EQUIVARIANT GENERALIZED COHOMOLOGY VIA STACKS

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ABSTRACT. We prove a general form of the statement that the cohomology of a quotient stack can be computed by the Borel construction. It also applies to the lisse extensions of generalized cohomology theories like motivic cohomology and algebraic cobordism. We use this to prove a (higher) equivariant Grothendieck–Riemann–Roch theorem, comparing Borel-equivariant G-theory and equivariant Chow groups. Finally, we give a Bernstein–Lunts-type gluing description of the ∞-category of equivariant sheaves on a scheme X, in terms of nonequivariant sheaves on X and sheaves on its Borel construction.

INTRODUCTION

The formalism of Grothendieck's six operations on (derived categories of) étale sheaves can be extended to algebraic stacks (see [LZ, LO]). Specialized to quotient stacks, this affords simple definitions of equivariant (co)homology. For example, if X is a variety with an action of an algebraic group G, we may define the G-equivariant Borel–Moore homology of X (with coefficients in a commutative ring Λ) as the hypercohomology of the complex $f^!(\Lambda_{BG})$, where Λ_{BG} is the constant sheaf on the classifying stack BG and $f : [X/G] \to BG$ is the projection from the quotient stack.

Classically, equivariant cohomology and (Borel–Moore) homology are defined via algebraic analogues of the Borel construction [Bor]. It is well-known that the two approaches should agree. In this paper, we prove a very general form of such a comparison that also applies to generalized motivic cohomology theories, and study the consequences for equivariant intersection theory.

On first pass, let us formulate the result for Betti or ℓ-adic sheaves. Given a locally of finite type Artin stack X over a field k, let $\mathbf{D}(X)$ denote the ∞-category of Betti or ℓ-adic sheaves on X. If G is an affine algebraic group over k acting on X, let $\mathrm{H}^*_G(X)$ and $\mathrm{H}^{\mathrm{BM},G}_{-*}(X)$ denote the hypercohomology groups of $f_* f^*(\Lambda_{BG})$ and $f_* f^!(\Lambda_{BG})$, respectively (where Λ denotes the constant sheaf $\mathbf{Q}$ or $\mathbf{Q}_\ell$, in the Betti or ℓ-adic cases respectively). We regard the Borel construction as a G-equivariant ind-scheme $U_\infty = \{U_\nu\}_\nu$, e.g. for $G = \mathrm{GL}_n$ this is the infinite Grassmannian $\mathrm{Gr}_{n,\infty}$ of rank n subspaces (see Sect. 3 for details). Then we claim (see Corollaries 5.2 and 5.4):

Theorem A. *Suppose G acts on a finite-dimensional Artin stack X of finite type over k. Then for every integer $n \in \mathbf{Z}$ there are canonical isomorphisms*

$$\mathrm{H}^n_G(X) \simeq \mathrm{H}^n(X \overset{G}{\times} U_\infty),$$

$$\mathrm{H}^{\mathrm{BM},G}_n(X) \simeq \mathrm{H}^{\mathrm{BM}}_n(X \overset{G}{\times} U_\infty).$$

Here we have written $X \times^G U_\infty$ for the quotient ind-stack $[(X \times U_\infty)/G]$, and we have (essentially by definition)
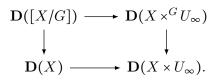
$$\mathrm{H}^n(X \overset{G}{\times} U_\infty) \simeq \varprojlim_\nu \mathrm{H}^n(X \overset{G}{\times} U_\nu),$$

$$\mathrm{H}^{\mathrm{BM}}_n(X \overset{G}{\times} U_\infty) \simeq \varprojlim_\nu \mathrm{H}^{\mathrm{BM}}_{n+2d_\nu}(X \overset{G}{\times} U_\nu)(-d_\nu),$$

where $d_\nu = \dim(U_\nu/G)$.

For X a quasi-projective k-scheme on which G acts linearly, each $X \times^G U_\nu$ is a scheme because G acts freely on U_ν. The right-hand sides of Theorem A are often taken as *definitions* of equivariant cohomology and Borel–Moore homology (see e.g. [Lus, §1]). They have also been considered in the case where X itself is a stack, e.g. the moduli stack of objects in an abelian or dg-category (see [Joy, §2.3]).

Our second main result provides a complete description of the ∞-category of sheaves on a quotient stack. It relates $\mathbf{D}([X/G])$ to the equivariant derived category of Bernstein and Lunts (compare [BL, Def. 2.1.3]).

Theorem B. *For every Artin stack X locally of finite type over k with G-action, there is a cartesian square of ∞-categories*

$$
\begin{array}{ccc}
\mathbf{D}([X/G]) & \longrightarrow & \mathbf{D}(X \times^G U_\infty) \\
\downarrow & & \downarrow \\
\mathbf{D}(X) & \longrightarrow & \mathbf{D}(X \times U_\infty).
\end{array}
$$

The ∞-category $\mathbf{D}(X \times^G U_\infty)$ is the limit over ν of $\mathbf{D}(X \times^G U_\nu)$. Thus Theorem B asserts in particular that a sheaf on the quotient stack $[X/G]$ amounts to the data of:

(i)   a sheaf $\mathcal{G} \in \mathbf{D}(X)$;

(ii)   a collection of sheaves $\mathcal{F}_\nu \in \mathbf{D}(X \times^G U_\nu)$ for every ν, with compatibility isomorphisms $\mathcal{F}_\nu|_{X \times^G U_{\nu+1}} \simeq \mathcal{F}_{\nu+1}$;

(iii) for every ν, an isomorphism $\mathcal{G}|_{X \times U_\nu} \simeq \mathcal{F}_\nu|_{X \times U_\nu}$;

and compatibilities between the isomorphisms of (ii) and (iii). See Corollary 9.5. Note that, in contrast with [BL], we work with unbounded derived categories.

To prove Theorem A, we will first consider the analogous statement at the level of derived global sections. We will see (Theorem 3.7 and Corollary 4.1) that for any sheaf $\mathcal{F} \in \mathbf{D}([X/G])$, the canonical map

$$R\Gamma([X/G], \mathcal{F}) \to \varprojlim_\nu R\Gamma(X \overset{G}{\times} U_\nu, \mathcal{F})$$

is invertible, where the limit is a *homotopy* limit in the ∞-category of spectra. The obstruction to passing from this statement to the analogous statement on hypercohomologies,

$$\mathrm{H}^i([X/G], \mathcal{F}) \to \varprojlim_\nu \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F}),$$

is the vanishing of Milnor's $\varprojlim^1$ term. We will show (Corollary 4.16) that this obstruction vanishes when $\mathcal{F}$ is eventually coconnective with respect to the cohomological t-structure (i.e., cohomologically bounded below).

As alluded to earlier, the main result of this paper is in fact an extension of Theorem A to generalized cohomology theories such as motivic cohomology ($\approx$ higher Chow groups), algebraic cobordism, and (variants of) algebraic K-theory. Similarly, we will prove a version of Theorem B for the stable motivic homotopy category. In particular, we will see:

Theorem C. *Let* $E \in \mathbf{SH}(k)_{<\infty}$ *be a motivic spectrum which is eventually coconnective for the homotopy t-structure. If* G *acts on an Artin stack* X *of finite type over* k *with separated diagonal, there are canonical isomorphisms*

$$\mathrm{H}^i([X/G], E)[\tfrac{1}{e}] \simeq \varprojlim_\nu \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, E)[\tfrac{1}{e}]$$

for every $i \in \mathbf{Z}$, *where* e *is the characteristic exponent of the field* k.

The work of Voevodsky, Ayoub, and Cisinski–Déglise provides the formalism of six operations on the ∞-categories $\mathbf{SH}(S)$ of motivic spectra over schemes S (see [Ayo, CD1]). Recently, this has been extended[1] to algebraic stacks[2] by various authors (see [RS, §2], [Kha2, App. A], [Cho], and [Kha5, §7]). We may thus use the above approach with quotient stacks to define equivariant Borel–Moore homology with coefficients in any motivic spectrum $E \in \mathbf{SH}(k)$:

$$\mathrm{H}_s^{\mathrm{BM},G}(X; E)(-r) := \pi_s R\Gamma([X/G], f^!(E|_{BG})(-r)),$$

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[1]We are referring here to the *lisse-extended* variant, as opposed to the genuine theory constructed in [KR]. For quotient stacks this corresponds to the difference between Borel-equivariant and genuine-equivariant cohomology theories. An example of the latter is algebraic K-theory, as discussed in Sect. 7.

[2]under very mild separation hypotheses, see Theorem 0.7

where X is an algebraic space[3] with G-action and $r, s \in \mathbf{Z}$. On the other hand, previous definitions via the Borel construction have long been considered in the literature already, notably by B. Totaro [Tot], Edidin–Graham [EG1], Deshpande [Des], Heller–Malagón-López [HML], and A. Krishna [Kri1]. We will show that these approaches coincide at least *at the spectrum level* (see Corollary 6.2). As stated above in Theorem C, we also have a comparison at the level of homotopy groups when E is eventually coconnective.

For example, in the case of the motivic cohomology spectrum, this yields a comparison of equivariant motivic Borel–Moore homology with the equivariant higher Chow groups of Edidin–Graham (Corollary 6.5):

$$\mathrm{H}^{\mathrm{BM},G}_{s+2n}(X; \Lambda^{\mathrm{mot}})(-n) \simeq \mathrm{A}^G_n(X, s) \otimes \Lambda \tag{0.1}$$

for all $n, s \in \mathbf{Z}$, where on the right-hand side are the G-equivariant higher Chow groups of X [EG1, §2.7].

For $E = \mathrm{KGL}$ the algebraic K-theory spectrum in $\mathbf{SH}(k)$, we get the spectrum-level computation of lisse-extended G-theory of a quotient stack (see Corollary 7.4):

$$\mathrm{G}^\lhd([X/G]) \simeq \varprojlim_\nu \mathrm{G}(X \overset{G}{\times} U_\nu). \tag{0.2}$$

Of course, $\mathrm{G}^\lhd(-)$ does *not* agree with the genuine extension of G-theory to stacks. In fact, the right-hand side can often be identified with the derived completion of the G-theory spectrum $\mathrm{G}([X/G])$ with respect to the augmention ideal in the representation ring of G, see [TVdB, CJ]. The formula (0.2) means that

$$\mathrm{G}^{G,\lhd}(-) := \mathrm{G}^\lhd([-/G]), \qquad \mathrm{G}^{G,\lhd}_s(-) := \pi_s \mathrm{G}^\lhd([-/G]), \ s \in \mathbf{Z}, \tag{0.3}$$

is Borel-type G-equivariant G-theory, and on algebraic stacks $\mathrm{G}^\lhd(-)$ may be regarded as a "globalization" thereof.[4]

Upon rationalization we deduce the following (higher) equivariant Grothendieck–Riemann–Roch theorem (where $(-)_{\mathbf{Q}} := (-) \otimes \mathbf{Q}$).

Theorem D (Equivariant GRR). *Let k be a field, G an affine algebraic group over k, and X a quasi-separated algebraic space of finite type over k with G-action. Then for every integer $s \in \mathbf{Z}$ there is a canonical isomorphism*

$$\mathrm{G}^{G,\lhd}_s(X)_{\mathbf{Q}} \simeq \prod_{n \in \mathbf{Z}} \mathrm{A}^G_n(X, s)_{\mathbf{Q}},$$

where on the right-hand side are the G-equivariant higher Chow groups of X [EG1, §2.7]. Moreover, this is compatible with equivariant proper push-forwards and equivariant quasi-smooth Gysin pull-backs.

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[3]Note that unlike in the Betti/étale cases, we avoid speaking of equivariant Borel–Moore homology of general Artin stacks. That is because in the context of $\mathbf{SH}$, no full account of the construction of the !-functors for non-representable morphisms exists in the literature at the time of writing (although see [Kha4, §4] for a sketch).

[4]Since KGL is not eventually coconnective (so that Theorem C does not apply), there is a potential $\varprojlim^1$ obstruction to computing $\mathrm{G}^{G,\lhd}_s(X)$ as $\varprojlim_\nu \mathrm{G}_s(X \times^G U_\nu)$.

For $s = 0$ the left-hand side becomes the completion of the (genuine) G-equivariant G_0 of X at the augmentation ideal I_G, so we get

$$\mathrm{G}_0^G(X)^{\wedge}_{\mathbf{Q},I_G} \simeq \prod_{n\in\mathbf{Z}} \mathrm{A}_n^G(X)_{\mathbf{Q}}. \tag{0.4}$$

This recovers the equivariant Grothendieck–Riemann–Roch theorem of Edidin–Graham [EG2, Thm. 4.1]. A higher generalization of the latter was previously obtained by A. Krishna for X *smooth* and quasi-projective [Kri2, Thm. 1.2].

Consider now the case of Voevodsky's algebraic cobordism spectrum $E = \mathrm{MGL} \in \mathbf{SH}(k)$. For simplicity, we write for all $n, s \in \mathbf{Z}$

$$\mathrm{MGL}_n^G(X) := \mathrm{H}_{2n}^{\mathrm{BM},G}(X; \mathrm{MGL})(-n)$$

and similarly for $\mathrm{MGL}_n(X)$. When k is of characteristic zero, a geometric model $\Omega_*(-)$ for "lower" algebraic bordism of quasi-projective k-schemes was given by Levine and Morel [LM]; that it agrees with $\mathrm{MGL}_n(X)$ was proven by Levine [Lev]. For the (lower) G-equivariant algebraic bordism theory

$$\Omega_n^{G,\mathrm{HML}}(X) := \varprojlim_{\nu} \Omega_{n+d_\nu-g}(X \overset{G}{\times} U_\nu),$$

considered by J. Heller and J. Malagón-López [HML] (where $d_\nu = \dim(U_\nu)$ and $g = \dim(G)$), we get surjections

$$\mathrm{MGL}_n^G(X) \twoheadrightarrow \Omega_n^{G,\mathrm{HML}}(X)$$

from G-equivariant motivic Borel–Moore homology with coefficients in MGL. But as Theorem C does not apply, because MGL is not eventually coconnective, we do not know whether this map is bijective in general. We will see that it is so with rational coefficients (see Theorem 8.4).

The issue seems related to the question, still open, of right-exactness of the localization sequence

$$\Omega_n^{G,\mathrm{HML}}(Z) \to \Omega_n^{G,\mathrm{HML}}(X) \to \Omega_n^{G,\mathrm{HML}}(U) \to 0 \tag{0.5}$$

for G-invariant closed subschemes $Z \subseteq X$ with open complement $U = X \smallsetminus Z$. In equivariant Chow homology the analogous property is obvious (see [EG1, Prop. 5]) since any homogeneous component of $\mathrm{A}_*^G(X)$ can be computed using a single approximation U_ν/G for large enough ν (as opposed to the entire tower $\{U_\nu/G\}_\nu$). In the case of $\Omega_*^G(-)$, A. Krishna[5] showed exactness at the end (i.e., surjectivity of restriction to an open) and explained why he did not believe exactness in the middle should hold (see Prop. 5.3 in [Kri1] and the discussion just before). While Heller–Malagón-López claimed that right-exact localization holds (see [HML, Thm. 20]), their argument does not in fact prove exactness in the middle (this gap is well-known in the area). We will show that $\mathrm{MGL}_*^G(-)$ does satisfy right-exact localization (see Theorem 8.12), and so therefore does $\Omega_*^{G,\mathrm{HML}}(-)_{\mathbf{Q}}$:

Theorem E. *Let k be a field of characteristic zero, G an affine algebraic group over k, and X a quasi-projective k-scheme with linearized G-action.*

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[5] Krishna [Kri1] used a slightly different definition of $\Omega_*^G(-)$, based on [Des]. It is shown however in [HML, Rem. 14] that the two are isomorphic.

Then for every G-invariant closed subscheme $Z \subseteq X$ with open complement $U = X \smallsetminus Z$ and every $n \in \mathbf{Z}$, there are exact sequences

$$\mathrm{MGL}_n^G(Z) \to \mathrm{MGL}_n^G(X) \to \mathrm{MGL}_n^G(U) \to 0$$

and

$$\Omega_n^{G,\mathrm{HML}}(Z)_\mathbf{Q} \to \Omega_n^{G,\mathrm{HML}}(X)_\mathbf{Q} \to \Omega_n^{G,\mathrm{HML}}(U)_\mathbf{Q} \to 0.$$

0.1. **Contents.** We begin in Sect. 1 by recalling the lisse extension construction from [KR, §12]. Our first key result (Proposition 2.5) is proven in Sect. 2: given a lisse-extended $\mathbf{A}^1$-invariant Nisnevich sheaf F, a stack $\mathfrak{X}$, and an ind-stack $\mathcal{U}_\infty$ over $\mathfrak{X}$ satisfying appropriate hypotheses, it states that $F(\mathfrak{X})$ can be computed as the limit of the pro-system $F(\mathcal{U}_\infty)$. This is inspired by arguments extracted from [MV].

