$H(Y/X, w) \otimes \mathbb{E}(X/S, v) \to \mathbb{E}(Y/S, w + q^*v)$$

which is defined by the same formula used to define products in Paragraph 2.2.7, except that the multiplication map $\mu_{\mathbb{E}} : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$ is replaced by the map $\mathbb{S}_S \otimes \mathbb{E} \to \mathbb{E}$ encoding the structure of

[9] See also [Dég18a, 1.2.1].

$\mathbb{S}_S$-module on $\mathbb{E}$. As a third approach, Gysin maps with arbitrary coefficients can be obtained from the purity transformation (Paragraph 4.3.3).

4.2. **(Refined) Gysin maps with coefficients.** Just as in Definition 2.4.1 and Proposition 2.4.2, Theorem 4.1.4 provides Gysin maps with coefficients:

Theorem 4.2.1. *Let $\mathbb{E} \in S\mathscr{H}(S)$ be a motivic spectrum equipped with a unital associative commutative multiplication. Then for any smoothable lci s-morphism $f : X \to Y$ of s-schemes over S, and any $v \in \mathrm{K}(Y)$, there is a* Gysin map

$$f^! : \mathbb{E}(Y/S, v) \to \mathbb{E}(X/S, \langle L_f \rangle + f^* v) \qquad x \mapsto \eta_f^{\mathbb{E}}.x. \tag{4.2.1.a}$$

These Gysin maps satisfy functoriality and transverse base change formulas that are exactly analogous to those of Proposition 2.4.2.

We also have an excess intersection formula with coefficients (generalizing Proposition 3.3.4):

Proposition 4.2.2. *Let $\mathbb{E} \in S\mathscr{H}(S)$ be a motivic spectrum equipped with a unital associative commutative multiplication. Suppose given a cartesian square of s-schemes over S of the form*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Delta"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where f and g are smoothable lci s-morphisms. Let ξ denote the excess bundle as in Paragraph 3.3.3. Then we have a canonical identification

$$\Delta^*(\eta_f) \simeq e(\xi, \mathbb{E}).\eta_g$$

in $\mathbb{E}(X'/Y', v^*\langle L_f \rangle)$. If u and v are proper, then we also have an identification

$$f^! \circ v_* \simeq u_* \circ \gamma_{e(\xi,\mathbb{E})} \circ g^!$$

of maps $\mathbb{E}(Y'/S, v^*(e)) \to \mathbb{E}(X/S, \langle L_f \rangle + e)$, for any $e \in \mathrm{K}(Y)$, where $\gamma_{e(\xi,\mathbb{E})}$ denotes multiplication by $e(\xi, \mathbb{E})$.

Applying Proposition 4.2.2 to the cartesian square

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Delta"] & Z \arrow[d, "i"] \\ Z \arrow[r, "i"'] & X, \end{tikzcd}$$

we obtain the self-intersection formula with coefficients:

Corollary 4.2.3. *Let $i : Z \to X$ be a regular closed immersion of s-schemes over S. Then we have canonical identifications*

$$\Delta^*(\eta_i) \simeq e(N_Z X, \mathbb{E})$$

in $\mathbb{E}(Z/Z, -\langle N_Z X \rangle)$, and for any $v \in \mathrm{K}(X)$ and identification

$$i^! i_* \simeq \gamma_{e(N_Z X,\mathbb{E})}$$

of maps $\mathbb{E}(Z/S, v) \to \mathbb{E}(Z/S, -\langle N_Z X \rangle + i^* v)$.

Applying Corollary 4.2.3 to the zero section $s : X \to E$ of a vector bundle, we obtain the following formula to compute Euler classes in $\mathbb{E}^0(X, \langle E \rangle)$:

Corollary 4.2.4. *Let X be a scheme and E a vector bundle over X with zero section $s : X \to E$. Then there is a canonical identification*

$$e(E, \mathbb{E}) \simeq s^! s_*(1) \tag{4.2.4.a}$$

in $\mathbb{E}(X/X, -\langle E \rangle) \simeq \mathbb{E}(X, \langle E \rangle)$.

We now introduce the notions of refined fundamental class and refined Gysin maps, following Fulton's treatment in intersection theory (cf. [Ful98, 6.2]).

Definition 4.2.5. Suppose given a cartesian square of s-schemes over S of the form

(4.2.5.a)
$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Delta"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where f is a smoothable lci s-morphism.

(i) The *refined fundamental class* of f, with respect to Δ and with coefficients in $\mathbb{E}$, is the class
$$\eta_\Delta^{\mathbb{E}} = \Delta^*(\eta_f^{\mathbb{E}})$$
in $\mathbb{E}(X'/Y', v^*\langle L_f\rangle)$.

(ii) The *refined Gysin map* of f, with respect to Δ and with coefficients in $\mathbb{E}$, is the Gysin map associated to the orientation $(\eta_\Delta^{\mathbb{E}}, u^*\langle L_f\rangle)$ of g, in the sense of Definition 2.4.1. That is, it is the induced map of bivariant spectra
$$g^!(\eta_\Delta^{\mathbb{E}}) : \mathbb{E}(Y'/S, e) \to \mathbb{E}(X'/S, u^*\langle L_f\rangle + e)$$
$$x \mapsto \Delta^*(\eta_f^{\mathbb{E}}).x$$
for any $e \in \mathrm{K}(Y')$. We sometimes denote it also by $g_\Delta^!$.

In terms of refined fundamental classes, we can reformulate the transverse base change and excess intersection formulas as follows:

Proposition 4.2.6. *Suppose given a cartesian square Δ as in* (4.2.5.a)*, with f is a smoothable lci s-morphism. Then we have:*

(i) Tautological base change formula. *If u and v are proper, then there is a canonical identification*
$$f^! \circ v_* \simeq u_* \circ g_\Delta^!$$
of maps $\mathbb{E}(Y'/S, u^*(e)) \to \mathbb{E}(X/S, \langle L_f\rangle + e)$, for any $e \in \mathrm{K}(Y)$.

(ii) *If g is also smoothable lci s-morphism, then we have an identification*
$$\eta_\Delta^{\mathbb{E}} \simeq e(\xi, \mathbb{E}).\eta_g$$
in $\mathbb{E}(X'/Y', u^*\langle L_f\rangle)$. In particular the refined Gysin map $g_\Delta^!$ is identified with the composite $\gamma_{e(\xi,\mathbb{E})} \circ g^!$, where the map $\gamma_{e(\xi,\mathbb{E})}$ is multiplication by $e(\xi, \mathbb{E})$.

(iii) *If the square Δ is tor-independent, so that in particular g is also smoothable lci s-morphism, then we have an identification*
$$\eta_\Delta^{\mathbb{E}} \simeq \eta_g^{\mathbb{E}}$$
in $\mathbb{E}(X'/Y', \langle L_g\rangle)$. In particular the refined Gysin map $g_\Delta^!$ is identified with the Gysin map $g^!$.

Proof. The first claim follows from the definitions (Paragraph 2.2.7). The second is precisely the excess intersection formula (Proposition 4.2.2). The third is the stability of fundamental classes under transverse base change (Theorem 4.1.4). □

Remark 4.2.7. Note that the excess intersection formula (Proposition 4.2.2) and transverse base change formula can be recovered by combining the tautological base change formula of Proposition 4.2.6(i) with parts (ii) and (iii).

4.3. **Purity, traces and duality.**

4.3.1. Let S be a scheme and let $f : X \to Y$ be a smoothable lci s-morphism of s-schemes over S. Then by Theorem 4.1.4 we obtain, by the construction of Paragraph 2.5.1, a *purity transformation*

$$\mathfrak{p}_f : \Sigma^{L_f} f^* \to f^!$$

of functors $S\mathscr{H}(Y) \to S\mathscr{H}(X)$, as well as trace and cotrace maps (Paragraph 2.5.3):

$$\mathrm{tr}_f : f_! \Sigma^{L_f} f^* \to \mathrm{Id}_{S\mathscr{H}(Y)}$$
$$\mathrm{cotr}_f : \mathrm{Id}_{S\mathscr{H}(Y)} \to f_* \Sigma^{-L_f} f^!.$$

These natural transformations satisfy 2-functoriality and transverse base change properties as described in Proposition 2.5.4 and Corollary 2.5.6.

Remark 4.3.2. There are some instances of purity transformations for non-smooth morphisms in the literature. One example is the case of Grothendieck duality in the setting of quasi-coherent or ind-coherent sheaves (see e.g. [Con00, Theorem 4.3.3], [GR17, Corollary 7.2.4]), where the transformation is indeed invertible for any Cohen-Macaulay morphism. A second one, much closer to the motivic context, can be found in [SGA4, Exposé XVIII, (3.2.1.2)] in the derived category of étale sheaves. This construction is valid for flat s-morphisms, and only involves Tate twists rather than arbitrary Thom spaces, which reflects the fact that the theory developed in SGA 4 is oriented (cf. Example 4.3.5 below).

4.3.3. The purity transformation induces Gysin maps on each of the four theories defined in Definition 2.2.1. That is, if $f : X \to Y$ is a smoothable lci s-morphism of s-schemes over S, then we get Gysin maps (for every $e \in \mathrm{K}(Y)$):

(i) *Bivariant theory:* $f^! : \mathbb{E}(Y/S, e) \to \mathbb{E}(X/S, \langle L_f\rangle + f^*(e))$.

(ii) *Cohomology with proper support:* $f_! : \mathbb{E}_c(X/S, \langle L_f\rangle + f^*(e)) \to \mathbb{E}_c(Y/S, e)$.

If f is *proper*, then we also get Gysin maps

(iii) *Cohomology:* $f_! : \mathbb{E}(X/S, \langle L_f\rangle + f^*(e)) \to \mathbb{E}(Y/S, e)$.

(iv) *Bivariant theory with proper support:* $f^! : \mathbb{E}^c(Y/S, e) \to \mathbb{E}^c(X/S, \langle L_f\rangle + f^*(e))$.

In particular, this is another (obviously equivalent) way to realize the Gysin maps considered in Theorem 4.2.1.

We now observe that the purity transformation can be defined for any motivic ∞-category of coefficients:

4.3.4. Let $\mathscr{T}$ be a motivic ∞-category of coefficients in the sense of [Kha16, Chap. 2, Def. 3.5.2], defined on the site $\mathscr{S}$ of (qcqs) schemes. That is, $\mathscr{T}$ is a presheaf of symmetric monoidal presentable ∞-categories on $\mathscr{S}$ satisfying certain axioms that guarantee (see [Kha16, Chap. 2, Cor. 4.2.3]) that $\mathscr{T}$ admits a full homotopy coherent formalism of six operations.