In Sect. 3 we specialize to $\mathfrak{X} = [X/G]$ and $\mathcal{U}_\infty = \{X \times^G U_\nu\}_\nu$. Proposition 2.5 implies in this case that for a lisse-extended $\mathbf{A}^1$-invariant sheaf of spectra F, its value on a quotient stack $[X/G]$ can be computed as the homotopy limit

$$F([X/G]) \simeq \varprojlim_\nu F(X \overset{G}{\times} U_\nu).$$

Our next main technical results, contained in Sect. 4, deal with the question of when we can pass to homotopy groups to get isomorphisms

$$\pi_i F([X/G]) \simeq \varprojlim_\nu \pi_i F(X \overset{G}{\times} U_\nu).$$

We will show that this is possible when F is the cohomology theory represented by a sheaf in an appropriate six functor formalism, using connectivity estimates to establish the Mittag–Leffler condition; see Corollary 4.16.[6]

Finally, we will apply this to Betti/étale cohomology Sect. 5 (proving Theorem A) and to generalized cohomology theories, i.e., cohomology theories representable in the stable motivic homotopy category, in Sect. 6 (proving Theorem C). This is applied in Sect. 7 to deduce equivariant Grothendieck–Riemann–Roch (Theorem D). The implications for equivariant algebraic (co)bordism are described in Sect. 8, which in particular contains the proof of Theorem E.

Finally, Sect. 9 is dedicated to the proof of Theorem B.

0.2. **Conventions on stacks.**

0.2.1. *The Nisnevich topology.* Recall that the Nisnevich topology on schemes is generated by families of étale morphisms $(Y_\alpha \to X)_\alpha$ such that the morphism $\coprod_\alpha Y_\alpha \to X$ is surjective on field-valued points (see e.g. [BH, App. A]).

A smooth morphism of schemes admits étale-local sections if and only if it is surjective (see [EGA, IV$_4$, Cor. 17.16.3(ii)]). Here is the Nisnevich analogue:

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[6]In a previous version, we had claimed the same for arbitrary F. However, this was based on a mistake in the formulation of Proposition 2.5.

Lemma 0.6. *Let $f : X \to Y$ be a smooth morphism of schemes. Then the following conditions are equivalent:*

(i) *The morphism f is surjective on field-valued points.*

(ii) *There exists a Nisnevich cover $Y' \twoheadrightarrow Y$ such that the base change $X \times_Y Y' \to Y'$ admits a section.*

Moreover, if Y is quasi-compact and quasi-separated, then Y' in (ii) can also be taken to be affine.

Proof. (ii) $\implies$ (i): We will show that for every field-valued point $y : \operatorname{Spec}(\kappa) \to Y$, the base change $X_{\{y\}} = X \times_Y \operatorname{Spec}(\kappa) \to \operatorname{Spec}(\kappa)$ admits a section. By base change, the condition implies that there is a Nisnevich cover $S \twoheadrightarrow \operatorname{Spec}(\kappa)$ such that $X_S \simeq X_{\{y\}} \times_{\operatorname{Spec}(\kappa)} S \to S$ admits a section. Since $S \twoheadrightarrow \operatorname{Spec}(\kappa)$ is surjective on field-valued points by definition, it admits a section. The composition of the section $\operatorname{Spec}(\kappa) \to S$, the section $S \to X_S$, and the morphism $X_S \to X_\kappa$ is then a section of $X_{\{y\}} \to \operatorname{Spec}(\kappa)$ as desired.

(i) $\implies$ (ii): Let $y : \operatorname{Spec}(\kappa) \to Y$ be a point and $x : \operatorname{Spec}(\kappa) \to X$ a lift. Since $f : X \to Y$ is smooth, there exists by [EGA, IV$_4$, 18.6.6(i), 18.5.17] a morphism $\tilde{x} : S \to X$ extending x, where S is the henselization of Y at y. Recall that S can be identified with the cofiltered limit of elementary étale neighbourhoods[7] of (Y, y). Since X is locally of finite presentation over Y, it follows that there exists an étale neighbourhood $Y'_y \to Y$ over y such that the Y-morphism $\tilde{x} : S \to X$ factors through $Y'_y \to X$. Then the disjoint union $Y' = \coprod_y Y'_y \twoheadrightarrow Y$ over all field-valued point y is an étale morphism which is surjective on field-valued points, i.e., a Nisnevich cover, with the desired property. If Y is quasi-compact, then there is a finite subcover refining Y' (see e.g. [EHIK, Lem. 2.1.2]), so in particular we may take Y' quasi-compact. We may then further replace Y' by a Zariski cover by an affine scheme. $\square$

0.2.2. *Stacks.* We work with *higher* stacks throughout the paper. Thus a *stack* is a presheaf of ∞-groupoids on the site of k-schemes that satisfies hyperdescent with respect to the étale topology.

Let $\tau \in \{\text{ét}, \text{Nis}\}$ stand for either the étale or Nisnevich topology. We say a morphism of stacks $f : X \to Y$ *admits τ-local sections* if, for any scheme T and any morphism $t : T \to Y$, there exists a τ-cover $T' \twoheadrightarrow T$ such that the base change $X \times_Y T' \to T'$ admits a section.

A morphism $f : X \to Y$ is *schematic* if for every scheme V and every morphism $V \to Y$, the fibred product $X \times_Y V$ is a scheme. A stack X is *$(\tau, 0)$-Artin* if it has schematic (-1)-truncated diagonal and there exists a scheme U and an étale morphism $U \to X$ with τ-local sections. For $\tau = \text{ét}$, these are the algebraic spaces; for $\tau = \text{Nis}$, these are the quasi-separated algebraic spaces by [Knu, Chap. 2, Thm. 6.4].

For $n > 0$, a morphism $f : X \to Y$ is *$(\tau, n-1)$-representable* if for every scheme V and every morphism $V \to Y$, the fibred product $X \times_Y V$ is $(\tau, n-1)$-Artin.

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[7] Here an elementary étale neighbourhood is an étale morphism $Y' \to Y$ along which y lifts to $y' : \operatorname{Spec}(\kappa) \to Y'$.

A stack X is (τ, n)-*Artin* if it has $(\tau, n-1)$-representable diagonal and there exists a scheme U and a smooth morphism $U \to X$ with τ-local sections.

A stack is τ-*Artin* if it is (τ, n)-Artin for some n.

The (ét, 1)-Artin stacks are Artin stacks (or algebraic stacks) as defined e.g. in [SP, Tag 026O]. More generally, the (ét, n)-Artin stacks and ét-Artin stacks are n-Artin stacks and higher Artin stacks as defined in [Gai, §4.2]. We will usually drop the "ét" from the notation.

A 1-Artin stack is (Nis, 1)-Artin if and only if it is quasi-separated with separated diagonal. The following is proven in [LMB, §6.7].

Theorem 0.7. *Let X be a quasi-separated 1-Artin stack with separated diagonal. Then there exists a scheme U and a smooth morphism $U \to X$ with Nisnevich-local sections. In particular, X is (Nis, 1)-Artin.*

0.2.3. *Fibre dimension.* We recall the notion of fibre dimension for a morphism of stacks, as studied in [EGA, IV$_3$, §13] and [SP, Tag 0DRQ].

Let $f : \mathcal{X} \to \mathcal{Y}$ be a locally of finite type 1-representable morphism of locally noetherian Artin stacks. Given a field κ and a κ-point x of $\mathcal{X}$, the *fibre dimension* of f at x is the dimension at x of the topological space underlying the fibre $f^{-1}(f(x)) = \mathcal{X}_{f(x)} = \mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}(\kappa)$. See [SP, Tag 0DRQ] for details, and [SP, Tag 04XG] for the definition of the underlying topological space of a 1-Artin stack.

The fibre dimension of $f : \mathcal{X} \to \mathcal{Y}$ determines a function on the underlying topological space of $\mathcal{X}$. This function is upper semi-continuous; see [EGA, IV$_3$, Thm. 13.1.3] and [SP, Tag 0DRE].

Lemma 0.8. *Let $f : Y \to X$ be a smooth morphism of locally noetherian schemes and $i : Z \hookrightarrow Y$ a closed immersion. Let $d_{Y/X}$ and $d_{Z/X}$ denote the fibre dimensions of f and $g = f \circ i$. For every point z of Z, we have $d_{Y/X}(z) - d_{Z/X}(z) = \mathrm{codim}(Z \times_X \{g(z)\}, Y \times_X \{g(z)\}).$*

Proof. Let $z : \mathrm{Spec}(k(z)) \to Z$ be a point of Z, with image $x = g(z)$ in X. By definition, the fibre dimensions of f and g at z are

$$d_{Y/X}(z) = \dim_z(Y \underset{X}{\times} \{x\}), \quad d_{Z/X}(z) = \dim_z(Z \underset{X}{\times} \{x\}),$$

respectively. Since Y is smooth over X, $Y \times_X \{x\}$ is in particular biequidimensional (combine [EGA, 0$_{\mathrm{IV}}$, Cor. 16.5.12], [EGA, IV$_2$, Cor. 5.10.9], and [Hei, Lem. 2.6]). It follows then from [Hei, Prop. 4.1] that

$$d_{Y/X}(z) - d_{Z/X}(z) = \mathrm{codim}(Z \underset{X}{\times} \{x\}, Y \underset{X}{\times} \{x\})$$

as claimed. $\qquad\square$

0.3. **Notation.** We denote by Ani the ∞-category of anima, a.k.a. homotopy types or ∞-groupoids. We work over a fixed commutative ring k, which we leave implicit in the notation. We denote by Sch (resp. Asp) the category of schemes (resp. quasi-separated algebraic spaces) of finite type over k.

The symbol $\tau \in \{\text{ét}, \text{Nis}\}$ will stand for the étale or Nisnevich topology. We denote by $^\tau\text{Stk}$ the ∞-category of τ-Artin stacks locally of finite type over k.[8] Given $S \in {}^\tau\text{Stk}$ we denote by $^\tau\text{Stk}_S$ the ∞-category of τ-Artin stacks locally of finite type over S. Note that any morphism in $^\tau\text{Stk}$ (resp. $^\tau\text{Stk}_S$) is automatically locally of finite type.

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1. Lisse extension

Given $\mathcal{X} \in {}^\tau\text{Stk}$, we denote by $\text{Lis}_{\mathcal{X}}$ (resp. $^\tau\text{LisStk}_{\mathcal{X}}$) the ∞-category of pairs (T, t) where $T \in \text{Sch}$ (resp. $T \in {}^\tau\text{Stk}$) and $t : T \to \mathcal{X}$ is a smooth morphism[9]. Let $F : \text{Lis}_{\mathcal{X}}^{\text{op}} \to \mathcal{V}$ be a presheaf, where $\mathcal{V}$ is an ∞-category with limits.

Definition 1.1. The *lisse extension* of F is the presheaf

$$F^{\triangleleft} : {}^\tau\text{LisStk}_{\mathcal{X}}^{\text{op}} \to \mathcal{V}$$

defined as the right Kan extension of F along the fully faithful functor $\text{Lis}_{\mathcal{X}} \hookrightarrow {}^\tau\text{LisStk}_{\mathcal{X}}$. In particular, we have

$$F^{\triangleleft}(\mathcal{X}) \simeq \varprojlim_{(T,t)} F(T)$$

where the limit is taken over $(T, t) \in \text{Lis}_{\mathcal{X}}$.

Given a presheaf $F : \text{Sch}^{\text{op}} \to \mathcal{V}$, we may restrict along the forgetful functor $\text{Lis}_{\mathcal{X}} \to \text{Sch}$ and form the lisse extension of the resulting presheaf $F_{\mathcal{X}}$ on $\text{Lis}_{\mathcal{X}}$. On the other hand, we may also describe $F_{\mathcal{X}}^{\triangleleft}$ more directly in terms of F. Moreover, this will work for presheaves with restricted functoriality (e.g. only for smooth or lci morphisms).

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[8]Thus every 1-Artin stack (in the usual sense) belongs to $^{\text{ét}}\text{Stk}$, and if it is quasi-separated with separated diagonal then also to $^{\text{Nis}}\text{Stk}$ (Theorem 0.7).

[9]Since $t : T \to \mathcal{X}$ is schematic for $T \in \text{Sch}$, it is smooth if and only if for any morphism $T' \to \mathcal{X}$ where T' is a scheme, the morphism of schemes $T \times_{\mathcal{X}} T' \to T'$ is smooth. See e.g. [Gai, §4.2].

Definition 1.2. Let $^\tau\mathrm{Stk}^? \subseteq {}^\tau\mathrm{Stk}$ be a wide subcategory containing all smooth morphisms, and denote by $\mathrm{Sch}^? \subseteq \mathrm{Sch}$ the intersection $\mathrm{Sch} \cap {}^\tau\mathrm{Stk}^?$. Given a presheaf $F : \mathrm{Sch}^{?,\mathrm{op}} \to \mathcal{V}$, its *lisse extension* $F^\lhd$ is its right Kan extension

$$F^\lhd : {}^\tau\mathrm{Stk}^{?,\mathrm{op}} \to \mathcal{V}$$

along $\mathrm{Sch}^? \hookrightarrow {}^\tau\mathrm{Stk}^?$.

Remark 1.3. Let $\mathrm{Sch}^?_{/\mathcal{X}}$ denote the full subcategory of the slice $^\tau\mathrm{Stk}^?_{/\mathcal{X}}$ spanned by pairs $(T, t : T \to \mathcal{X})$ where T is a scheme (and t is a morphism in $^\tau\mathrm{Stk}^?$). Note that morphisms $(T, t) \to (T', t')$ are morphisms $T \to T'$ in $\mathrm{Sch}^?$ which are compatible with t and t'. Then we have

$$F^\lhd(\mathcal{X}) \simeq \varprojlim_{(T,t) \in \mathrm{Sch}^?_{/\mathcal{X}}} F(T).$$

Proposition 1.4. *Let $F : \mathrm{Sch}^{?,\mathrm{op}} \to \mathcal{V}$ be a presheaf. Assume that $\mathrm{Sch}^?$ contains all lci morphisms and that its morphisms are stable under smooth base change[10]. If F satisfies τ-descent, then for every $\mathcal{X} \in {}^\tau\mathrm{Stk}$ there is a canonical isomorphism*

$$F^\lhd|_{{}^\tau\mathrm{LisStk}_{\mathcal{X}}} \to (F|_{\mathrm{Lis}_{\mathcal{X}}})^\lhd$$

of presheaves on $^\tau\mathrm{LisStk}_{\mathcal{X}}$.

For example, we will apply Proposition 1.4 to presheaves on Sch and the subcategories

$$\mathrm{Sch}^{\mathrm{sm}}, \ \mathrm{Sch}^{\mathrm{lci}} \quad \subseteq \quad \mathrm{Sch}$$

containing only the smooth and lci morphisms, respectively.

Lemma 1.5. *Let $F : \mathrm{Sch}^{?,\mathrm{op}} \to \mathcal{V}$ be a presheaf where $\mathrm{Sch}^?$ satisfies the conditions of Proposition 1.4. If F satisfies τ-descent, then for every $\mathcal{X} \in {}^\tau\mathrm{Stk}$, every scheme U and every smooth morphism $p : U \to \mathcal{X}$ admitting τ-local sections, the canonical map*

$$F^\lhd(\mathcal{X}) \to \mathrm{Tot}(F(U_\bullet)) \tag{1.6}$$

is invertible, where $U_\bullet$ is the Čech nerve of p, "Tot" denotes the totalization of a cosimplicial object.

Proof. Assume first that $X = \mathcal{X}$ is a scheme. By assumption, there exists a scheme V and a τ-cover $V \twoheadrightarrow X$ over which p admits a section. Since F satisfies descent for the Čech nerve of $V \twoheadrightarrow X$, we may replace X by V and thereby assume that p admits a section. This section (which is lci, hence determines a morphism in $\mathrm{Sch}^?$) gives rise to a splitting of the augmented simplicial object $U_\bullet \to X$, so that the map (1.6) is invertible by [Lur1, Lem. 6.1.3.16].

Now we consider the general case. For every pair (T, t) where $T \in \mathrm{Sch}$ and $t : T \to \mathcal{X}$ is a morphism in $^\tau\mathrm{Stk}^?$, denote by $U_T \twoheadrightarrow T$ the base change of p

---

[10]i.e., if $X \to Y$ is a morphism in $\mathrm{Sch}^?$ and $Y' \to Y$ is a smooth morphism in Sch, then the base change $X \times_Y Y' \to Y'$ belongs to $\mathrm{Sch}^?$

and by $U_{T,\bullet}$ its Čech nerve. The canonical map

$$F(T) \to \mathrm{Tot}(F(U_{T,\bullet})) \tag{1.7}$$

is invertible by above.

Note that for every smooth morphism $Y \to \mathcal{X}$ from a scheme, the base change functor $\mathrm{Sch}^{?,\mathrm{op}}_{/\mathcal{X}} \to \mathrm{Sch}^{?,\mathrm{op}}_{/Y}$ is cofinal. Indeed, given $(T,t) \in \mathrm{Sch}^{?}_{/Y}$ we have a section $s : T \to T \times_{\mathcal{X}} Y$ over Y of the projection $T \times_{\mathcal{X}} Y \to T$. Since the latter is smooth, s is lci and determines a morphism in $\mathrm{Sch}^{?}_{/Y}$ whose target lies in the essential image of the functor in question. Thus, the morphism (1.6) is the limit over $(T,t) \in \mathrm{Sch}^{?}_{/\mathcal{X}}$ of the isomorphisms (1.7). $\qquad\square$

Proof of Proposition 1.4. It will suffice to show that for every $\mathcal{Y} \in {}^{\tau}\mathrm{LisStk}_{\mathcal{X}}$, the projection map

$$F^{\triangleleft}(\mathcal{Y}) \simeq \varprojlim_{(T,t)\in \mathrm{Sch}^{?}_{/\mathcal{Y}}} F(T) \to \varprojlim_{(T,t)\in (\mathrm{Lis}_{\mathcal{X}})_{/\mathcal{Y}}} F(T) \tag{1.8}$$

is invertible. Here $(\mathrm{Lis}_{\mathcal{X}})_{/\mathcal{Y}}$ is the ∞-category of pairs (T,t) where $T \in \mathrm{Lis}_{\mathcal{X}}$ and $t : T \to \mathcal{Y}$ is a morphism in ${}^{\tau}\mathrm{LisStk}_{\mathcal{X}}$.

Let $p : Y \twoheadrightarrow \mathcal{Y}$ be a smooth morphism admitting τ-local sections where Y is a scheme. Denote by $Y_{\bullet}$ the Čech nerve of p, so that there is an equivalence $\mathrm{Tot}(F(Y_{\bullet})) \simeq F^{\triangleleft}(\mathcal{Y})$ by Lemma 1.5. This defines a diagram $\mathbf{\Delta}^{\mathrm{op}} \to (\mathrm{Lis}_{\mathcal{X}})_{/\mathcal{Y}}$, so by projection there is a canonical map

$$\varprojlim_{(T,t)\in (\mathrm{Lis}_{\mathcal{X}})_{/\mathcal{Y}}} F(T) \to \varprojlim_{[n]\in \mathbf{\Delta}} F(Y_n) \simeq F^{\triangleleft}(\mathcal{Y}).$$

One verifies that this is inverse to (1.8). $\qquad\square$

2. A PRO-APPROXIMATION LEMMA

This section contains the first key technical result of the paper (Proposition 2.5). We begin by recalling some preliminaries about pro-objects.

Remark 2.1. Let $\mathscr{V}$ be an ∞-category with limits. We denote by $\mathrm{Pro}(\mathscr{V})$ the ∞-category of pro-objects in $\mathscr{V}$ (see e.g. [BHH]). There is a canonical fully faithful functor $\mathscr{V} \hookrightarrow \mathrm{Pro}(\mathscr{V})$ sending an object $V \in \mathscr{V}$ to the constant pro-object $\{V\}$, whose essential image generates $\mathrm{Pro}(\mathscr{V})$ under cofiltered limits (see [BHH, Cor. 3.2.14]). The functor $\mathscr{V} \hookrightarrow \mathrm{Pro}(\mathscr{V})$ preserves finite limits, which are computed levelwise in $\mathrm{Pro}(\mathscr{V})$; indeed, this is easily verified by universal properties using the formula for mapping anima

$$\mathrm{Maps}_{\mathrm{Pro}(\mathscr{V})}(\{X_\alpha\}_\alpha, \{Y_\beta\}_\beta) \simeq \varprojlim_\beta \varinjlim_\alpha \mathrm{Maps}_{\mathscr{V}}(X_\alpha, Y_\beta). \tag{2.2}$$

In particular, it follows that $\mathrm{Pro}(\mathscr{V})$ admits arbitrary limits by [Lur1, Prop. 4.4.2.6]. Moreover, formation of limits defines a functor

$$\mathrm{Pro}(\mathscr{V}) \to \mathscr{V}, \quad \{X_\alpha\}_\alpha \mapsto \varprojlim_\alpha X_\alpha, \tag{2.3}$$

which is right adjoint to $\mathscr{V} \hookrightarrow \mathrm{Pro}(\mathscr{V})$ (again using (2.2)), and in particular limit-preserving.

The following definition is inspired by [MV, §4, Def. 2.1] (see also Remark 2.11 below).

Notation 2.4. Let $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$ and $\{\mathcal{U}_\nu\}_\nu$ a filtered diagram in ${}^{\tau}\mathrm{Stk}_{\mathfrak{X}}$ with monomorphisms as transition maps. Suppose that for every index ν there is a vector bundle $\mathcal{V}_\nu$ over $\mathfrak{X}$, an open immersion $\mathcal{U}_\nu \hookrightarrow \mathcal{V}_\nu$ over $\mathfrak{X}$, and a closed substack $\mathcal{W}_\nu \subseteq \mathcal{V}_\nu$ complementary to $\mathcal{U}_\nu$ containing the zero section, such that the following conditions hold:

(i)   For every affine scheme T and every morphism $T \to \mathfrak{X}$, there exists an index ν_0 such that the morphism $\mathcal{U}_{\nu_0} \times_{\mathfrak{X}} T \to T$ admits Nisnevich-local sections.

(ii) For every index ν, there exists $\mu > \nu$ such that the transition map $\mathcal{U}_\nu \to \mathcal{U}_\mu$ factors as follows:

$$
\begin{array}{ccc}
\mathcal{V}_\nu \smallsetminus \mathcal{W}_\nu & \xrightarrow{(0,\mathrm{id})} & \mathcal{V}_\nu \times_{\mathfrak{X}} \mathcal{V}_\nu \smallsetminus \mathcal{W}_\nu \times_{\mathfrak{X}} \mathcal{W}_\nu \\
\| & & \vdots \\
\mathcal{U}_\nu & \hooklongrightarrow & \mathcal{U}_\mu.
\end{array}
$$

A presheaf $F : \mathrm{Lis}_{\mathfrak{X}}^{\mathrm{op}} \to \mathscr{V}$ is $\mathbf{A}^1$-*invariant* if for every $T \in \mathrm{Lis}_{\mathfrak{X}}$, the canonical map $F(T) \to F(T \times \mathbf{A}^1)$ is invertible.

Proposition 2.5. *Let* $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$ *and* $\{\mathcal{U}_\nu\}_\nu$ *as in Notation 2.4. Let* $F : \mathrm{Lis}_{\mathfrak{X}}^{\mathrm{op}} \to \mathscr{V}$ *be an* $\mathbf{A}^1$-*invariant* τ-*sheaf and write* $\overline{F} : \mathrm{Lis}_{\mathfrak{X}}^{\mathrm{op}} \to \mathscr{V} \hookrightarrow \mathrm{Pro}(\mathscr{V})$ *for the composite with the canonical embedding (Remark 2.1). Then there is a canonical isomorphism*

$$\overline{F}^{\lhd}(\mathfrak{X}) \simeq \{F^{\lhd}(\mathcal{U}_\nu)\}_\nu \tag{2.6}$$

in $\mathrm{Pro}(\mathscr{V})$. *In particular, the canonical morphisms* $F^{\lhd}(\mathfrak{X}) \to F^{\lhd}(\mathcal{U}_\nu)$ *determine an isomorphism*

$$F^{\lhd}(\mathfrak{X}) \simeq \varprojlim_{\nu} F^{\lhd}(\mathcal{U}_\nu) \tag{2.7}$$

in $\mathscr{V}$.

Remark 2.8. Note that $\overline{F}^{\lhd}(\mathfrak{X})$ need not be isomorphic to the constant pro-system $\{F^{\lhd}(\mathfrak{X})\}$. In particular, Proposition 2.5 does not imply the existence of an isomorphism

$$\{F^{\lhd}(\mathfrak{X})\} \simeq \{F^{\lhd}(U_\nu)\}_\nu.$$

In fact, $\overline{F}^{\lhd}(\mathfrak{X})$ may not be essentially constant at all, see Remark 4.18 for an example.

The proof of Proposition 2.5 will require the following lemmas:

Lemma 2.9. *Let* $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$ *and* $F : \mathrm{Lis}_{\mathfrak{X}}^{\mathrm{op}} \to \mathscr{V}$ *be a presheaf. Denote by*

$$F_! : \mathrm{Fun}(\mathrm{Lis}_{\mathfrak{X}}^{\mathrm{op}}, \mathrm{Ani}) \to \mathrm{Pro}(\mathscr{V})^{\mathrm{op}}$$

the unique colimit-preserving functor which restricts to $\overline{F} : \mathrm{Lis}_{\mathfrak{X}}^{\mathrm{op}} \to \mathscr{V} \hookrightarrow \mathrm{Pro}(\mathscr{V})$ *(notation as in Proposition 2.5). Let* $A \to B$ *be an* $(\mathbf{A}^1, \mathrm{Nis})$-*local*

equivalence *in* $\mathrm{Fun}(\mathrm{Lis}_{\mathcal{X}}^{\mathrm{op}}, \mathrm{Ani})$, *i.e. a morphism such that the induced map*

$$\mathrm{Maps}(B, G) \to \mathrm{Maps}(A, G)$$

is invertible for every $\mathbf{A}^1$*-invariant Nisnevich sheaf* $G \in \mathrm{Fun}(\mathrm{Lis}_{\mathcal{X}}^{\mathrm{op}}, \mathrm{Ani})$*. If* F *satisfies* $\mathbf{A}^1$*-invariance and Nisnevich descent, then* $F_!(A) \to F_!(B)$ *is invertible.*

Proof. By [Lur1, Prop. 5.5.4.15(4)], a morphism $A \to B$ in $\mathrm{Fun}(\mathrm{Lis}_{\mathcal{X}}^{\mathrm{op}}, \mathrm{Ani})$ is an $(\mathbf{A}^1, \mathrm{Nis})$-local equivalence if and only if it belongs to the strongly saturated class generated by the morphisms

(i)   for every $T \in \mathrm{Lis}_{\mathcal{X}}$, the projection $T \times \mathbf{A}^1 \to T$;

(ii)   for every Nisnevich covering family $(T'_\alpha \to T)_\alpha$ in $\mathrm{Lis}_{\mathcal{X}}$, the morphism $\varinjlim T'_\bullet \to T$ where $T'_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Lis}_{\mathcal{X}}^{\mathrm{op}}, \mathrm{Ani})$ is the Čech nerve of $\coprod_\alpha T'_\alpha \to T$.

By (the proof of) [Kha1, Thm. 2.2.7], (ii) may be replaced by the following class:

(ii *bis*)   the morphism from the initial presheaf to the presheaf represented by the empty scheme; and for every étale morphism $V \to T$ in $\mathrm{Lis}_{\mathcal{X}}$ which is an isomorphism away from a cocompact closed subset $K \subseteq |T|$, the morphism $V \coprod_W U \to T$ in $\mathrm{Fun}(\mathrm{Lis}_{\mathcal{X}}^{\mathrm{op}}, \mathrm{Ani})$, where $U = T \smallsetminus K$ and $W = V \times_T U$.

Since $F_!$ preserves colimits, it will thus suffice to show that it inverts morphisms of type (i) and (ii *bis*). For $T \in \mathrm{Lis}_{\mathcal{X}}$, $F_!$ sends $T \times \mathbf{A}^1 \to T$ to the morphism of constant pro-objects

$$\{F(T)\} \to \{F(T \times \mathbf{A}^1)\}$$

which is invertible since F is $\mathbf{A}^1$-invariant. Similarly, $F_!$ sends $V \coprod_W U \to T$ as in (ii *bis*) to the morphism of pro-objects

$$\{F(T)\} \to \{F(V)\} \underset{\{F(W)\}}{\times} \{F(U)\}.$$

Since finite limits in $\mathrm{Pro}(\mathcal{V})$ are computed levelwise (Remark 2.1), this is identified with the morphism of constant pro-objects

$$\{F(T)\} \to \{F(V) \underset{F(W)}{\times} F(U)\},$$

which is invertible since F satisfies Nisnevich descent (and by [Kha1, Thm. 2.2.7]). $\square$

Lemma 2.10. *Let* $X = \mathcal{X}$ *be an affine scheme and* $\{U_\nu\}_\nu = \{\mathcal{U}_\nu\}_\nu$ *be as in Notation 2.4. Suppose there exists an index* ν_0 *such that* $U_{\nu_0} \to X$ *admits a section. Then the presheaf* $U_\infty := \varinjlim_\nu U_\nu$ *(where the colimit is taken in presheaves) is* $\mathbf{A}^1$*-contractible on smooth affine* X*-schemes.*

Proof. The following argument is extracted from the proof of [MV, §4, Prop. 2.3]. The claim is that the anima $R\Gamma(T, \mathrm{L}_{\mathbf{A}^1} U_\infty)$ is contractible for every affine $T \in \mathrm{Lis}_X$, where $\mathrm{L}_{\mathbf{A}^1}$ denotes the $\mathbf{A}^1$-localization functor (see e.g.

[Hoy1, Proof of Prop. C.6]), i.e. that the simplicial set

$$\mathrm{Maps}_{\mathrm{Fun}(\mathrm{Lis}_X^{\mathrm{op}},\mathrm{Ani})}(T \times \mathbf{A}^\bullet, U_\infty) \simeq \varinjlim_\nu \mathrm{Maps}_{\mathrm{Lis}_X}(T \times \mathbf{A}^\bullet, U_\nu)$$

is a contractible Kan complex. By [EHKSY, Lem. A.2.6] and closed gluing
for the presheaf U_∞, it is enough to show that for every $n \geqslant 0$ and every affine
$T \in \mathrm{Lis}_X$, the restriction map

$$\mathrm{Maps}(T \times \mathbf{A}^n, U_\infty) \to \mathrm{Maps}(T \times \partial\mathbf{A}^n, U_\infty)$$

is surjective on π_0, where we identify $\mathbf{A}^n$ with the closed subscheme of
$\mathbf{A}^{n+1} = \mathrm{Spec}(\mathbf{Z}[T_0,\ldots,T_n])$ defined by $\sum_i T_i = 1$, and $\partial\mathbf{A}^n$ is the closed
subscheme defined by the further equation $T_0 \ldots T_n = 0$.

Let $\nu \geqslant \nu_0$ be an index. Denote by $s : X \to U_{\nu_0} \to U_\nu$ the induced section
and by $t : T \to X \to U_\nu$ its composite with the structural morphism. The
existence of t shows the surjectivity for $n = 0$.

Let $n > 0$ and $f : T \times \partial\mathbf{A}^n \to U_\nu$ a morphism over X. We claim that this
extends to a morphism $g : T \times \mathbf{A}^n \to U_\mu$ for some index $\mu \geqslant \nu$. Since T and
X are affine and V_ν is a vector bundle over X, there exists an X-morphism
$g' : T \times \mathbf{A}^n \to V_\nu$ which restricts to f on $T \times \partial\mathbf{A}^n$. Since $T \times \partial\mathbf{A}^n$ and $g'^{-1}(W_\nu)$
are disjoint as closed subschemes of $T \times \mathbf{A}^n$, there exists for the same reason
an X-morphism $g'' : T \times \mathbf{A}^n \to V_\nu$ which restricts to 0 on $T \times \partial\mathbf{A}^n$ and to

$$g'^{-1}(W_\nu) \to X \xrightarrow{s} U_\nu \subseteq V_\nu$$

on $g'^{-1}(W_\nu)$. By construction, the induced X-morphism

$$(g'', g') : T \times \mathbf{A}^n \to V_\nu \underset{X}{\times} V_\nu$$

restricts to $(0, f) : T \times \partial\mathbf{A}^n \to V_\nu \times_X V_\nu$, and factors through the complement
of $W_\nu \times_X W_\nu$. Let $\mu > \nu$ and the morphism $V_\nu \times_X V_\nu \smallsetminus W_\nu \times_X W_\nu \to U_\mu$ be as
in assumption (ii). Then the composite

$$g : T \times \mathbf{A}^n \xrightarrow{(g'', g')} V_\nu \underset{T}{\times} V_\nu \smallsetminus W_\nu \underset{X}{\times} W_\nu \to U_\mu$$

fits into the commutative diagram

$$
\begin{array}{ccc}
T \times \partial\mathbf{A}^n & \xrightarrow{\ f\ } & U_\mu \\
\downarrow & \overset{g}{\nearrow} & \downarrow \\
T \times \mathbf{A}^n & \longrightarrow & X
\end{array}
$$

as desired. $\square$

Proof of Proposition 2.5. The second isomorphism (2.7) follows from (2.6)
by applying the limit-preserving functor $\mathrm{Pro}(\mathscr{V}) \to \mathscr{V}$ (2.3). By Lemma 2.9
it will suffice to show that $\varinjlim_\nu \mathcal{U}_\nu \to \mathcal{X}$ is an $(\mathbf{A}^1, \mathrm{Nis})$-local equivalence in
$\mathrm{Fun}(\mathrm{Lis}_{\mathcal{X}}^{\mathrm{op}}, \mathrm{Ani})$. By universality of colimits, this morphism is identified with
the colimit over $T \in \mathrm{Lis}_{\mathcal{X}}$ of the base changes

$$\varinjlim_\nu \mathcal{U}_\nu \underset{\mathcal{X}}{\times} T \to T.$$

Since local equivalences are preserved by the colimit-preserving functor $\mathrm{Fun}(\mathrm{Lis}_T^{\mathrm{op}}, \mathrm{Ani}) \to \mathrm{Fun}(\mathrm{Lis}_{\mathcal{X}}^{\mathrm{op}}, \mathrm{Ani})$ sending $U \in \mathrm{Lis}_T$ to $U \in \mathrm{Lis}_{\mathcal{X}}$, and because the data and assumptions in Notation 2.4 are stable under base change, we may thus replace $\mathcal{X}$ by T to assume that it is a scheme. We can moreover assume that $X := \mathcal{X}$ is affine, arguing similarly using the fact that it can be written as a colimit of affines up to local equivalence. Finally, we may also assume up to local equivalence that $U_{\nu_0} \to X$ admits a section for some index ν_0 (by condition (i)). Now the claim follows from Lemma 2.10. $\qquad\square$

Remark 2.11. In Notation 2.4, a sufficient condition for (i) is that for every field κ and every κ-valued point $s : \mathrm{Spec}(\kappa) \to \mathcal{X}$, there exists an index ν_s and a lift $\mathrm{Spec}(\kappa) \to \mathcal{U}_{\nu_s}$. Indeed, let us show that if $\mathcal{X}$ is affine then there exists an index ν_0 such that $\mathcal{U}_\nu \to \mathcal{X}$ admits Nisnevich-local sections. The assumption implies that the disjoint union $\coprod_\nu \mathcal{U}_\nu \to \mathcal{X}$ is a smooth morphism (not necessarily of finite type) which is surjective on field-valued points. By Lemma 0.6 (and its étale analogue [EGA, Cor. 17.16.3(ii)]), there exists an affine scheme X and a Nisnevich cover $X \twoheadrightarrow \mathcal{X}$ along which the base change $\coprod_\nu \mathcal{U}_\nu \times_{\mathcal{X}} X \to X$ admits a section. Since X is quasi-compact, there is a finite subset I of indices through which the section factors. Any section of $\mathcal{U}_\nu$ gives rise to a section of $\mathcal{U}_\mu$ for any $\mu > \nu$ (by composition with the transition map), so we may assume that I consists of a single index ν_0.