At this point we note that all the definitions and constructions in Sections 2 and 3 make sense in the setting of $\mathscr{T}$ (and not only $S\mathscr{H}$), as they only use the six operations. In particular:

(1) One can define the four theories (Definition 2.2.1) in this setting. For example, the bivariant theory represented by any $\mathbb{E} \in \mathscr{T}(S)$ is given by:

$$\mathbb{E}(X/S, v, \mathscr{T}) := \mathrm{Maps}_{\mathscr{T}(S)}(\mathbb{1}_S, p_*(p^!(\mathbb{E}) \otimes \mathrm{Th}_X(-v, \mathscr{T})))$$

where $p : X \to S$ is an s-morphism and $v \in \mathrm{K}(X)$. Here we have written $\mathbb{1}_S \in \mathscr{T}(S)$ for the monoidal unit, and $\mathrm{Th}_X(-v, \mathscr{T})$ for the Thom space[10] internal to $\mathscr{T}$.

[10] Thom spaces can be defined in terms of the six operations by relative purity (Paragraph 2.1.8).

(2) We have fundamental classes
$$\eta_f^{\mathscr{T}} \in \mathbb{E}(X/Y, -\langle L_f \rangle, \mathscr{T})$$
for any smoothable lci s-morphism $f : X \to Y$ of s-schemes over S, with coefficients in any $\mathbb{E} \in \mathscr{T}(S)$ for arbitrary $\mathscr{T}$. These again form a system of fundamental classes as in Theorem 3.3.2, satisfying stability under transverse base change.

(3) We have Gysin maps in bivariant theory with coefficients in any $\mathbb{E} \in \mathscr{T}(S)$ (as well as in the other three theories) for arbitrary $\mathscr{T}$. These Gysin maps are functorial, satisfy transverse base change and excess intersection formulas.

(4) We have natural transformations
$$\begin{aligned} \mathfrak{p}_f^{\mathscr{T}} &: \Sigma^{L_f} f^* \to f^! \\ \mathrm{tr}_f^{\mathscr{T}} &: f_! \Sigma^{L_f} f^* \to \mathrm{Id} \\ \mathrm{cotr}_f^{\mathscr{T}} &: \mathrm{Id} \to f_* \Sigma^{-L_f} f^! \end{aligned}$$
in the setting of any $\mathscr{T}$.

Example 4.3.5. Suppose that $\mathscr{T}$ is *oriented* in the sense that there are Thom isomorphisms
$$\mathrm{Th}_X(v, \mathscr{T}) \to \mathbb{1}_X(r)[2r],$$
for any $v \in \mathrm{K}(X)$ of virtual rank r, which are functorial and respect the $\mathcal{E}_\infty$-group structure on $\mathrm{K}(X)$ up to a homotopy coherent system of compatibilities. Then for any smoothable lci s-morphism f of relative virtual dimension d, the purity transformation takes the form:
$$\mathfrak{p}_f^{\mathscr{T}} : f^*(-)(d)[2d] \to f^! \tag{4.3.5.a}$$
and similarly for the trace and cotrace maps.

Example 4.3.6. Let $S = \mathrm{Spec}(\mathbb{Z}[1/\ell])$ for a prime ℓ, and let Λ be one of $\mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$. Then, as X varies over S-schemes, the stable ∞-category of étale Λ-sheaves $D(X_{\text{ét}}, \Lambda)$ defines a motivic ∞-category of coefficients (see [SGA4], [Eke90], [CD16], [LZ12]). In particular, we obtain purity transformations of the form (4.3.5.a) generalizing the previously known constructions.

Definition 4.3.7. Let S be a scheme and $f : X \to S$ a smoothable lci s-morphism of S-schemes. Let $\mathscr{T}$ be a motivic ∞-category of coefficients. We say that $\mathbb{E} \in \mathscr{T}(S)$ is *f-pure* if the canonical morphism
$$(\mathfrak{p}_f^{\mathscr{T}})_{\mathbb{E}} : \Sigma^{L_f} f^*(\mathbb{E}) \to f^!(\mathbb{E})$$
is invertible.

Remark 4.3.8.

(i) If f is *smooth*, then every object $\mathbb{E} \in \mathscr{T}(S)$ is f-pure. This is because the purity theorem (Paragraph 2.1.7) is valid in any motivic ∞-category of coefficients $\mathscr{T}$.

(ii) Variants of Definition 4.3.7 have been considered previously (for specific examples of $\mathscr{T}$) by several authors (see e.g. [ILO14, XVI, 3.1.5], [BD17, 4.4.2], [Pep15, 1.7]).

(iii) Given a smoothable lci s-morphism $f : X \to Y$, the full subcategory of $\mathcal{SH}(S)$ spanned by the f-pure objects satisfies good formal properties: it is stable under direct factors, extensions, and tensor products with strongly dualizable objects.

Note in particular that, for $\mathscr{T} = \mathcal{SH}$, the orientation η_f is universally strong (Definition 2.3.2) if and only if $\mathbb{S}_S$ is f-pure. We have the following variant of Lemma 2.3.14:

Lemma 4.3.9. *Let $\mathscr{T}$ be a motivic ∞-category of coefficients. Suppose that $f : X \to S$ is a smoothable lci s-morphism and that $\mathbb{E} \in \mathscr{T}(S)$ is an f-pure object. Then there are duality isomorphisms*
$$\mathbb{E}(X, v) \to \mathbb{E}(X/S, \langle L_f \rangle - v),$$
$$\mathbb{E}_c(X/S, v) \to \mathbb{E}^c(X/S, \langle L_f \rangle - v),$$

for every $v \in \mathrm{K}(X)$.

Recall that an ∞-category of coefficients $\mathscr{T}$ is called *continuous* if, whenever a scheme S can be written as the limit of a filtered diagram $(S_\alpha)_\alpha$ of schemes with affine dominant transition maps, then the canonical functor

$$\varinjlim_{\alpha} \mathscr{T}(S_\alpha) \to \mathscr{T}(S)$$

is an equivalence, where the colimit is taken in the ∞-category of *presentable* ∞-categories (and colimit-preserving functors).

Proposition 4.3.10. *Let S be a scheme, $\mathscr{T}$ a motivic ∞-category of coefficients, and $\mathbb{E} \in \mathscr{T}(S)$ an object. Let $f : X \to Y$ be a smoothable s-morphism of S-schemes and denote by $p : X \to S$ and $q : Y \to S$ the structure morphisms. Assume that one of the following conditions is satisfied:*

(i) *X and Y are smooth over S.*

(ii) *X and Y are regular, and S is the spectrum of a field k. The ∞-category of coefficients $\mathscr{T}$ is continuous. The object $\mathbb{E}$ is defined*[11] *over a perfect subfield of k.*

Then the morphism f is lci, and $q^*\mathbb{E}$ is f-pure.

Proof. Since f factors through a closed immersion and a smooth morphism, we may reduce to the case of closed immersions, using the associativity formula and the fact that $\mathfrak{p}_p^{\mathscr{T}}$ is invertible for p smooth. Moreover, in both cases f is automatically a *regular* closed immersion. The second case reduces to the first by using the continuity property of $\mathscr{T}$ together with Popescu's theorem [Swa98]. For the first case, the morphisms $p : X \to S$ and $q : Y \to S$ are smooth. By construction we have a commutative diagram

$$\begin{tikzcd}
\mathbb{E}_X \otimes \mathrm{Th}_X(L_f) \arrow[r, "\eta_f^{\mathbb{E}}"] \arrow[d, "\mathrm{Id} \otimes \eta_f"'] & f^!(\mathbb{E}_Y) \\
\mathbb{E}_X \otimes f^!(\mathbb{S}_Y) \arrow[ur, "Ex^{!*}_{\otimes}"'] &
\end{tikzcd}$$

where the left-hand vertical arrow is invertible by Lemma 3.2.15 and the fact that η_p is an isomorphism for p smooth (Definition 2.3.5). Therefore it suffices to note that the morphism induced by the exchange transformation $Ex^{!*}_{\otimes}$ (Paragraph 2.1.10) is invertible. After writing $\mathbb{E}_X = p^*(\mathbb{E})$ and $\mathbb{E}_Y = q^*(\mathbb{E})$, and using the purity isomorphisms $\mathfrak{p}_p : p^* \simeq \Sigma^{-T_p} p^!$ and $\mathfrak{p}_q : q^* \simeq \Sigma^{-T_q} q^!$ (Paragraph 2.1.7), this follows from the the $\otimes$-invertibility of Thom spaces. $\square$

The following definition first appears (as a conjecture) in the context of étale cohomology in [SGA5, I, 3.1.4]. In our setting it was already introduced in [Dég18b, Dég18a]. The following could be regarded as a more precise formulation, though in fact it is not difficult to see that both definitions are equivalent (cf. [Dég18a, Prop. 4.2.2]).

Definition 4.3.11. Let S be a scheme, $\mathscr{T}$ be a motivic ∞-category of coefficients, and $\mathbb{E} \in \mathscr{T}(S)$ an object. We say that $\mathbb{E}$ satisfies *absolute purity* if the following condition holds: given any commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

where f, p and q are s-morphisms, f is smoothable lci, and X and Y are regular, then the inverse image $q^*(\mathbb{E}) \in \mathscr{T}(Y)$ is f-pure.

Remark 4.3.12.

[11] That is, $\mathbb{E} \in \mathscr{T}(\mathrm{Spec}(k))$ is isomorphic to $g^*(\mathbb{E}')$, where $g : \mathrm{Spec}(k) \to \mathrm{Spec}(k')$ with $k' \subset k$ a perfect subfield, and $\mathbb{E}' \in \mathscr{T}(\mathrm{Spec}(k'))$.

(i) In view of the functoriality property of the purity transformation (Proposition 2.5.4), and by part (i) of Remark 4.3.8, it suffices to check the absolute purity property for diagrams as above where f is a *closed immersion*.

(ii) If S is the spectrum of a field and $\mathscr{T}$ is continuous, then it follows from Proposition 4.3.10 that every $\mathbb{E} \in \mathscr{T}(S)$ satisfies absolute purity.

(iii) Given the previous definition, the absolute purity property is stable under direct factors, extensions, tensor product with strongly dualizable objects (as in Remark 4.3.8). One also deduces from the projection formula that absolutely pure objects are stable under direct image p_* for p smooth and proper.

Example 4.3.13. It is known that the motivic spectra **KGL**, $\mathbf{H}\mathbb{Q}$, and $\mathbf{MGL} \otimes \mathbb{Q}$ satisfy absolute purity over Spec($\mathbb{Z}$) (see [Dég18b, Rem. 1.3.5]). It was conjectured in [Dég18b, Conjectures B and C] that **MGL** and $\mathbb{S}$ also satisfy absolute purity. In a work in preparation, we will show that this is also the case for the hermitian K-theory spectrum **BO** in Example 4.4.5.

Remark 4.3.14. Note that our definition of absolute purity is formally advantageous. For example, recall it was deduced in [CD19] that **KGL** satisfies absolute purity. This implies that $\mathbf{KGL}_{\mathbb{Q}}$ satisfies absolute purity. We know that $\mathbf{H}_{\mathrm{B}}$ is a direct factor of $\mathbf{KGL}_{\mathbb{Q}}$. According to our definition, this implies that $\mathbf{H}_{\mathrm{B}}$ is absolutely pure. This argument allows to bypass the proof of [CD19, Th. 14.4.1].