3. The Borel construction

We now specialize our general results from the previous sections to the case of the Borel construction. By the latter we mean the following (compare [Lus, 1.1], [Tot, §1], [MV, §4.2], [HML, Def. 10]).

Let S be a quasi-separated algebraic space, locally of finite type over k, and G an fppf group scheme over S.

Notation 3.1. We construct an $\mathbf{N}_{>0}$-indexed tower $\{U_\nu\}_\nu$ where each U_ν is an algebraic space of finite type over S with a free G-action, and the transition maps are G-equivariant closed immersions. The construction will make use of the following auxiliary choices:

(i) An embedding of G as a closed subgroup scheme of $\mathrm{GL}_S(\mathcal{E})$ for some finite locally free sheaf $\mathcal{E}$ on S. Write $V = \mathbf{V}_S(\mathcal{E})$ for the associated vector bundle over S with its induced G-action.

(ii) A open $U \subseteq V$ on which G acts freely, whose (reduced) complement $W \subseteq V$ is of fibre dimension over S (0.2.3) strictly smaller than that of V over S.

For every integer $\nu > 0$, set $V_\nu := V^{\times \nu}$ and $W_\nu := W^{\times \nu}$ (where both powers are fibred over S), and let $U_\nu \subseteq V_\nu$ be the complement of W_ν. The diagonal action of G on $V^{\times \nu}$ restricts to a free action on U_ν. The transition map $U_\nu \hookrightarrow U_{\nu+1}$ is defined by restricting the G-equivariant closed immersion $(\mathrm{id}, 0) : V^{\times \nu} \hookrightarrow V^{\times \nu} \times_S V = V^{\times \nu+1}$.

Since the quotients $[U_\nu/G]$ are algebraic spaces (as the actions are free), we will also write $U_\nu/G := [U_\nu/G]$.

Example 3.2. Suppose G is embeddable, in the sense that some embedding $G \subseteq \mathrm{GL}_S(\mathcal{E})$ as in (i) exists (e.g. S is the spectrum of a field and G is a linear algebraic group). Then the choice of some U as in (ii) also exists, see e.g. [Tot, Rem. 1.4].

Remark 3.3. The following two features of the construction in Notation 3.1 will be useful later:

(i)   For each ν consider the function

$$c_\nu := d_{V_\nu/S}|_{W_\nu} - d_{W_\nu/S}$$

where $d_{V_\nu/S}$ and $d_{W_\nu/S}$ denote the fibre dimensions of V_ν and W_ν over S, respectively.[11]   We have $c_\nu = \nu \cdot c_1$ (this follows from [EGA, IV$_2$, Cor. 4.1.5]), so in particular c_ν tends to ∞ as ν increases.

(ii)   For all indices $\mu > \nu$, the open immersion $U_\nu \times_S V_{\mu-\nu} \hookrightarrow V_\nu \times_S V_{\mu-\nu} = V_\mu$ factors through the open $U_\mu \subseteq V_\mu$. In other words, the transition map $U_\nu \hookrightarrow U_\mu$ factors as follows:

$$
\begin{array}{ccc}
U_\nu & \xrightarrow{\ (\mathrm{id},0)\ } & U_\nu \times_S V_{\mu-\nu} \\
\downarrow & \nearrow & \downarrow \\
U_\mu & \hookrightarrow & V_\mu = V_\nu \times_S V_{\mu-\nu}.
\end{array}
$$

Notation 3.4. Let $^\tau\mathrm{Stk}_S^G$ denote the ∞-category of locally of finite type τ-Artin stacks X over S with G-action such that the quotient $[X/G]$ is τ-Artin. Note that this is equivalent to the ∞-category $^\tau\mathrm{Stk}_{BG}$ of locally of finite type τ-Artin stacks $\mathcal{X}$ over $BG = [S/G]$ (via the assignment $X \mapsto \mathcal{X} = [X/G]$). For $X \in {}^\tau\mathrm{Stk}_S^G$, we write

$$X \overset{G}{\underset{S}{\times}} U_\nu := [X/G] \underset{BG}{\times} (U_\nu/G)$$

for each ν. This is representable and of finite type over $[X/G]$ (hence in particular is τ-Artin).

Remark 3.5. The ∞-category $^\tau\mathrm{Stk}_S^G$ contains every locally of finite type quasi-separated algebraic space over S with G-action. More generally, let X be a locally of finite type (ét, 1)-Artin stack over S with G-action. If X is quasi-separated with separated diagonal, then the quotient $[X/G]$ is again quasi-separated (ét, 1)-Artin with separated diagonal, hence X belongs to $^\tau\mathrm{Stk}_S^G$ by Theorem 0.7.

Remark 3.6. If X is a quasi-projective scheme over S with a linearized G-action, then each $X \times_S^G U_\nu$ is a quasi-projective S-scheme (see [MFK, Prop. 7.1]).

Applying the results of Sect. 2, we obtain:

Theorem 3.7. *Let* $X \in {}^\tau\mathrm{Stk}_S^G$ *and let* $F : \mathrm{Lis}_{[X/G]}^{\mathrm{op}} \to \mathscr{V}$ *be an* $\mathbf{A}^1$*-invariant* τ*-sheaf with values in an* ∞*-category* $\mathscr{V}$ *with limits. Then there is a canonical*

---

[11]See (0.2.3). By Lemma 0.8, c_ν can be computed at any point of W_ν as the fibrewise codimension.

isomorphism

$$F^{\triangleleft}([X/G]) \simeq \varprojlim_{\nu} F^{\triangleleft}(X \overset{G}{\underset{S}{\times}} U_{\nu}) \tag{3.8}$$

in $\mathscr{V}$.

Proof. The filtered diagram $\{U_{\nu}\}_{\nu}$ over S satisfies the assumptions of Notation 2.4 (compare [MV, §4, Ex. 2.2] and Remark 2.11). Moreover, the transition maps and the maps $U_{\nu} \hookrightarrow V_{\nu}$ are all G-equivariant, so the same holds for the quotient $\{U_{\nu}/G\}_{\nu}$ over $BG = [S/G]$. By base change it also holds for $\{X \times_S^G U_{\nu}\}_{\nu}$ over $[X/G]$. Thus the claim follows from Proposition 2.5. $\square$

Corollary 3.9. *Let $F : \mathrm{Sch}^{\mathrm{lci,op}} \to \mathrm{Spt}$ be an $\mathbf{A}^1$-invariant τ-sheaf of spectra. Then for every $X \in {}^{\tau}\mathrm{Stk}_S^G$, there is a canonical isomorphism of spectra*

$$F^{\triangleleft}([X/G]) \to \varprojlim_{\nu} F^{\triangleleft}(X \overset{G}{\underset{S}{\times}} U_{\nu}),$$

where $F^{\triangleleft} : {}^{\tau}\mathrm{Stk}^{\mathrm{lci,op}} \to \mathrm{Spt}$ denotes the lisse extension (Definition 1.2).

Proof. Given $X \in {}^{\tau}\mathrm{Stk}_S^G$, we may replace F by its restriction to $\mathrm{Lis}_{[X/G]}$ in view of Proposition 1.4. Then we conclude by applying Theorem 3.7. $\square$

4. Pro-approximation in weaves

4.1. **Sheaf cohomology.** We now turn our attention to cohomology theories represented by a sheaf in some category of coefficients. That is, let $\mathbf{D} : \mathrm{Sch}^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}$ be a τ-sheaf of ∞-categories and consider its lisse extension $\mathbf{D}^{\triangleleft} : {}^{\tau}\mathrm{Stk}^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}$ as in Definition 1.2, so that

$$\mathbf{D}^{\triangleleft}(\mathfrak{X}) = \varprojlim_{(T,t) \in \mathrm{Lis}_{\mathfrak{X}}} \mathbf{D}(T)$$

for any $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$. In this situation, Theorem 3.7 yields:

Corollary 4.1. *Let G and $\{U_{\nu}\}_{\nu}$ be as in Notation 3.1. For every $X \in {}^{\tau}\mathrm{Stk}_S^G$ and $\mathcal{F} \in \mathbf{D}^{\triangleleft}([X/G])$, the canonical morphism of spectra*

$$R\Gamma([X/G], \mathcal{F}) \to \varprojlim_{\nu} R\Gamma(X \overset{G}{\underset{S}{\times}} U_{\nu}, \mathcal{F})$$

is invertible.

Proof. Given $X \in {}^{\tau}\mathrm{Stk}_S^G$ and $\mathcal{F} \in \mathbf{D}^{\triangleleft}([X/G])$, consider the presheaf $F : {}^{\tau}\mathrm{LisStk}_{[X/G]}^{\mathrm{op}} \to \mathrm{Spt}$ defined by the assignment

$$(T, t : T \to [X/G]) \mapsto R\Gamma(T, \mathcal{F}) := \mathrm{Maps}_{\mathbf{D}^{\triangleleft}(T)}(\mathbf{1}_T, t^{\star}(\mathcal{F})).$$

This is lisse-extended from its restriction $F|_{\mathrm{Lis}_{[X/G]}}$, which is an $\mathbf{A}^1$-invariant τ-sheaf (since $\mathbf{D}$ is topological). The claim now follows from Theorem 3.7. $\square$

In this section, our goal is to address the analogue of Corollary 4.1 at the level of hypercohomology, i.e., invertibility of the morphisms

$$\mathrm{H}^i([X/G], \mathcal{F}) \to \varprojlim_{\nu} \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_{\nu}, \mathcal{F})$$

for $i \in \mathbf{Z}$. This will require us to pass to a slightly more involved setup.

4.2. **Weaves.** Let $\mathbf{D}$ be a *weave* on Sch in the sense of [Kha4, Kha5]. We will assume that $\mathbf{D}$ is *topological* as in [Kha4, §2], meaning that it satisfies homotopy invariance for vector bundles and localization for closed-open decompositions. Sometimes, notably in Corollary 4.16, we will also assume that $\mathbf{D}$ satisfies *topological invariance*, i.e., that for a finite radicial surjection f, the functor f^* is an equivalence (cf. [EK]). When $\mathbf{D}$ also satisfies continuity, this implies that that f^* is an equivalence for any universal homeomorphism.

We set $\tau = \mathrm{Nis}$, or $\tau = \mathrm{\acute{e}t}$ if $\mathbf{D}$ has étale descent. By [Kha5], the lisse extension $\mathbf{D}^{\lhd}$ admits a canonical structure of weave on $^{\tau}\mathrm{Stk}$, where all locally of finite type morphisms are shriekable. Roughly speaking, $\mathbf{D}^{\lhd}$ amounts to a collection of ∞-categories $\mathbf{D}^{\lhd}(\mathfrak{X})$ for every $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$, an adjoint pair of functors (f^*, f_*) for every morphism f in $^{\tau}\mathrm{Stk}$, and an adjoint pair of functors $(f_!, f^!)$ for every locally of finite type morphism f, and homotopy coherent base change and projection formulas.

The classical examples are the Betti and étale weaves (Sect. 5), which were extended to stacks by Liu–Zheng [LZ]. For more general weaves, such as the stable motivic homotopy weave (Sect. 6), we have appeal to [Kha5, §7].

4.3. **The cohomological t-structure.**

Definition 4.2. Let $X \in \mathrm{Sch}$. Denote by $\mathbf{D}(X)_{\geqslant n} \subseteq \mathbf{D}(X)$ the full subcategory generated under colimits and extensions by objects of the form $a_! a^!(\mathbf{1})(q)[n]$, where $a : T \to X$ is a smooth morphism from a scheme and $q \in \mathbf{Z}$, and by $\mathbf{D}(X)_{\leqslant n} \subseteq \mathbf{D}(X)$ the full subcategory spanned by $\mathcal{F} \in \mathbf{D}(X)$ for which the spectrum of derived global sections $R\Gamma(T, \mathcal{F}(q))$ is n-coconnective for all $q \in \mathbf{Z}$ and all smooth X-schemes T. The pair $(\mathbf{D}(X)_{\geqslant 0}, \mathbf{D}(X)_{\leqslant -1})$ of orthogonal subcategories defines a t-structure on $\mathbf{D}(X)$ by [Lur2, Prop. 1.4.4.11].

We say that $\mathcal{F} \in \mathbf{D}(X)$ is *n-connective*, resp. *n-coconnective*, if it belongs to $\mathbf{D}(X)_{\geqslant n}$, resp. $\mathbf{D}(X)_{\leqslant n}$. We say that $\mathcal{F}$ is *eventually connective*, resp. *eventually coconnective*, if it belongs to $\mathbf{D}(X)_{> -\infty} = \bigcup_n \mathbf{D}(X)_{\geqslant n}$, resp. $\mathbf{D}(X)_{<\infty} = \bigcup_n \mathbf{D}(X)_{\leqslant n}$.

Note that, for a morphism $f : X' \to X$ in Sch, the functor f^* is right t-exact, i.e., preserves connectivity. If f is smooth, $f^* \simeq f^!\langle -\Omega_f \rangle$ has a left adjoint $f_!\langle \Omega_f \rangle$. By the defintions, the latter is right t-exact. Hence by adjunction f^* is also *left* t-exact in this case, i.e., also preserves coconnectivity.

We extend the cohomological t-structure to stacks as follows:

Proposition 4.3. *Let $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$. There exists a unique t-structure on the stable ∞-category $\mathbf{D}^{\lhd}(\mathfrak{X})$ such that $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathfrak{X})$ belongs to $\mathbf{D}^{\lhd}(\mathfrak{X})_{\leqslant n}$, resp. $\mathbf{D}^{\lhd}(\mathfrak{X})_{\geqslant n}$, if and only if for every $(T, t : T \to \mathfrak{X}) \in \mathrm{Lis}_{\mathfrak{X}}$, the object $t^*(\mathcal{F})$ belongs to $\mathbf{D}(T)_{\leqslant n}$, resp. $\mathbf{D}(T)_{\geqslant n}$.*

Proof. By definition, we have equivalences

$$\mathbf{D}^{\triangleleft}(\mathcal{X}) \simeq \varprojlim_{t} \mathbf{D}(T), \qquad \mathbf{D}^{\triangleleft}(\mathcal{X})_{\geqslant 0} \simeq \varprojlim_{t} \mathbf{D}(T)_{\geqslant 0}$$

where the limits are taken over pairs $(T, t : T \to \mathcal{X}) \in \mathrm{Lis}_{\mathcal{X}}$ and the transition functors are t^*. Since the latter are t-exact, they restrict to left exact functors on the subcategories $\mathbf{D}(T)_{\geqslant 0}$. Moreover, each $\mathbf{D}(T)_{\geqslant 0}$ is a Grothendieck prestable ∞-category by [Lur3, Prop. C.1.4.1]. In this situation [Lur3, Prop. C.3.2.4] implies that the limit $\mathbf{D}^{\triangleleft}(\mathcal{X})_{\geqslant 0}$ is also Grothendieck prestable and that the functors $t^* : \mathbf{D}^{\triangleleft}(\mathcal{X})_{\geqslant 0} \to \mathbf{D}(T)_{\geqslant 0}$ are left exact and jointly conservative. Passing back to stabilizations, it follows from [Lur3, Cor. C.3.2.5, Prop. C.1.4.1] that $\mathbf{D}^{\triangleleft}(\mathcal{X})$ admits a t-structure whose connective part is $\mathbf{D}^{\triangleleft}(\mathcal{X})_{\geqslant 0}$ and such that the functors $t^* : \mathbf{D}^{\triangleleft}(\mathcal{X}) \to \mathbf{D}(T)$ are t-exact and jointly conservative. The latter implies that an object $\mathcal{F} \in \mathbf{D}^{\triangleleft}(\mathcal{X})$ belongs to the coconnective part of the t-structure if and only if $t^*(\mathcal{F}) \in \mathbf{D}(T)_{\leqslant 0}$ for every $(T, t) \in \mathrm{Lis}_{\mathcal{X}}$. $\square$

Theorem 4.4. *Let* $f : \mathcal{X}' \to \mathcal{X}$ *be a morphism in* $^{\tau}\mathrm{Stk}$.