4.4. **Examples.** We first consider the oriented case, which was already worked out in [Dég18b, Dég18a, Nav18]. We will show that we do in fact recover the constructions of *op.cit.* in this case.

Definition 4.4.1. Let S be a scheme and $\mathbb{E} \in S\mathscr{H}(S)$ a motivic spectrum. An *orientation* of $\mathbb{E}$ consists of the data of *Thom isomorphisms*

$$\tau_v^c : \mathbb{E}(X/S, v) \simeq \mathbb{E}(X/S, r), \tag{4.4.1.a}$$

for every s-scheme X over S and every class $v \in \mathrm{K}(X)$ of virtual rank r, which are functorial and respect the $\mathcal{E}_\infty$-group structure on $\mathrm{K}(X)$ up to a homotopy coherent system of compatibilities. Here we write simply $r \in \mathrm{K}(X)$ for the class of the trivial bundle of rank r.

Example 4.4.2. An **MGL**-module structure on $\mathbb{E}$ (where **MGL** is viewed as an $\mathcal{E}_\infty$-ring spectrum) gives rise to an orientation of $\mathbb{E}$.

4.4.3. Let $\mathbb{E} \in S\mathscr{H}(S)$ be an oriented motivic spectrum. Then for any smoothable lci s-morphism $f : X \to Y$ of s-schemes over S, denote by d_f the virtual dimension of f (i.e., d_f is the rank of $\langle L_f \rangle \in \mathrm{K}(X)$). Under the Thom isomorphism (4.4.1.a), the fundamental class $\eta_f \in \mathbb{E}(X/Y, \langle L_f \rangle)$ corresponds to a class

$$\eta_f^c = \tau_{L_f}^c(\eta_f) \in \mathbb{E}(X/Y, d_f).$$

This latter class can be viewed as another orientation (η_f^c, d_f), and coincides with the fundamental class defined in [Dég18a, 2.5.3].[12] These orientations also form a system of fundamental classes as in Definition 2.3.6, and they define Gysin maps

$$f_c^! = f^!(\eta_f^c) : \mathbb{E}(X/S, r) \to \mathbb{E}(Y/S, d_f + r), \quad x \mapsto \eta_f^c.x,$$

which are related to the Gysin maps of Theorem 4.2.1 via a commutative diagram:

$$\begin{tikzcd}
\mathbb{E}(X/S, v) \arrow[r, "f^!"] \arrow[d, "\tau_v^c"', "\wr"] & \mathbb{E}(Y/S, \langle L_f \rangle + f^*v) \arrow[d, "\tau^c_{L_f + f^*v}", "\wr"'] \\
\mathbb{E}(X/S, r) \arrow[r, "f_c^!"] & \mathbb{E}(Y/S, d_f + r).
\end{tikzcd}$$

[12] One reduces to the case of regular closed immersion and smooth morphisms. The case of smooth morphisms is obvious (which reduces to reduces to the six functors formalism). For the case of regular closed immersions, using the deformation to the normal cone and the compatibility of the two fundamental classes with transverse base change, we reduce to the case of the zero section of a vector bundle. This case follows because both fundamental classes gives the (refined) Thom class of the vector bundle, by design.

Therefore, Theorem 4.1.4 gives in particular a homotopy coherent refinement of the construction of [Dég18a]. Note, by the way, that the diagram above gives a simple proof of the Grothendieck–Riemann–Roch formula (cf. [Dég18a, 3.2.6 and 3.3.10] for the formulation in $\mathbb{A}^1$-homotopy). Indeed, it boils down to the definition of the Todd class (cf. [Dég18a, 3.2.4 and 3.3.5]). Compared to *op. cit.*, this proof does not require choosing a factorization of f.[13]

Example 4.4.4. *Higher Chow groups.* Suppose that S is the spectrum of a field k of characteristic exponent p. Taking $\mathbb{E} \in \mathscr{SH}(S)$ to be the (oriented) motivic cohomology spectrum $\mathbf{H}\mathbb{Z}$, we may identify the resulting bivariant theory with Bloch's higher Chow groups, up to inverting p [Dég18a, Example 1.2.10(1)]. The construction of Paragraph 4.4.3 then gives Gysin maps in higher Chow groups (with p inverted) for arbitrary smoothable lci s-morphisms. By the results of Sect. 4.2, these Gysin maps are functorial and satisfy transverse base change and excess intersection formulas.

We now proceed to consider some new examples.

Example 4.4.5. *Hermitian K-theory.* Let $S = \mathrm{Spec}(\mathbb{Z}[1/2])$. According to [PW18b][14], for any regular S-scheme X, there exists a motivic ring spectrum $\mathbf{BO}_X \in \mathscr{SH}(X)$ that represents hermitian K-theory of smooth X-schemes. In view of its geometric model (denoted by $\mathbf{BO}^{geom}$ in *op.cit.*), $\mathbf{BO}$ is defined over S (in the sense that there are canonical isomorphisms $\mathbf{BO}_X \simeq f^*(\mathbf{BO}_S)$ for every $f : X \to S$). Note that for non-regular schemes, $\mathbf{BO}$-cohomology is a *homotopy invariant* version of hermitian K-theory (on the model of [Cis13]), though this notion has not yet been introduced and worked out as far as we know.

The twisted bivariant theory associated with $\mathbf{BO}$ as above is new. The Gysin morphisms that one gets on $\mathbf{BO}$-cohomology are also new, at least in the generality of arbitrary proper smoothable lci s-morphisms, between arbitrary schemes (possibly singular and not defined over a base field). In the case of regular schemes, our construction for some part of hermitian K-theory (namely, that which compares to Balmer's higher Witt groups) should be compared to that of [CH11]. This would require a similar discussion to that of Paragraph 4.4.3 as, according to Panin and Walter, hermitian K-theory has a special kind of orientation which allows to consider only twists by line bundles (see also the next example). We intend to come back to these questions in a future work.