(i)   *The functor* f^* *is right t-exact. If* f *is smooth,* f^* *is also left t-exact.*

(ii)   *The functor* f_* *is left t-exact.*

(iii) *If* f *is smooth of relative dimension* d*, then* $f^![-2d]$ *is left t-exact and* $f_![2d]$ *is right t-exact.*

(iv)   *Suppose* $\mathbf{D}$ *satisfies topological invariance and continuity. If* f *is of fibre dimension* $\leqslant d$ *(see (0.2.3)), then* $f^![-2d]$ *is left t-exact and* $f_![2d]$ *is right t-exact.*

(v)   *For every K-theory class* $v \in \mathrm{K}(\mathcal{X})$ *of virtual rank* r*, the shifted Thom twist* $\langle v \rangle[-2r] : \mathbf{D}^{\triangleleft}(\mathcal{X}) \to \mathbf{D}^{\triangleleft}(\mathcal{X})$ *is t-exact.*

Proof. That f^* is right t-exact (so that f_* is left t-exact by adjunction) follows easily from the case of schemes. We deduce by Poincaré duality that f^* is also left t-exact when f is smooth.

The statement about Thom twists $\langle v \rangle[-2r]$ can be checked smooth-locally on $\mathcal{X}$ (since $*$-inverse image along smooth morphisms is t-exact), so we may assume the K-theory class v can be represented as a difference of finite locally frees. In this case we reduce to showing that $\langle r \rangle[-2r] = (r)$ (Tate twist) is t-exact. This is clear since it is evidently left t-exact, and admits a left adjoint $(-r)$ which is also left t-exact.

For $f : \mathcal{X}' \to \mathcal{X}$ smooth of relative dimension d we deduce that $f^! \simeq f^*\langle \Omega_f \rangle$ sends n-coconnective objects to $(n + 2d)$-coconnective objects, and its left adjoint $f_!$ sends n-connective objects to $(n - 2d)$-connective objects.

If $f : \mathcal{X}' \to \mathcal{X}$ is a morphism of fibre dimension $\leqslant d$, then the same holds assuming that $\mathbf{D}$ satisfies topological invariance and continuity. Indeed, take an object $\mathcal{F} \in \mathbf{D}(\mathcal{X}')_{\geqslant n}$ and let us show that $f_!(\mathcal{F})$ is $(n - 2d)$-connective. Replacing f by its base change along some smooth atlas $X \twoheadrightarrow \mathcal{X}$, we may assume that $X = \mathcal{X}$ is a scheme. By [BH, Prop. B.3] it will suffice to show that $x^* f_!(\mathcal{F})$ is $(n - 2d)$-connective for every point $x : \mathrm{Spec}(k(x)) \to X$. By

the base change formula we may thus reduce further to the case where X is the spectrum of a field κ, and by topological invariance and continuity that it is moreover perfect. By nilpotent invariance we may also assume that $\mathcal{X}'$ is reduced. Then $\mathcal{X}'$ admits a dense open which is smooth over $\mathrm{Spec}(\kappa)$ (take a smooth atlas by a scheme and take the image of the smooth locus of the latter), so by noetherian induction and the localization triangle we reduce to the case where $f : \mathcal{X}' \to \mathrm{Spec}(\kappa)$ is smooth. □

Note that an immersion $i : \mathcal{Y} \hookrightarrow \mathcal{X}$ has fibre dimension $\leqslant 0$, so Theorem 4.4(iv) only yields that $i_!$ is right t-exact and $i^!$ is left t-exact. Nevertheless, we have:

Corollary 4.5. *Suppose* $\mathbf{D}$ *satisfies topological invariance and continuity. Let* $f : \mathcal{Y} \to \mathcal{X}$ *be a smooth morphism in* $^{\tau}\mathrm{Stk}$ *and* $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ *a closed immersion. Let* $d_{\mathcal{Y}/\mathcal{X}}$ *and* $d_{\mathcal{Z}/\mathcal{X}}$ *denote the fibre dimensions and write* $c = d_{\mathcal{Y}/\mathcal{X}}|_{\mathcal{Z}} - d_{\mathcal{Z}/\mathcal{X}}$ *(regarded as a locally constant function on* $\mathcal{Z}$ *). Then* $i^! f^*[2c]$ *is left t-exact.*

Proof. By Poincaré duality for the smooth morphism f, we have $i^! f^*[2c] \simeq g^!\langle -\Omega_f \rangle[2c]$, where $g = f \circ i$. The claim is that for every 0-coconnective $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathcal{X})$, $g^!(\mathcal{F})\langle -\Omega_f \rangle$ is $(-2c)$-coconnective. By Theorem 4.4, $g^![-2d_{\mathcal{Z}/\mathcal{X}}]$ and $\langle -\Omega_f \rangle[2d_{\mathcal{Y}/\mathcal{X}}]$ are left t-exact. In particular, $g^!(\mathcal{F})\langle -\Omega_f \rangle$ is $(2d_{\mathcal{Z}/\mathcal{X}} - 2d_{\mathcal{Y}/\mathcal{X}})$-coconnective if $\mathcal{F}$ is 0-coconnective. □

4.4. Acyclic morphisms.

The following terminology is inspired by [SGA4, Exp. XV, Déf. 1.7]:

Definition 4.6.

(i) A morphism $f : \mathcal{Y} \to \mathcal{X}$ in $^{\tau}\mathrm{Stk}$ is *n-acyclic*, for some $n \in \mathbf{Z}$, if the unit morphism $\mathcal{F} \to f_* f^*(\mathcal{F})$ is *n-coconnective* for every $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathcal{X})_{<\infty}$.[12]

(ii) A morphism $f : \mathcal{Y} \to \mathcal{X}$ in $^{\tau}\mathrm{Stk}$ is *acyclic* if it is *n-acyclic* for every n.

For example, any vector bundle projection is acyclic (by homotopy invariance).

Remark 4.7. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism in $^{\tau}\mathrm{Stk}$.

(i) If f is *n-acyclic*, then for every $\mathcal{F} \in \mathbf{D}(\mathcal{X})_{<\infty}$ the map
$$f^* : R\Gamma(\mathcal{Y}, \mathcal{F}) \to R\Gamma(\mathcal{X}, \mathcal{F})$$
is injective on π_{n+1} and bijective on π_i for $i \geqslant n + 2$. In other words, the map
$$f^* : \mathrm{H}^i(\mathcal{Y}, \mathcal{F}) \to \mathrm{H}^i(\mathcal{X}, \mathcal{F})$$
is injective is injective for $i = -n - 1$ and bijective for $i \leqslant -n - 2$.

(ii) If the cohomological t-structure on $\mathbf{D}^{\lhd}(\mathcal{X})$ is left-complete, then f is *n-acyclic* if and only if $\mathcal{F} \to f_* f^*(\mathcal{F})$ is an isomorphism for *every* $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathcal{X})$.

(iii) If the cohomological t-structure on $\mathbf{D}^{\lhd}(\mathcal{X})$ is right-separated (e.g. right-complete), then f is acyclic if and only if $\mathcal{F} \to f_* f^*(\mathcal{F})$ is an isomorphism for every $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathcal{X})_{<\infty}$.

---

[12]We say a morphism is *n-(co)connective* if its fibre is *n*-(co)connective.

4.5. **Pro-acyclic morphisms.**

Definition 4.8. Let $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$ and let $\mathcal{Y} := \{\mathcal{Y}_\nu\}_\nu$ be a sequential diagram in ${}^{\tau}\mathrm{Stk}_{\mathfrak{X}}$ with structural morphisms $f_\nu : \mathcal{Y}_\nu \to \mathfrak{X}$. We say that $f := \{f_\nu\}_\nu : \mathcal{Y} \to \mathfrak{X}$ is:

(i) n-*acyclic* if for every $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathfrak{X})_{<\infty}$, the canonical morphism in $\mathbf{D}^{\lhd}(\mathfrak{X})$

$$\mathcal{F} \to \varprojlim_\nu f_{\nu,*} f_\nu^*(\mathcal{F})$$

is n-coconnective. It is *acyclic* if it is n-acyclic for all $n \in \mathbf{Z}$.

(ii) n-*pro-acyclic* if for every $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathfrak{X})_{<\infty}$, there exists an index $\nu(n)$ such that the morphisms

$$\mathcal{F} \to f_{\nu,*} f_\nu^*(\mathcal{F})$$

are n-coconnective for all $\nu \geqslant \nu(n)$.

(iii) *pro-acyclic* if it is n-pro-acyclic for all $n \in \mathbf{Z}$.

Since coconnectivity is stable under limits, n-pro-acyclic morphisms are n-acyclic.

Example 4.9. If each $f_\nu : \mathcal{Y}_\nu \to \mathfrak{X}$ is n-acyclic for all ν, then $f = \{f_\nu\}_\nu$ is n-pro-acyclic, hence n-acyclic. In fact, it suffices that for every $n \in \mathbf{Z}$ there exists an index $\nu(n)$ such that f_ν is n-acyclic for all $\nu \geqslant \nu(n)$.

Remark 4.10. Suppose $f := \{f_\nu\}_\nu : \mathcal{Y} \to \mathfrak{X}$ is n-pro-acyclic. For every $\mathcal{F} \in \mathbf{D}^{\lhd}(\mathfrak{X})_{<\infty}$, it follows then from the definition of the cohomological t-structure that there exists an index $\nu(n)$ such that for all $\nu \geqslant \nu(n)$, the map

$$f_\nu^* : R\Gamma(\mathfrak{X}, \mathcal{F}) \to R\Gamma(\mathfrak{X}, f_{\nu,*} f_\nu^* \mathcal{F}) \simeq R\Gamma(\mathcal{Y}_\nu, \mathcal{F}) \tag{4.11}$$

has n-coconnective fibre, for all $\nu \geqslant \nu(n)$. In other words, (4.11) is injective on π_{n+1} and bijective on π_i for $i \geqslant n+2$, i.e., that

$$f_\nu^* : \mathrm{H}^i(\mathfrak{X}, \mathcal{F}) \to \mathrm{H}^i(\mathcal{Y}_\nu, \mathcal{F})$$

is injective for $i = -n-1$ and bijective for $i \leqslant -n-2$. Thus if f is pro-acyclic, then for every integer $i \in \mathbf{Z}$ the maps

$$f_\nu^* : \mathrm{H}^i(\mathfrak{X}, \mathcal{F}) \to \mathrm{H}^i(\mathcal{Y}_\nu, \mathcal{F}) \tag{4.12}$$

become invertible for sufficiently large ν, and in particular

$$\mathrm{H}^i(\mathfrak{X}, \mathcal{F}) \to \varprojlim_\nu \mathrm{H}^i(\mathcal{Y}_\nu, \mathcal{F}) \tag{4.13}$$

is invertible for every $i \in \mathbf{Z}$.

Proposition 4.14. *Let $\mathfrak{X} \in {}^{\tau}\mathrm{Stk}$ and let $\{f_\nu : \mathcal{Y}_\nu \to \mathfrak{X}\}_\nu$ be a sequential diagram in ${}^{\tau}\mathrm{Stk}_{\mathfrak{X}}$. If $\{f_\nu\}_\nu$ is n-acyclic (resp. n-pro-acyclic), then for every smooth morphism $\mathfrak{X}' \to \mathfrak{X}$ in ${}^{\tau}\mathrm{Stk}$, so is the base change $\{f_\nu' : \mathcal{Y}_\nu \times_{\mathfrak{X}} \mathfrak{X}' \to \mathfrak{X}'\}_\nu$.*

Proof. Follows from the smooth base change formula and the fact that $*$-inverse image along smooth morphisms is t-exact. $\square$

4.6. **Pro-acyclicity of the Borel construction.** We adopt again the setup of Sect. 3. In this subsection we will prove:

Proposition 4.15. *Let* $\mathbf{D}$ *be a lisse-extended topological weave on* $^\tau\mathrm{Stk}$, *and assume* $\mathbf{D}$ *satisfies topological invariance and continuity. Then we have:*

(i)   *The morphism* $\{U_\nu/G\}_\nu \to [S/G] = BG$ *is pro-acyclic.*

(ii)   *For every* $X \in {}^\tau\mathrm{Stk}_S^G$, *the morphism*

$$\{X \overset{G}{\underset{S}{\times}} U_\nu\}_\nu \to [X/G]$$

is pro-acyclic, where $X \times_S^G U_\nu := [X/G] \times_{BG} (U_\nu/G)$.

Corollary 4.16. *Let* $\mathbf{D}$ *be as in Proposition 4.15. For every* $X \in {}^\tau\mathrm{Stk}_S^G$ *and every* $\mathcal{F} \in \mathbf{D}^\lhd(S)_{<\infty}$, *the morphism*

$$\mathrm{H}^i([X/G], \mathcal{F}) \to \varprojlim_\nu \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F}) \tag{4.17}$$

is invertible for all $i \in \mathbf{Z}$. *Moreover, there exists a sufficiently large index* ν *such that*

$$\mathrm{H}^i([X/G], \mathcal{F}) \to \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F})$$

is invertible.

Proof. Combine Proposition 4.15 with Remark 4.10.   □

Remark 4.18. Corollary 4.16 says in other words that for any eventually coconnective $\mathcal{F} \in \mathbf{D}^\lhd([X/G])_{<\infty}$, the pro-system

$$\{\pi_i R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F})\}_\nu$$

is essentially constant for all $i \in \mathbf{Z}$. This may fail without the eventually coconnective hypothesis. For example, take $X = S = \mathrm{Spec}(k)$ with k a field, $G = \mathbf{G}_{m,k}$, $\mathbf{D} = \mathbf{SH}$ as in Sect. 6, $E = \mathrm{KGL} \in \mathbf{SH}(k)$ the algebraic K-theory spectrum, and $\mathcal{F} = a^*\mathrm{KGL}$ where $a : B\mathbf{G}_{m,k} \to \mathrm{Spec}(k)$. Then the pro-system

$$\{\pi_0 R\Gamma(U_\nu/\mathbf{G}_{m,k}, \mathrm{KGL})\}_\nu \simeq \{\mathrm{K}_0(\mathbf{P}_k^\nu)\}_\nu \simeq \{\mathbf{Z}[t]/(t^\nu)\}_\nu$$
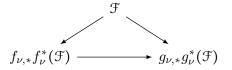
is not isomorphic to the constant pro-system $\{\mathbf{Z}[[t]]\}$. We thank Marc Levine for pointing out this example.

The proof of Proposition 4.15 will use the following lemma:

Lemma 4.19. *Suppose that* $\mathbf{D}$ *satisfies topological invariance and continuity. Let* $\mathfrak{X} \in {}^\tau\mathrm{Stk}$ *and let* $\{j_\nu : \mathcal{U}_\nu \to \mathcal{Y}_\nu\}_\nu$ *be a sequential diagram of open immersions in* $^\tau\mathrm{Stk}_\mathfrak{X}$ *with* $\mathcal{Y}_\nu$ *smooth over* $\mathfrak{X}$. *Write* $c_\nu := d_{\mathcal{Y}_\nu/\mathfrak{X}}|_{\mathfrak{z}_\nu} - d_{\mathfrak{z}_\nu/\mathfrak{X}}$, *where* $\mathcal{Z}_\nu := \mathcal{Y}_\nu \smallsetminus \mathcal{U}_\nu$ *are the reduced complements and* $d_{\mathcal{Y}_\nu/\mathfrak{X}}$ *and* $d_{\mathcal{Z}_\nu/\mathfrak{X}}$ *are the fibre dimensions (see (0.2.3)). Assume that for every* $c \in \mathbf{Z}$ *there exists an index* $\nu(c)$ *for which* $c_\nu \geqslant c$ *for all* $\nu \geqslant \nu(c)$. *If* $\{f_\nu : \mathcal{Y}_\nu \to \mathfrak{X}\}_\nu$ *is pro-acyclic, then* $\{g_\nu : \mathcal{U}_\nu \to \mathfrak{X}\}_\nu$ *is pro-acyclic.*

Proof. By definition of the lisse extension and of the cohomological t-structure on $\mathbf{D}^\lhd(\mathfrak{X})$, we may assume that $X = \mathfrak{X}$, $Y_\nu = \mathcal{Y}_\nu$, and $U_\nu = \mathcal{U}_\nu$ are schemes.

For every ν and every $\mathcal{F} \in \mathbf{D}(X)_{<\infty}$ we have a commutative triangle

$$
\begin{array}{ccc}
 & \mathcal{F} & \\
 \swarrow & & \searrow \\
 f_{\nu,*} f_\nu^*(\mathcal{F}) & \longrightarrow & g_{\nu,*} g_\nu^*(\mathcal{F})
\end{array}
$$

where the horizontal arrow is induced by the unit $\mathrm{id} \to j_{\nu,*} j_\nu^*$. This gives rise to the exact triangle

$$\mathcal{K}(f_\nu) \to \mathcal{K}(g_\nu) \to \mathcal{K}_\nu$$

where $\mathcal{K}(f_\nu)$, $\mathcal{K}(g_\nu)$, and $\mathcal{K}_\nu$ are the fibres of the left-hand diagonal, right-hand diagonal, and horizontal arrows respectively. It will thus suffice to show the following claim:

($*$) For every integer $n \in \mathbf{Z}$ and every $\mathcal{F} \in \mathbf{D}^\triangleleft(\mathcal{X})_{<\infty}$, choose an integer $c \geqslant (C - n)/2$; then $\mathcal{K}_\nu$ is n-coconnective for all $\nu \geqslant \nu(c)$.

By the localization triangle, we have

$$\mathcal{K}_\nu \simeq f_{\nu,*} i_{\nu,*} i_\nu^! f_\nu^*(\mathcal{F})$$

where $i_\nu : Z_\nu \to Y_\nu$ is the inclusion of the reduced complement of U_ν. Let $C \in \mathbf{Z}$ such that $\mathcal{F}$ is C-coconnective. For every fixed $n \in \mathbf{Z}$, if $c \geqslant (C - n)/2$ then $d_{Y_\nu/X}|_{Z_\nu} - d_{Z_\nu/X} \geqslant c$ for all $\nu \geqslant \nu(c)$. It follows by Theorem 4.4 and Corollary 4.5 that $\mathcal{K}_\nu$ is $(C - 2c)$-coconnective. As $(C - 2c) \leqslant n$, $\mathcal{K}_\nu$ is in particular n-coconnective for every $\nu \geqslant \nu(c)$. $\qquad\square$

Proof of Proposition 4.15. Given $X \in {}^\tau\mathrm{Stk}_S^G$, consider the tower of open immersions

$$\{X \overset{G}{\underset{S}{\times}} U_\nu \to X \overset{G}{\underset{S}{\times}} V_\nu\}_{\nu>0}$$

over $[X/G]$, where the notation is as in 3.1. For every ν, $X \times_S^G V_\nu$ is the total space of a vector bundle over $[X/G]$, hence is acyclic (by homotopy invariance for $\mathbf{D}$). It will therefore suffice to check the condition on fibre dimensions in Lemma 4.19. Since fibre dimension is stable under base change, this follows from the fact that $c_\nu = d_{V_\nu/S}|_{W_\nu} - d_{W_\nu/S}$ tends to ∞, where $W_\nu \subseteq V_\nu$ is the reduced complement of U_ν (see Remark 3.3). $\qquad\square$

5. Betti and étale (co)homology

In this section we specialize the results of the previous section to the following weaves:

(i) *Betti:* Suppose $k = \mathbf{C}$. For every locally of finite type k-scheme X, let $\mathbf{D}(X) := \mathrm{D}(X(\mathbf{C}), \Lambda)$ denote the derived ∞-category of sheaves of Λ-modules on the topological space $X(\mathbf{C})$, for some commutative ring Λ. This satisfies topological invariance by definition.

(ii) *Étale (torsion coefficients):* For every locally of finite type k-scheme X, let $\mathbf{D}(X) := \mathrm{D}_{\text{ét}}(X, \Lambda)$ denote the derived ∞-category of sheaves of Λ-modules on the small étale site of X, where Λ is a commutative ring of positive characteristic n, with n invertible in k. Topological invariance holds by [SGA4, Exp. VIII, Thm. 1.1].

(iii) *Étale (adic coefficients):* For every locally of finite type k-scheme S, let
$\mathbf{D}(X)$ denote the limit of ∞-categories

$$\varprojlim_{n>0} \mathrm{D}_{\text{ét}}(X, \Lambda/\mathfrak{m}^n)$$

where Λ is a discrete valuation ring whose residue characteristic is
invertible in k. Topological invariance follows from the case of torsion
coefficients.

In each case the unit $\mathbf{1}_X \in \mathbf{D}(X)$ is just the constant sheaf with coefficients
in Λ. These all satisfy étale descent, so one may take $\tau = $ ét; in this case the
lisse extension $\mathbf{D}^{\lhd}$ is the unique étale sheaf on $^{\tau}\mathrm{Stk}$ which restricts to $\mathbf{D}$ on
Sch and in particular coincides with the extension to stacks considered in
[LZ]. More generally the following discussion goes through for topological
weaves which are oriented and satisfy topological invariance and continuity,
and for which the units $\mathbf{1}_X \in \mathbf{D}(X)$ lie in the heart of the cohomological
t-structure for every $X \in \mathrm{Sch}$.

Let S, G, and $\{U_\nu\}_\nu$ be as in Notation 3.1. Corollaries 4.1 and 4.16 specialize
to:

Corollary 5.1. *For every $X \in {}^{\tau}\mathrm{Stk}_S^G$ and $\mathcal{F} \in \mathbf{D}^{\lhd}([X/G])$, there are canonical
isomorphisms*

$$R\Gamma([X/G], \mathcal{F}) \simeq \varprojlim_{\nu} R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F}).$$

*Moreover, if $\mathcal{F} \in \mathbf{D}^{\lhd}([X/G])_{<\infty}$ is eventually coconnective, then there are
canonical isomorphisms*

$$\mathrm{H}^s([X/G], \mathcal{F}) \simeq \varprojlim_{\nu} \mathrm{H}^s(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F})$$

for every $s \in \mathbf{Z}$.

Taking coefficients in the constant sheaf $\Lambda_{BG} = \mathbf{1}_{BG} \in \mathbf{D}^{\lhd}(BG)$ we deduce:

Corollary 5.2. *For every $X \in {}^{\tau}\mathrm{Stk}_S^G$, there are canonical isomorphisms*

$$\mathrm{H}_G^s(X) \simeq \varprojlim_{\alpha} \mathrm{H}^s(X \times^G U_\alpha)$$

for all $s \in \mathbf{Z}$.

Proof. The object $f^*(\Lambda_{BG}) = \Lambda_{[X/G]} \in \mathbf{D}^{\lhd}([X/G])$ is 0-coconnective (Theorem 4.4(i)). $\square$

For $X \in {}^{\tau}\mathrm{Stk}^G$, define the equivariant Borel–Moore homology spectrum
(relative to the base S)

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(X; \Lambda) = R\Gamma([X/G], f^!(\Lambda_{BG}))$$

where $f : [X/G] \to [S/G] = BG$ is the projection.

Corollary 5.3. *We have*

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(X; \Lambda) \simeq \varprojlim_{\nu} \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\underset{S}{\times}} U_\nu; \Lambda)(-d_\nu + g)[-2d_\nu + 2g],$$

where d_ν is the relative dimension of $U_\nu \to S$ and g is the relative dimension of $G \to S$.

Proof. By Corollary 5.1 we have

$$R\Gamma([X/G], f^!(\Lambda_{BG})) \simeq \varprojlim_{\nu} R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, q_\nu^* f^!(\Lambda_{BG}))$$

where $q_\nu : X \times_S^G U_\nu \to [X/G]$ is the base change of $U_\nu/G \to BG$. The latter are smooth of relative dimension d_ν. By the Poincaré duality isomorphisms $q_\nu^* \simeq q_\nu^!(-d_\nu)[-2d_\nu]$ and $\Lambda_{BG} \simeq a_{BG}^!(\Lambda_S)(g)[2g]$, where $a_{BG} : BG \to S$, the right-hand side is identified with the limit of the (shifted and Tate twisted) Borel–Moore chains on $X \times_S^G U_\nu$ as claimed. $\square$

We define the equivariant Borel–Moore homology groups by

$$\mathrm{H}_s^{\mathrm{BM},G}(X; \Lambda) = \pi_s R\Gamma([X/G], f^!(\Lambda_{BG})) \simeq \mathrm{H}^{-s}([X/G], f^!(\Lambda_{BG}))$$

for $s \in \mathbf{Z}$.

Corollary 5.4. *If X is of finite dimension (in the sense of [SP, 0AFL] or [LMB, Eq. (11.14)]), then*

$$\mathrm{H}_s^{\mathrm{BM},G}(X; \Lambda) \simeq \varprojlim_{\nu} \mathrm{H}_{s+2d_\nu-2g}^{\mathrm{BM}}(X \overset{G}{\underset{S}{\times}} U_\nu; \Lambda)(-d_\nu + g).$$

for every $s \in \mathbf{Z}$.

Proof. If X is of dimension $\leqslant d$, then $f : [X/G] \to BG$ is of relative dimension $\leqslant d$, so $f^!(\Lambda_{BG})$ is $2d$-coconnective by Theorem 4.4. We conclude by the second part of Corollary 5.1. $\square$

6. Generalized cohomology theories

In this section we consider the case where $\mathbf{D}$ is a general topological weave. For concreteness, we take the universal case $\mathbf{D} = \mathbf{SH}$ (and $\tau = \mathrm{Nis}$).

Given a scheme X, let $\mathbf{SH}(X)$ denote the stable ∞-category of motivic spectra over X (see e.g. [Hoy1, App. C]), and consider the lisse extension $\mathbf{SH}^\lhd(-)$ with respect to $*$-inverse image.

Let S, G, and $\{U_\nu\}_\nu$ be as in Notation 3.1. By Corollary 4.1 we deduce the following, a vast generalization of [KR, Thm. 12.9].

Corollary 6.1. *For every $X \in {}^\tau\mathrm{Stk}_S^G$ and $\mathcal{F} \in \mathbf{SH}^\lhd([X/G])$, there are canonical isomorphisms*

$$R\Gamma([X/G], \mathcal{F}) \simeq \varprojlim_{\nu} R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F}).$$

The six functor formalism on $\mathbf{SH}(-)$ persists to the lisse extension $\mathbf{SH}^\lhd(-)$ by [Cho, Kha5]. In particular, for any locally of finite type morphism f in ${}^\tau\mathrm{Stk}$ one has the adjoint pair of functors $(f_!, f^!)$.

Given a motivic spectrum $E \in \mathbf{SH}(S)$, let $E_{BG} = E|_{BG}$ denote its $*$-inverse image in $\mathbf{SH}^\lhd(BG)$. For $X \in {}^\tau\mathrm{Stk}_S^G$, define the equivariant Borel–Moore

homology spectrum (relative to the base S)

$$\mathrm{C}^{\mathrm{BM},G}_{\bullet}(X;E) = R\Gamma([X/G], f^{!}(E_{BG}))$$

where $f : [X/G] \to [S/G] = BG$ is the projection. We denote by $\langle n \rangle \simeq (n)[2n]$ the Thom twist by the trivial bundle of rank n.

Corollary 6.2. *For every* $X \in {}^{\tau}\mathrm{Stk}^{G}_{S}$ *there is a canonical isomorphism*

$$\mathrm{C}^{\mathrm{BM},G}_{\bullet}(X;E) \simeq \varprojlim_{\nu} \mathrm{C}^{\mathrm{BM}}_{\bullet}(X \overset{G}{\underset{S}{\times}} U_{\nu};E)\langle -\Omega_{U_{\nu}/S} + \Omega_{G/S}\rangle.$$

If E *is oriented, then moreover*

$$\mathrm{C}^{\mathrm{BM},G}_{\bullet}(X;E) \simeq \varprojlim_{\nu} \mathrm{C}^{\mathrm{BM}}_{\bullet}(X \overset{G}{\underset{S}{\times}} U_{\nu};E)\langle -d_{\nu} + g\rangle,$$

where d_{ν} *(resp.* g*) is the relative dimension of* $U_{\nu} \to S$ *(resp.* $G \to S$*).*

Proof. Follows from Corollary 6.1 as in the proofs of Corollaries 5.3 and 5.4, using the (unoriented) Poincaré duality isomorphisms

$$q^{*}_{\nu} \simeq q^{!}_{\nu}\langle -\Omega_{U_{\nu}/S}\rangle, \quad E_{BG} \simeq a^{!}_{BG}(E)\langle\Omega_{G/S}\rangle,$$

where $\Omega_{U_{\nu}/S}$ is the (G-equivariant) relative cotangent sheaf of $U_{\nu} \to S$, and similarly for $\Omega_{G/S}$, and $a_{BG} : BG \to S$ is the projection. $\square$

Suppose k is a field, $S = \mathrm{Spec}(k)$. If k has characteristic exponent e, then $\mathbf{SH}[1/e]$ (and more generally $\mathbf{D}[1/e]$ for any topological weave $\mathbf{D}$) satisfies topological invariance by [EK] as well as continuity. The cohomological t-structure of Subsect. 4.3 is the homotopy t-structure in this case. By Corollary 4.16 we obtain:

Corollary 6.3. *If* $E \in \mathbf{SH}(k)_{<\infty}$ *is eventually coconnective, then for every* $X \in {}^{\tau}\mathrm{Stk}^{G}_{S}$ *there are canonical isomorphisms*

$$\mathrm{H}^{s}([X/G], E)[\tfrac{1}{e}] \simeq \varprojlim_{\nu} \mathrm{H}^{s}(X \overset{G}{\underset{S}{\times}} U_{\nu}, E)[\tfrac{1}{e}]$$

for every $s \in \mathbf{Z}$.

Consider the equivariant Borel–Moore homology groups

$$\mathrm{H}^{\mathrm{BM},G}_{s}(X;E) = \pi_{s}R\Gamma([X/G], f^{!}(E_{BG})) \simeq \mathrm{H}^{-s}([X/G], f^{!}(E_{BG}))$$

for $s \in \mathbf{Z}$, where $f : [X/G] \to BG$. The argument of Corollary 5.4 shows:

Corollary 6.4. *If* X *is of finite dimension and* $E \in \mathbf{SH}(S)_{<\infty}$ *is eventually coconnective, then*

$$\mathrm{H}^{\mathrm{BM},G}_{s}(X;E)[\tfrac{1}{e}] \simeq \varprojlim_{\nu} \mathrm{H}^{\mathrm{BM}}_{s}(X \overset{G}{\underset{S}{\times}} U_{\nu};E)[\tfrac{1}{e}]\langle -\Omega_{U_{\nu}/S} + \Omega_{G/S}\rangle$$

for every $s \in \mathbf{Z}$. *If* E *is oriented, then moreover*

$$\mathrm{H}^{\mathrm{BM},G}_{s}(X;E)[\tfrac{1}{e}] \simeq \varprojlim_{\nu} \mathrm{H}^{\mathrm{BM}}_{s+2d_{\nu}-2g}(X \overset{G}{\underset{S}{\times}} U_{\nu};E)[\tfrac{1}{e}](-d_{\nu} + g)$$

for every $s \in \mathbf{Z}$.

Let Λ be a commutative ring in which e is invertible, and let $E = \Lambda^{\mathrm{mot}} \in \mathbf{SH}(k)$ be the Λ-linear motivic cohomology spectrum. Since the latter is 0-connective, we obtain using the comparison between motivic Borel–Moore homology of schemes and higher Chow groups (see [MVW, Prop. 19.18] and [CD2, Cor. 8.12]):

Corollary 6.5. *For every quasi-separated algebraic space X of finite type over k with G-action, there are canonical isomorphisms*

$$\mathrm{H}_{s+2n}^{\mathrm{BM},G}(X; \Lambda^{\mathrm{mot}})(-n) \simeq \mathrm{A}_n^G(X, s) \otimes \Lambda$$

for all $n, s \in \mathbf{Z}$, where on the right-hand side are the G-equivariant higher Chow groups of X [EG1, §2.7].

Similarly, one has

$$\mathrm{H}_{s+2n}^{\mathrm{BM}}(\mathfrak{X}; \Lambda^{\mathrm{mot}})(-n) \simeq \mathrm{A}_n(\mathfrak{X}, s) \otimes \Lambda,$$

where $\mathfrak{X} = [X/G]$ and the right-hand side is defined in [EG1, §5.3] or [Kre]. We expect this comparison to generalize to all Artin stacks $\mathfrak{X}$ of finite type over k with affine stabilizers, cf. [BP].

Remark 6.6. The right-hand side of Corollary 6.4 indicates a definition of G-equivariant (higher) Chow–Witt groups in terms of the Borel construction, in such a way that the resulting theory identifies with the generalized Borel–Moore homology theory associated with the Milnor–Witt motivic cohomology spectrum (in a manner parallel to Corollary 6.5).