Example 4.4.6. *Higher Chow–Witt groups.* Let k be a perfect field. Introduced by Barge and Morel, the theory of Chow–Witt groups was fully developed by Fasel [Fas07, Fas08]. More recently, the theory was extended to "higher Chow–Witt groups" in a series of works [CF14, DF17a, DF17b]. In particular, given any coefficient ring R, there exists a motivic ring spectrum $\mathbf{H}_{\mathrm{MW}}\, R$ in $\mathscr{SH}(k)$ called the *Milnor–Witt spectrum* (cf. [DF17b, 3.1.2]). We denote by $H^{MW}(X/k, v, R)$ (resp. $H_{MW}(X, v, R)$) its associated bivariant theory (resp. cohomology).

For any smooth s-scheme X over k and any $v \in \mathrm{K}(X)$ of virtual rank r, one has a canonical isomorphism

$$H^0_{MW}(X, v, R) \simeq \widetilde{\mathrm{CH}}^r(X, \det(v)) \otimes R,$$

which is contravariantly functorial in X and covariantly functorial in v [DF17a, 4.2.6, 4.2.7]. In particular, the ring spectrum $\mathbf{H}_{\mathrm{MW}}$ is symplectically oriented in the sense of Panin and Walter [PW18b]. When X is possibly non-smooth, the bivariant theory $H_0^{MW}(X/k, v)$ can be computed by a Gersten complex with coefficients in the Milnor–Witt ring of the residue fields, so we can put:

$$H_0^{MW}(X/k, v, R) = \widetilde{\mathrm{CH}}_r(X, \det(v)) \otimes R$$

and view this as the Chow–Witt group of the scheme X. Similarly, the groups $H_i^{MW}(X/k, v)$ for $i \geq 0$ can be viewed as the *higher Chow–Witt groups*. In fact, we have canonical maps

$$\varphi_X : H_i^{MW}(X/k, v, R) \to \mathrm{CH}_n(X, i) \otimes R,$$

[13] For the record, Grothendieck mentioned that such a direct proof of his formula, without going through a factorisation and the use of a blow-up, should exist.

[14] The paper of Panin and Walter is not yet published. If one instead sets $S = \mathrm{Spec}(k)$, with k a field of characteristic different from 2, then one can take the ring spectrum constructed in [Hor05].

where n is the rank of the virtual bundle v, which are functorial in X with respect to proper pushforward (resp. pullback along open immersions).

The construction of Theorem 4.2.1 gives Gysin maps on these higher Chow–Witt groups, for any smoothable lci s-morphisms. These Gysin maps are functorial and satisfy transverse base change and excess intersection formulas. Furthermore, the maps φ_X are compatible with Gysin morphisms by construction. All in all, we get a robust bivariant theory of higher Chow–Witt groups.

Example 4.4.7. $\mathbb{A}^1$*-homology.* Let S be a scheme. Recall that for any commutative ring R, there is a motivic ring spectrum $\mathsf{N}R_S \in S\mathscr{H}(S)$ representing $\mathbb{A}^1$-homology with coefficients. This is nothing else than the R-linearization $\mathbb{S}_S \otimes R$ of the motivic sphere spectrum (see [CD19, 5.3.35] for another description). It is clear that $\mathsf{N}R$ is stable under base change in the sense that there are tautological isomorphisms $f^*(\mathsf{N}R_S) \simeq \mathsf{N}R_T$ for every morphism $f : T \to S$.

By Theorem 4.2.1 we obtain Gysin morphisms for the associated bivariant theories and cohomologies. Note in particular that this gives a very general notion of transfer maps in cohomology, along arbitary finite lci morphisms, extending the definitions of Morel in [Mor12].[15]

4.5. **Application: specializations.** In this subsection we investigate two of the many applications of the theory of refined Gysin maps (Definition 4.2.5). Throughout this subsection, we fix a motivic ∞-category of coefficients $\mathscr{T}$, a scheme S, and an object $E \in \mathscr{T}(S)$.

4.5.1. Let S be a scheme. For an s-scheme X over S, any section $s : S \to X$ which is a *regular* closed immersion, and any s-morphism $p : Y \to X$, consider the cartesian square

$$\begin{tikzcd} Y_s \arrow[r, "t"] \arrow[d, "p_s"'] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "p"] \\ S \arrow[r, "s"'] & X. \end{tikzcd}$$

The associated refined Gysin map takes the form

$$t^!_\Delta : \mathbb{E}(Y/S, e) \to \mathbb{E}(Y_s/S, -p_s^*\langle N_S X\rangle + e)$$

for any $e \in \mathrm{K}(Y)$. Thus any class $\alpha \in \mathbb{E}(Y/S, e)$ determines a family of *specializations*

$$\alpha_s = t^!_\Delta(\alpha) \in \mathbb{E}(Y_s/S, -p_s^*\langle N_S X\rangle + e)$$

for every s.

Example 4.5.2. In the case where $S = \operatorname{Spec}(k)$ is the spectrum of a field, we can take $s : S \to X$ to be any regular k-rational point.

Remark 4.5.3. If $\mathbb{E}$ is oriented, then we can identify $\mathbb{E}(Y_s/S, e - p_s^*\langle N_S X\rangle) \simeq \mathbb{E}(Y_s/S, e - d)$, where d is the codimension of the point s, and thus view α_s as a class in $\mathbb{E}(Y_s/S, e - d)$. The same can be accomplished in general up to some choice of a trivialization of the normal bundle $N_S X$.