7. Algebraic K-theory

Consider the Nisnevich sheaf of spectra $\mathrm{K} : \mathrm{Asp}^{\mathrm{op}} \to \mathrm{Spt}$ which sends a quasi-separated algebraic space X of finite type over k to its Bass–Thomason–Trobaugh K-theory spectrum (see e.g. [Kha3, Def. 2.6, Rem. 2.15]). Write $\mathrm{KH} : \mathrm{Asp}^{\mathrm{op}} \to \mathrm{Spt}$ for its $\mathbf{A}^1$-invariant version, defined by

$$\mathrm{KH}(X) = \varinjlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}(X \times \mathbf{A}^n)$$

for every $X \in \mathrm{Asp}$, see e.g. [Kha3, §4.2]. The canonical map $\mathrm{K}(X) \to \mathrm{KH}(X)$ is invertible when X is regular.

We study the lisse extensions $\mathrm{K}^{\lhd}$ and $\mathrm{KH}^{\lhd}$. Note that with rational coefficients, the presheaf $\mathrm{K}(-)_{\mathbf{Q}} : \mathrm{Asp}^{\mathrm{op}} \to \mathrm{Spt}$ sending $X \in \mathrm{Asp}$ to its rationalized K-theory spectrum $\mathrm{K}(X) \otimes \mathbf{Q}$, satisfies étale descent (see e.g. [Kha3, Thm. 5.1]). Thus its lisse extension $\mathrm{K}(-)_{\mathbf{Q}}^{\lhd}$ is the unique extension of $\mathrm{K}_{\mathbf{Q}}$ to an étale sheaf on Artin stacks; in particular, it coincides with the construction $\mathrm{K}^{\mathrm{\acute{e}t}}(-)_{\mathbf{Q}}$ in [Kha3, §5.2].

Let S, G, and $\{U_\nu\}_\nu$ be as in Notation 3.1. Applying Corollary 3.9 to the presheaf $\mathrm{KH}|_{\mathrm{Sch}^{\mathrm{lci}}}$ yields:

Corollary 7.1. *For every quasi-separated algebraic space X of finite type over S with G-action, there is a canonical isomorphism*

$$\mathrm{KH}^{\lhd}([X/G]) \simeq \varprojlim_{\nu} \mathrm{KH}(X \overset{G}{\underset{S}{\times}} U_\nu).$$

Remark 7.2. More generally, we get the same computation for any localizing invariant of stable ∞-categories which satisfies $\mathbf{A}^1$-invariance. For example, this also applies to topological K-theory (over the complex numbers) and periodic cyclic homology in characteristic zero.

Corollary 7.3. *If X is regular, then moreover*

$$\mathrm{K}^{\lhd}([X/G]) \simeq \varprojlim_{\nu} \mathrm{K}(X \overset{G}{\underset{S}{\times}} U_\nu).$$

Suppose the base ring k is noetherian. Since lci morphisms are of finite Tor-amplitude, G-theory (= algebraic K-theory of coherent sheaves) defines a presheaf of spectra $\mathrm{G} : \mathrm{Asp}^{\mathrm{lci,op}} \to \mathrm{Spt}$ on the category of quasi-separated algebraic spaces of finite type over k and lci morphisms (see e.g. [Kha3, §3]). Thus Corollary 3.9 yields:

Corollary 7.4. *For every quasi-separated algebraic space X of finite type over S with G-action, there is a canonical isomorphism*

$$\mathrm{G}^{\lhd}([X/G]) \simeq \varprojlim_{\nu} \mathrm{G}(X \overset{G}{\underset{S}{\times}} U_\nu).$$

Definition 7.5. With notation as above, we define *Borel-type G-equivariant K-theory*, *KH-theory*, and *G-theory* by

$$\mathrm{K}^{G,\lhd}(X) := \mathrm{K}^{\lhd}([X/G]),$$
$$\mathrm{KH}^{G,\lhd}(X) := \mathrm{KH}^{\lhd}([X/G]),$$
$$\mathrm{G}^{G,\lhd}(X) := \mathrm{G}^{\lhd}([X/G])$$

for all quasi-separated algebraic spaces X of finite type over S with G-action.

Proposition 7.6. *For every quasi-separated algebraic space X of finite type over S with G-action, the spectrum $\mathrm{G}^{G,\lhd}(X)$ is connective. If X is regular, then the same holds for $\mathrm{K}^{G,\lhd}(X)$.*

Proof. Recall that the claim is true for G trivial (i.e., $\mathrm{G}(X)$ is connective), and note that the second claim follows from the first since $\mathrm{K}(T) \simeq \mathrm{G}(T)$ for every $(T,t) \in \mathrm{Lis}_X$ when X (hence T) is regular.

For every $s > 0$, we have using Corollary 7.4 the canonical surjections

$$\pi_{-s}\mathrm{G}^{G,\lhd}(X) \twoheadrightarrow \varprojlim_{\nu} \pi_{-s}\mathrm{G}(X \overset{G}{\times} U_\nu) \simeq 0 \qquad (7.7)$$

with kernel $\varprojlim_{\nu}^1 \pi_{-s+1}\mathrm{G}(X \times^G U_\nu)$. We claim that the pro-system

$$\{\pi_{-s+1}\mathrm{G}(X \overset{G}{\times} U_\nu)\}_\nu \qquad (7.8)$$

satisfies the Mittag–Leffler condition for every $s > 0$. This will imply that the $\varprojlim^1$ terms vanish, hence $\pi_{-s}\mathrm{G}^{G,\lhd}(X)$ all vanish.

By Remark 3.3 the transition map for $\mu > \nu$ is the composite of the restriction map along an open,

$$\pi_{-s+1}\mathrm{G}(X \overset{G}{\times} U_\mu) \to \pi_{-s+1}\mathrm{G}(X \overset{G}{\times} (U_\nu \underset{S}{\times} V_{\mu-\nu}))$$

and pull-back along the zero section of a vector bundle,

$$\pi_{-s+1}\mathrm{G}(X \overset{G}{\times} (U_\nu \underset{S}{\times} V_{\mu-\nu})) \to \pi_{s+1}\mathrm{G}(X \overset{G}{\times} U_\nu).$$

The former is surjective if $s = 1$ and bijective if $s > 1$ (by the long-exact localization sequence and the connectivity of G-theory of algebraic spaces), and the latter is bijective for all s (by homotopy invariance). Thus (7.8) has surjective transition maps and in particular satisfies the Mittag–Leffler condition. □

We now prove Theorem D. From now on, we take S to be the spectrum of the base commutative ring k (so that G is a group scheme over k). We begin with the following spectrum-level formulation of equivariant Grothendieck–Riemann–Roch:

Theorem 7.9. *Suppose k is regular noetherian. Let X be a quasi-separated algebraic space of finite type over k with G-action. Then there is a canonical isomorphism of spectra*

$$\mathrm{G}^{G,\triangleleft}(X)_{\mathbf{Q}} \simeq \prod_{i \in \mathbf{Z}} \mathrm{C}^{\mathrm{BM},G}_{\bullet}(X; \mathbf{Q}^{\mathrm{mot}})\langle -i \rangle.$$

Moreover, this isomorphism commutes with equivariant proper push-forwards and equivariant quasi-smooth Gysin pull-backs.

Taking homotopy groups and combining with Corollary 6.5, we deduce:

Corollary 7.10. *Suppose k is a field. Then for every quasi-separated algebraic space X of finite type over k with G-action, there are canonical isomorphisms*

$$\mathrm{G}^{G,\triangleleft}_s(X)_{\mathbf{Q}} \simeq \prod_{i \in \mathbf{Z}} \mathrm{A}^{G}_i(X, s)_{\mathbf{Q}}$$

for all $s \in \mathbf{Z}$. Moreover, these isomorphisms commute with equivariant proper push-forwards and equivariant quasi-smooth Gysin pull-backs.

Rationally, we can also show that the $\underleftarrow{\lim}^1$ obstruction vanishes:

Corollary 7.11. *Suppose k is regular noetherian. Then for every quasi-separated algebraic space X of finite type over k with G-action, the canonical morphisms*

$$\mathrm{G}^{G,\triangleleft}_s(X)_{\mathbf{Q}} \to \underleftarrow{\lim_\nu} \mathrm{G}_s(X \overset{G}{\times} U_\nu)_{\mathbf{Q}}$$

are bijective for all $s \in \mathbf{Z}$.

Proof. In view of Corollary 7.4 there is a canonical surjection

$$\mathrm{G}^{G,\triangleleft}_s(X)_{\mathbf{Q}} \twoheadrightarrow \underleftarrow{\lim_\nu} \mathrm{G}_s(X \overset{G}{\times} U_\nu)_{\mathbf{Q}}.$$

Under the isomorphisms of Theorem 7.9 this is identified with the product over $i \in \mathbf{Z}$ of the maps

$$\pi_s \mathrm{C}_\bullet^{\mathrm{BM},G}(X; \mathbf{Q}^{\mathrm{mot}})\langle -i \rangle \twoheadrightarrow \varprojlim_\nu \pi_s \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\times} U_\nu; \mathbf{Q}^{\mathrm{mot}})\langle -i - d_\nu + g \rangle,$$

since products commute with π_s and $\varprojlim$. Each of these maps are bijective by Corollary 6.4, so the claim follows. $\qquad\square$

Corollary 7.12. *Suppose k is a field and G is a smooth affine algebraic group over k. Then for every quasi-projective scheme X over k with linearized G-action, there is a canonical isomorphism*

$$\mathrm{G}_0^G(X)_{I_G}^\wedge \simeq \prod_{i \in \mathbf{Z}} \mathrm{A}_i^G(X)_{\mathbf{Q}}$$

where the left-hand side is the completion of the G-equivariant G-theory of X at the augmentation ideal $I_G \subseteq \mathrm{K}_0(BG)$.

Proof. Krishna shows in [Kri3, Thm. 9.10] that under the assumptions, the completion $\mathrm{G}_0^G(X)_{I_G}^\wedge$ agrees with the right-hand side of Corollary 7.11 for $s = 0$. $\qquad\square$

Remark 7.13. In [CJ] G. Carlsson and R. Joshua showed (under some technical hypotheses) that the right-hand side of Corollary 7.4, and hence $\mathrm{G}^{G,\triangleleft}(X)$, agrees with the "Adams completion" of $\mathrm{G}^G(X) = \mathrm{G}([X/G])$ with respect to the augmentation map $\mathrm{K}(BG) = \mathrm{K}^G(S) \to \mathrm{K}(S)$.

At the level of homotopy groups, A. Krishna studied in [Kri3] the question of bijectivity of the map (the "Atiyah–Segal completion problem")

$$\mathrm{G}_s^G(X)_{I_G}^\wedge \to \varprojlim_\nu \mathrm{G}_s(X \overset{G}{\times} U_\nu) \qquad\qquad (7.14)$$

for X smooth quasi-projective. He showed that this holds when X is moreover projective and G is connected split reductive, but that it may fail for X non-projective (even with $G = \mathbf{G}_m$). In general, we can say that in the smooth but non-projective case the Atiyah–Segal map (7.14) is invertible with rational coefficients. Indeed, rationally both sides are isomorphic to $\prod_{i \in \mathbf{Z}} \mathrm{A}_i^G(X, s)_{\mathbf{Q}}$: the right-hand side by Corollaries 7.10 and Corollary 7.11, and the left-hand side by Krishna's version of equivariant GRR in [Kri2].

We begin the proof of Theorem 7.9, which involves some stable motivic homotopy theory. Let $\mathrm{KGL} \in \mathbf{SH}(k)$ denote the algebraic K-theory spectrum over k (see [CD1, §13.1]). Set

$$\mathrm{KGL}_{\mathcal{X}}^\triangleleft := a^*(\mathrm{KGL}), \quad \mathbf{Q}_{\mathcal{X}}^{\mathrm{mot},\triangleleft} := a^*(\mathbf{Q}^{\mathrm{mot}}) \in \mathbf{SH}^\triangleleft(\mathcal{X})$$

for every $\mathcal{X} \in {}^\tau\mathrm{Stk}$ with structural morphism $a : \mathcal{X} \to \mathrm{Spec}(k)$, where $\mathbf{Q}^{\mathrm{mot}} \in \mathbf{SH}(k)$ is the rational motivic cohomology spectrum as in Corollary 6.5. We begin by generalizing the Adams decomposition of $\mathrm{KGL}_{\mathbf{Q}}$ (see [Ri], [CD1, §14.1]) to stacks.

Proposition 7.15.

(i)   *For every $\mathcal{Y} \in {}^{\tau}\mathrm{Stk}$, there is a canonical isomorphism*

$$\mathrm{KGL}_{\mathcal{Y},\mathbf{Q}}^{\lhd} \simeq \bigoplus_{i\in\mathbf{Z}} \mathbf{Q}_{\mathcal{Y}}^{\mathrm{mot},\lhd}\langle i\rangle. \tag{7.16}$$

If $\mathcal{Y}$ is smooth then there is also a canonical isomorphism

$$\mathrm{KGL}_{\mathcal{Y},\mathbf{Q}}^{\lhd} \simeq \prod_{i\in\mathbf{Z}} \mathbf{Q}_{\mathcal{Y}}^{\mathrm{mot},\lhd}\langle i\rangle. \tag{7.17}$$

(ii) *For every morphism $f : \mathcal{X} \to \mathcal{Y}$ in ${}^{\tau}\mathrm{Stk}$ with $\mathcal{Y}$ smooth, there is a canonical isomorphism*

$$f_* f^! (\mathrm{KGL}_{\mathcal{Y},\mathbf{Q}}^{\lhd}) \to \prod_{i\in\mathbf{Z}} f_* f^! (\mathbf{Q}_{\mathcal{Y}}^{\mathrm{mot},\lhd})\langle i\rangle \tag{7.18}$$

in $\mathbf{SH}^{\lhd}(\mathcal{Y})$.

Proof. Consider the canonical morphisms

$$\bigoplus_{i\in\mathbf{Z}} \mathbf{Q}^{\mathrm{mot}}\langle i\rangle \to \mathrm{KGL}_{\mathbf{Q}} \to \prod_{i\in\mathbf{Z}} \mathbf{Q}^{\mathrm{mot}}\langle i\rangle$$

in $\mathbf{SH}(k)$. The first is invertible by [Ri, Thm. 5.3.10]. The composite is also invertible: for every smooth k-scheme X and $s \geqslant 0$, it induces an isomorphism on $\mathrm{H}^{-s}(X,-)$ since

$$\mathrm{H}^{-s}(X, \mathbf{Q}^{\mathrm{mot}}\langle i\rangle) \simeq \mathrm{H}^{2i-s}(X; \mathbf{Q}(i)) \simeq \mathrm{Gr}_{\gamma}^i \mathrm{K}_s(X)_{\mathbf{Q}}$$

vanishes for $i < 0$ and $i \gg 0$. The second isomorphism uses the fact that X is smooth over k and hence regular (see [CD1, Cor. 14.2.14]).

The isomorphisms in the first claim follow by $*$-inverse image along $a : \mathcal{Y} \to \mathrm{Spec}(k)$, which commutes with colimits (resp. limits when $\mathcal{Y}$ is smooth). The second claim then follows since $f^!$ and f_* commute with limits. $\qquad\square$

Proof of Theorem 7.9. For every X as in the statement we have by Proposition 7.15 a canonical isomorphism $f_* f^! (\mathrm{KGL}_{BG,\mathbf{Q}}^{\lhd}) \simeq \prod_{i\in\mathbf{Z}} f_* f^! (\mathbf{Q}_{BG}^{\mathrm{mot},\lhd})\langle i\rangle$ where $f : [X/G] \to BG$ is the projection. Formation of derived global sections commutes with limits, so as X varies this gives the canonical isomorphism

$$\mathrm{C}_{\bullet}^{\mathrm{BM},G}(-; \mathrm{KGL}_{\mathbf{Q}}) \simeq \prod_{i\in\mathbf{Z}} \mathrm{C}_{\bullet}^{\mathrm{BM},G}(-; \mathbf{Q}^{\mathrm{mot}})\langle i\rangle.$$

The claim follows by combining this with the canonical isomorphism

$$\mathrm{G}^{G,\lhd}(-) \simeq \mathrm{C}_{\bullet}^{\mathrm{BM},G}(-; \mathrm{KGL})$$

obtained by lisse-extending the isomorphism $\mathrm{G}(-) \simeq \mathrm{C}_{\bullet}^{\mathrm{BM}}(-; \mathrm{KGL})$ of sheaves of spectra on $\mathrm{Sch}^{\mathrm{lci}}$ (see [Jin]). The first displayed isomorphism commutes with equivariant proper push-forwards and equivariant quasi-smooth Gysin pull-backs by construction. The second displayed isomorphism commutes with equivariant proper push-forwards by the arguments of [Jin, §3.1] and with equivariant quasi-smooth Gysin pull-backs by [Kha3, §6.2]. $\qquad\square$

8. Algebraic bordism

Let $S = \mathrm{Spec}(k)$, G an affine algebraic group over k, and $\{U_\nu\}_\nu$ as in Notation 3.1. Corollary 6.2 applied to the algebraic cobordism spectrum $\mathrm{MGL} \in \mathbf{SH}(k)$ (and its Tate twists) shows that the associated equivariant Borel–Moore theory can be computed, at the level of spectra, by the Borel construction:

Corollary 8.1. *For every* $X \in {}^\tau\mathrm{Stk}_S^G$ *and* $r, s \in \mathbf{Z}$ *there are canonical isomorphisms of spectra*

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(X;\mathrm{MGL}) \simeq \varprojlim_\nu \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\times} U_\nu;\mathrm{MGL})\langle -d_\nu + g\rangle$$

where $d_\nu = \dim(U_\nu)$ *and* $g = \dim(G)$.