Example 4.5.4. In the case where $S = \operatorname{Spec}(k)$ is the spectrum of a field and $\mathbb{E} = \mathbf{H}\mathbb{Z}$ (Example 4.4.4), the construction of Paragraph 4.5.1 generalizes Fulton's construction in [Ful98, Sect. 10.1] to higher Chow groups.

Example 4.5.5. Suppose that $S = \operatorname{Spec}(k)$ is the spectrum of a perfect field and take now $\mathbb{E}$ to be the Milnor–Witt spectrum (Example 4.4.6). In this case the construction of Paragraph 4.5.1 gives a refinement of Example 4.5.4 with "coefficients in quadratic forms". In particular, we can specialize Chow–Witt cycles: let X be a smooth and connected scheme over k of dimension d and let $p : Y \to X$ be an s-morphism. Then Y can be considered as a family of k-schemes parametrized by X, and given any Chow–Witt cycle $\alpha \in \widetilde{\mathrm{CH}}_d(Y)$ we get specializations $\alpha_s \in \widetilde{\mathrm{CH}}_0\left(Y_s, p_s^* \det(-N_S X)\right)$.

[15] Morel defines transfer only for finite field extensions, but he works unstably.

Beware however that the theory is more complicated than the case of usual Chow groups. For instance, assume that the morphism $p : Y \to X$ is proper. Then each $p_s : Y_s \to S$ is proper and we may consider the *degree* of α_s in the (twisted) Grothendieck–Witt group, i.e.:

$$\deg(\alpha_s) = (p_s)_*(\alpha_s) \in \widetilde{\mathrm{CH}}_0\left(S, \det(-N_S X)\right) \simeq \mathrm{GW}(k, \det(-N_S X)) \simeq \mathrm{GW}(k).$$

Now unlike in the Chow groups, these degrees depend in general on the point s. In fact, the Chow–Witt cycle $p_*(\alpha) \in \widetilde{\mathrm{CH}}_n(X)$ corresponds to the class of an unramified quadratic form φ in $\mathrm{GW}(\kappa(X))$, and the class $\deg(\alpha_s)$ is the specialization of φ at s.

4.5.6. We now construct an analogue of Fulton's specialization map in [Ful98, Sect. 20.3]. Suppose given cartesian squares

$$\begin{tikzcd} X_Z \arrow[r, hook, "i_X"] \arrow[d, "f_Z"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "f"] & X_U \arrow[l, hook', "j_X"'] \arrow[d, "f_U"] \\ Z \arrow[r, hook, "i"'] & S & U \arrow[l, hook', "j"] \end{tikzcd}$$

where i is a regular closed immersion of codimension d, j is the inclusion of the open complement, and f is an s-morphism. Suppose also given the choice of a null-homotopy $e(N_Z S) \simeq 0$ in $H(Z, \langle N_Z S\rangle)$ (for example when the bundle $N_Z S$ is trivial, any choice of trivialization gives rise to such a null-homotopy by Proposition 3.1.7). For any object $A \in \mathscr{T}(X)$, the composition

$$i_{X*}(i_X^* A \otimes f_Z^* Th(-N_Z S)) \to i_{X*} i_X^! A \to A \to i_{X*} i_X^* A$$

where the first map is induced by the refined fundamental class of the square Δ as in 4.3.1 and the other maps are obtained from adjunctions, agrees with the multiplication by the class $f_Z^* e(N_Z S)$ by the self-intersection formula (Corollary 4.2.3), which is then null-homotopic by our hypothesis. Therefore using the localization triangle (Proposition 2.2.10), we obtain a natural transformation of the form

$$i_{X*}(i_X^* A \otimes f_Z^* Th(-N_Z S)) \to j_{X!} j_X^! A. \tag{4.5.6.a}$$

Now let $\mathbb{E} \in \mathscr{T}(S)$ be an i-pure spectrum. Then for any $e \in \mathrm{K}(X)$, the map (4.5.6.a) induces the following *specialization map*:

$$\sigma : \mathbb{E}(X_U/U, e) \simeq \mathbb{E}(X_U/S, e) \to \mathbb{E}(X_Z/S, e - f_Z^*\langle N_Z S\rangle) \simeq \mathbb{E}(X_Z/Z, e). \tag{4.5.6.b}$$

Remark 4.5.7. In the case where $S = \mathrm{Spec}(k)$ is the spectrum of a field and $\mathbb{E} = \mathbf{H}\mathbb{Z}$ (Example 4.4.4), the map (4.5.6.b) generalizes Fulton's specialization map in [Ful98, Sect. 20.3] to higher Chow groups.

Remark 4.5.8. The construction of Paragraph 4.5.6 is compatible with Ayoub's motivic nearby cycle functor in the following sense. Let S be the spectrum of a field k of characteristic 0, $i : S \to \mathbb{A}^1_S$ the inclusion of the origin, and $j : U = \mathbb{G}_{m,S} \to \mathbb{A}^1_S$ the complement. Let $f : X \to \mathbb{A}^1_S$ be a smooth morphism, $e \in \mathrm{K}(X)$. Note that any $\mathbb{E} \in S\mathscr{H}(S)$ is i-pure by Proposition 4.3.10, and that there is a canonical trivialization of the normal bundle $N_S(\mathbb{A}^1_S)$. Therefore the requirements of Paragraph 4.5.6 are satisfied and we obtain a specialization map $\sigma : \mathbb{E}(X_U/U, e) \to \mathbb{E}(X_S/S, e)$ as in (4.5.6.b). Then this map is induced by the canonical natural transformation

$$i_X^* j_{X*} \to \Psi_f$$

where $\Psi_f : S\mathscr{H}(X_U) \to S\mathscr{H}(X_S)$ is the motivic nearby cycle functor in [Ayo07, 3.5.6].

4.6. **Application: the motivic Gauss-Bonnet formula.** Let $p : X \to S$ be a smooth proper morphism. Recall that the spectrum $\Sigma^\infty_+(X) \simeq p_! p^!(\mathbb{S}_S)$ is a strongly dualizable object of $S\mathscr{H}(S)$ [CD19, Prop. 2.4.31], so that we may consider the *trace* of its identity endomorphism. This is an endomorphism $\chi^{cat}(X/S) \in \mathrm{Maps}_{S\mathscr{H}(S)}(\mathbb{S}_S, \mathbb{S}_S)$ that we refer to as the *categorical Euler characteristic*; see [Hoy15, § 3] for details. In this subsection, we view $\chi^{cat}(X/S)$ as a class in $H(S/S, 0)$, and compute it as the "degree of the Euler class of the tangent bundle":

Theorem 4.6.1. *Let $p : X \to S$ be a smooth proper morphism. Then there is an identification $\chi^{cat}(X/S) \simeq p_*(e(T_{X/S}))$ in the group $H(S,0)$.*

Remark 4.6.2. Let S be the spectrum of a field of characteristic different from 2, and let $p : X \to S$ be a smooth projective morphism. Under these assumptions, a version of Theorem 4.6.1 was proven recently by Levine [Lev17b, Theorem 1]. The formulation of *loc. cit.* can be recovered from Theorem 4.6.1 by applying the $\mathbb{A}^1$-regulator map (Definition 4.1.2).

4.6.3. In order to prove Theorem 4.6.1, we begin by giving a useful intermediate description of $\chi^{cat}(X/S)$. Consider the cartesian square

$$\begin{tikzcd} X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & S \end{tikzcd}$$

and let $\delta : X \to X \times_S X$ denote the diagonal (a regular closed immersion).

Lemma 4.6.4. *The endomorphism $\chi^{cat}(X/S) : \mathbb{S}_S \to \mathbb{S}_S$ is obtained by evaluating the following natural transformation at the monoidal unit $\mathbb{S}_S$:*

$$\begin{tikzcd} \mathrm{Id} \arrow[rrrr, dashed] \arrow[d, "\mathrm{unit}"'] & & & & \mathrm{Id} \\ p_*p^* \arrow[r, equal] & p_*\delta^!(\pi_2)^!p^* & p_*\delta^!(\pi_1)^*p^! \arrow[l, "\sim", "Ex^{*!}"'] \arrow[r, "\theta"'] & p_*\delta^*(\pi_1)^*p^! \arrow[r, equal] & p_*p^! \arrow[u, "\mathrm{counit}"'] \end{tikzcd}$$

where $\theta : \delta^! \to \delta^*$ is the exchange transformation $Ex^{*!} : \mathrm{Id}^*\,\delta^! \to \mathrm{Id}^!\,\delta^*$.

Proof. This follows from the description given in [Hoy15, Prop. 3.6], in view of the commutativity of the diagram

$$\begin{tikzcd} \delta^!(\pi_2)^!p^* \arrow[r, equal] & p^* \arrow[r, equal] & (\pi_1)_*\delta_*\delta^*(\pi_2)^!p^* \arrow[r, "\varepsilon"] & (\pi_1)_*(\pi_2)^!p^* \\ \delta^!(\pi_1)^*p^! \arrow[u, "Ex^{*!}", "\wr"'] \arrow[r, "\theta"] & p^! \arrow[r, equal] & (\pi_1)_*\delta_*\delta^*(\pi_1)^*p^! & (\pi_1)_*(\pi_1)^*p^! \arrow[l, "\eta"'] \arrow[u, "Ex^{*!}", "\wr"'] \end{tikzcd}$$

which the reader will easily verify. □

Proof of Theorem 4.6.1. By Lemma 4.6.4, it will suffice to show that the following diagram commutes:

$$\begin{tikzcd} p^* = \delta^!(\pi_2)^!p^* \arrow[dr, "\Sigma^{-T_p}*\mathfrak{p}_p"'] & \delta^!(\pi_1)^*p^! \arrow[l, "Ex^{*!}"'] \arrow[r, "\theta"] & \delta^*(\pi_1)^*p^! = p^! \\ & \Sigma^{-T_p}p^! \arrow[u, "\mathfrak{p}_\delta"] \arrow[ur, "\mathfrak{e}_p * p^!"'] & \end{tikzcd}$$

Here we have written $\mathfrak{e}_p$ for the natural transformation $\Sigma^{-T_p} \to \mathrm{Id}$ induced by the Euler class $e(T_p) : \mathbb{S}_X \to \mathrm{Th}_X(T_p)$. The commutativity of the left-hand triangle follows by construction of the fundamental class of p (Example 2.3.4) and Corollary 3.2.17. For the right-hand triangle, commutativity follows immediately from the self-intersection formula (Example 3.2.9((ii))), which asserts the commutativity of the square

$$\begin{tikzcd} \mathrm{Th}_X(-T_p) \arrow[r, "\eta_\delta"] \arrow[d, equal] & \delta^!(\pi_1)^*(\mathbb{S}_X) \arrow[r, "\theta"] & \delta^*(\pi_1)^*(\mathbb{S}_X) \arrow[d, equal] \\ \mathrm{Th}_X(-T_p) \arrow[rr, "e(T_p)"] & & \mathbb{S}_X. \end{tikzcd}$$

□

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