Recall that if X is a quasi-projective k-scheme with linearized G-action, then each $X \times^G U_\nu$ is a quasi-projective k-scheme (Remark 3.6). Thus in this case Corollary 8.1 computes the G-equivariant bordism of X in terms of non-equivariant bordism of schemes.

When k is a field of characteristic zero, there is an identification

$$\mathrm{H}_{2n}^{\mathrm{BM}}(X;\mathrm{MGL})(-n) \simeq \Omega_n(X),$$

for all quasi-projective k-schemes X, of the $(2*, *)$-graded part of Borel–Moore homology with coefficients in MGL, with the (lower) algebraic bordism theory defined by Levine and Morel (see [LM, Lev]).

Building on the theory $\Omega_*(-)$, Heller and Malagón-López defined in [HML] a (lower) G-equivariant algebraic bordism theory by the formula

$$\Omega_n^{G,\mathrm{HML}}(X) := \varprojlim_\nu \Omega_{n+d_\nu - g}(X \overset{G}{\times} U_\nu) \tag{8.2}$$

for X quasi-projective with linearized G-action. By Corollary 8.1, the canonical surjection from the homotopy groups of a homotopy limit to the limit of its homotopy groups reads in this case

$$\pi_0 \mathrm{C}_\bullet^{\mathrm{BM},G}(X;\mathrm{MGL})\langle -n\rangle$$
$$\twoheadrightarrow \varprojlim_\nu \pi_0 \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\times} U_\nu;\mathrm{MGL})\langle -n - d_\nu + g\rangle \simeq \Omega_n^{G,\mathrm{HML}}(X) \tag{8.3}$$

for every $n \in \mathbf{Z}$. Since $\mathrm{MGL} \in \mathbf{SH}(k)$ is not eventually coconnective, we cannot apply Corollary 6.4 to deduce that the above map is invertible. Nevertheless, we can show this holds after rationalization:

Theorem 8.4. *Let* k *be a field of characteristic zero, * G *an affine algebraic group, and* X *a quasi-projective* k-scheme *with linearized* G-action. *Then the canonical map*

$$\pi_0 \mathrm{C}_\bullet^{\mathrm{BM},G}(X;\mathrm{MGL}_\mathbf{Q})\langle -n\rangle \twoheadrightarrow \Omega_n^{G,\mathrm{HML}}(X)_\mathbf{Q}$$

is invertible for every $n \in \mathbf{Z}$.

We begin with the analogue of Proposition 7.15 for MGL. For $\mathcal{X} \in {}^\tau\mathrm{Stk}$ with structural morphism $a : \mathcal{X} \to \mathrm{Spec}(k)$, set $\mathrm{MGL}_{\mathcal{X}}^{\lhd} := a^*(\mathrm{MGL}) \in \mathbf{SH}^{\lhd}(\mathcal{X})$ and $\mathbf{Q}_{\mathcal{X}}^{\mathrm{mot},\lhd} := a^*(\mathbf{Q}^{\mathrm{mot}})$.

Proposition 8.5.

(i)   *For every $\mathcal{Y} \in {}^\tau\mathrm{Stk}$, there is a canonical isomorphism*

$$\mathrm{MGL}_{\mathcal{Y},\mathbf{Q}}^{\lhd} \simeq \bigoplus_{i \geqslant 1} \mathbf{Q}_{\mathcal{Y}}^{\mathrm{mot},\lhd}\langle i \rangle. \tag{8.6}$$

If $\mathcal{Y}$ is smooth then there is also a canonical isomorphism

$$\mathrm{MGL}_{\mathcal{Y},\mathbf{Q}}^{\lhd} \simeq \prod_{i \geqslant 1} \mathbf{Q}_{\mathcal{Y}}^{\mathrm{mot},\lhd}\langle i \rangle. \tag{8.7}$$

(ii)   *For every morphism $f : \mathcal{X} \to \mathcal{Y}$ in ${}^\tau\mathrm{Stk}$ with $\mathcal{Y}$ smooth, there is a canonical isomorphism*

$$f_* f^!(\mathrm{MGL}_{\mathcal{Y},\mathbf{Q}}^{\lhd}) \to \prod_{i \geqslant 1} f_* f^!(\mathbf{Q}_{\mathcal{Y}}^{\mathrm{mot},\lhd})\langle i \rangle \tag{8.8}$$

in $\mathbf{SH}^{\lhd}(\mathcal{Y})$.

Proof. Recall that there is a canonical decomposition $\mathrm{MGL}_{\mathbf{Q}} \simeq \bigoplus_{i \geqslant 1} \mathbf{Q}\langle i \rangle$ in $\mathbf{SH}(k)$ (see [NSØ, Cor. 10.6]). Hence one may argue exactly as in the proof of Proposition 7.15, using the vanishing of $\mathrm{Gr}_\gamma^i \mathrm{K}_s(X)_{\mathbf{Q}}$ for $i \gg 0$. $\qquad\square$

Proof of Theorem 8.4. For every finite type k-scheme Y with action of an affine algebraic group H, we have natural isomorphisms

$$\mathrm{C}_\bullet^{\mathrm{BM},H}(Y; \mathrm{MGL}_{\mathbf{Q}}) \simeq \prod_{i \geqslant 1} \mathrm{C}_\bullet^{\mathrm{BM},H}(Y; \mathbf{Q})\langle i \rangle \tag{8.9}$$

by Proposition 8.5 applied to $f : [Y/H] \to BH$. Under these isomorphisms, the surjection

$$\begin{aligned}
&\pi_0 \mathrm{C}_\bullet^{\mathrm{BM},G}(X; \mathrm{MGL}_{\mathbf{Q}})\langle -n \rangle \\
&\quad \twoheadrightarrow \varprojlim_\nu \pi_0 \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\times} U_\nu; \mathrm{MGL}_{\mathbf{Q}})\langle -n - d_\nu + g \rangle \simeq \Omega_n^{G,\mathrm{HML}}(X)_{\mathbf{Q}}
\end{aligned} \tag{8.10}$$

is identified with the product over $i \geqslant 1$ of the maps

$$\pi_0 \mathrm{C}_\bullet^{\mathrm{BM},G}(X; \mathbf{Q}^{\mathrm{mot}})\langle -n + i \rangle \twoheadrightarrow \varprojlim_\nu \pi_0 \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\times} U_\nu; \mathbf{Q}^{\mathrm{mot}})\langle -n + i - d_\nu + g \rangle,$$

since products commute with π_0 and $\varprojlim$. Each of these maps are bijective by Corollary 6.4, so the claim follows. $\qquad\square$

With integral coefficients, we suspect that (8.3) may not be invertible in general.[13] We propose that the "correct" definition[14] of $\Omega_n^G(-)$ should satisfy

$$\Omega_n^G(X) \simeq \pi_0 \mathrm{C}_\bullet^{\mathrm{BM},G}(X;\mathrm{MGL})\langle -n\rangle. \tag{8.11}$$

By the general properties of the construction $\mathrm{C}_\bullet^{\mathrm{BM}}(-;E)$, for $E \in \mathbf{SH}(k)$, $\Omega_*^G(-)$ admits proper push-forwards and quasi-smooth Gysin pull-backs, and satisfies homotopy invariance and the projective bundle formula. We will now show that it also satisfies the right-exact localization property[15]. In [AKLPR1, AKLPR2] it is shown that it also satisfies the equivariant concentration and localization theorems.

Theorem 8.12. *Let k be a perfect field, G an affine algebraic group over k, and X a quasi-separated algebraic space of finite type over k with G-action. Then we have:*

(i)   *For every G-equivariant closed immersion $i : Z \to X$ with complementary open immersion $j : U \to X$, the localization sequence*

$$\mathrm{H}_{2n}^{\mathrm{BM},G}(Z;\mathrm{MGL})(-n)$$
$$\xrightarrow{i_*} \mathrm{H}_{2n}^{\mathrm{BM},G}(X;\mathrm{MGL})(-n) \xrightarrow{j^*} \mathrm{H}_{2n}^{\mathrm{BM},G}(U;\mathrm{MGL})(-n) \to 0.$$

   is exact.

(ii) *For every integer $n \in \mathbf{Z}$, the spectrum $\mathrm{C}_\bullet^{\mathrm{BM},G}(X;\mathrm{MGL})\langle -n\rangle$ is connective. That is, we have*

$$\mathrm{H}_s^{\mathrm{BM},G}(X;\mathrm{MGL})(-r) = 0$$

   for all $r, s \in \mathbf{Z}$ with $s < 2r$.

Proof. The localization exact triangle

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(Z;\mathrm{MGL}) \to \mathrm{C}_\bullet^{\mathrm{BM},G}(X;\mathrm{MGL}) \to \mathrm{C}_\bullet^{\mathrm{BM},G}(U;\mathrm{MGL})$$

induces a long exact sequence in $\mathrm{H}_*^{\mathrm{BM},G}(-;\mathrm{MGL})(-n)$ for every $n \in \mathbf{Z}$. In particular we have the exact sequence

$$\mathrm{H}_{2n}^{\mathrm{BM},G}(Z;\mathrm{MGL})(-n) \xrightarrow{i_*} \mathrm{H}_{2n}^{\mathrm{BM},G}(X;\mathrm{MGL})(-n) \xrightarrow{j^*} \mathrm{H}_{2n}^{\mathrm{BM},G}(U;\mathrm{MGL})(-n)$$
$$\xrightarrow{\partial} \mathrm{H}_{2n-1}^{\mathrm{BM},G}(Z;\mathrm{MGL})(-n).$$

This shows that the second claim implies the first.

---

[13]For X smooth projective and G connected and split reductive, we expect that (8.3) is invertible by arguments similar to those in [Kri3, §§6-7].

[14]Of course, one could take this as a definition, but by convention $\Omega_*^G(-)$ should be a "geometrically" defined theory as in [LM]. More precisely, for every n there should be geometrically defined sheaves of spectra on quasi-projective k-schemes, analogous to the Bloch cycle complexes, whose hypercohomologies compute $\Omega_n(-)$. (For smooth schemes X, such has been constructed in [EHKSY2, Thm. 3.4.1(ii)], using finite quasi-smooth derived schemes over X as generators.) Then the G-equivariant versions of these spectra should be defined by the Borel construction, i.e., via a homotopy limit as in Corollary 8.1. Finally, $\Omega_n^G(-)$ should be defined as the π_0 of this spectrum-level G-equivariant bordism theory.

[15]Although it is claimed in [HML, Thm. 20] that the same holds for $\Omega_*^{G,\mathrm{HML}}(-)$, there is a well-known gap in their proof.

We now demonstrate claim (ii) in the case of G the trivial group. If X is a smooth scheme, then by Poincaré duality the claim is equivalent to the vanishing of $\mathrm{H}^q(X; \mathrm{MGL}(p))$ for $q > 2p$, see e.g. [BKWX, Thm. B.1]. In general, the schematic locus defines a dense open $U \subseteq X$ (see [SP, Tag 06NH]). Since the field k is perfect, the smooth locus V of U is a further dense open. Using the localization sequence

$$\mathrm{H}^{\mathrm{BM}}_s(Z; \mathrm{MGL})(-r) \to \mathrm{H}^{\mathrm{BM}}_s(X; \mathrm{MGL})(-r) \to \mathrm{H}^{\mathrm{BM}}_s(V; \mathrm{MGL})(-r),$$

where $Z \subseteq X$ is the reduced closed complement, we conclude by noetherian induction.

For the case of general G, the proof is similar to that of Proposition 7.6. We claim that for every $n \in \mathbf{Z}$ the pro-system

$$\{\pi_0 \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times} U_\nu; \mathrm{MGL})\langle -n - d_\nu\rangle\}_\nu \tag{8.13}$$

satisfies the Mittag–Leffler condition. Indeed, by Remark 3.3 the transition map for $\mu > \nu$ is by construction the composite of the restriction map along an open,

$$\pi_0 \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times} U_\mu; \mathrm{MGL})\langle -n - d_\mu\rangle \to \pi_0 \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times}(U_\nu \times V_{\mu-\nu}); \mathrm{MGL})\langle -n - d_\mu\rangle$$

and Gysin pull-back along the zero section of a vector bundle,

$$\pi_0 \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times}(U_\nu \times V_{\mu-\nu}); \mathrm{MGL})\langle -n - d_\mu\rangle \to \pi_0 \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times} U_\nu; \mathrm{MGL})\langle -n - d_\nu\rangle.$$

By claim (i) in the case of G trivial, the former is surjective. By homotopy invariance, the latter is invertible. Thus the pro-system (8.13) has surjective transition maps and in particular satisfies the Mittag–Leffler condition.

Using Corollary 8.1, we have the canonical surjections

$$\pi_{-1} \mathrm{C}^{\mathrm{BM},G}_\bullet(X; \mathrm{MGL})\langle -n\rangle \twoheadrightarrow \varprojlim_\nu \pi_{-1} \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times} U_\nu; \mathrm{MGL})\langle -n - d_\nu + g\rangle \tag{8.14}$$

with kernel $\varprojlim^1_\nu \pi_0 \mathrm{C}^{\mathrm{BM}}_\bullet(X \times^G U_\nu; \mathrm{MGL})\langle -n - d_\nu\rangle$. The latter vanishes by the Mittag–Leffler condition verified above. Hence in that case, Claim (i) for G trivial implies that the target vanishes, hence so does the source.

Now, by induction we see that the pro-system $\{\pi_{-s+1} \mathrm{C}^{\mathrm{BM}}_\bullet(X \times^G U_\nu; \mathrm{MGL})\langle *\rangle\}_\nu$ vanishes for all $s > 1$. Arguing by the Milnor exact sequence again, we have bijectivity of (8.14) for all lower homotopy groups as well, and we conclude again by Claim (i) for G trivial. This shows claim (ii). $\qquad\square$

Combining with Theorem 8.4, we deduce that the theory of [HML] satisfies the right-exact localization property with rational coefficients:

Corollary 8.15. *Let k be a field of characteristic zero, G an affine algebraic group over k, and X a quasi-projective k-scheme with linearized G-action. Then for every G-equivariant closed immersion $i : Z \to X$ with complementary open immersion $j : U \to X$, the localization sequence*

$$\Omega^{G,\mathrm{HML}}_n(Z)_{\mathbf{Q}} \xrightarrow{i_*} \Omega^{G,\mathrm{HML}}_n(X)_{\mathbf{Q}} \xrightarrow{j^*} \Omega^{G,\mathrm{HML}}_n(U)_{\mathbf{Q}} \to 0.$$

is exact.

9. Categorification

Let $\mathbf{D} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pres}$ be a τ-sheaf with values in the ∞-category Pres of presentable ∞-categories and left adjoint functors. Denote by $\mathbf{D}^{\lhd} : {}^{\tau}\mathrm{Stk}^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}$ its lisse extension as in Definition 1.2. Given a morphism $f : \mathcal{X} \to \mathcal{Y}$ we denote the induced functor by $f^* = \mathbf{D}(f) : \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$, and by $f_* : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y})$ its right adjoint.

We will assume that $\mathbf{D}$ satisfies the following two properties:

(i)   *Local $\mathbf{A}^1$-invariance*: for every $X \in \mathrm{Sch}$, the unit morphism

$$\mathrm{id} \to \pi_*\pi^*$$

is fully faithful, where $\pi : X \times \mathbf{A}^1 \to X$ is the projection. In other words, $\pi^* : \mathbf{D}(X) \to \mathbf{D}(X \times \mathbf{A}^1)$ is fully faithful.

(ii) *Smooth base change formula*: for every cartesian square in Sch

$$
\begin{array}{ccc}
X' & \xrightarrow{\ g\ } & Y' \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
X & \xrightarrow{\ f\ } & Y,
\end{array}
$$

the base change transformation

$$v^* f_* \xrightarrow{\ \mathrm{unit}\ } g_* g^* v^* f_* \simeq g_* u^* f^* f_* \xrightarrow{\ \mathrm{counit}\ } g_* u^*$$

is invertible.

For example, we may take $\mathbf{D} = \mathbf{SH}$ (with $\tau = \mathrm{Nis}$, see Sect. 6) or more generally any topological weave in the sense of [Kha4].

Fix S, G, and $\{U_\nu\}_\nu$ as in Notation 3.1. For every $X \in {}^{\tau}\mathrm{Stk}_S^G$ we consider the square

$$
\begin{array}{ccc}
X \times_S U_\nu & \xrightarrow{\ p_\nu\ } & X \\
{\scriptstyle v_\nu}\downarrow & & \downarrow{\scriptstyle u} \\
X \times_S^G U_\nu & \xrightarrow{\ q_\nu\ } & [X/G]
\end{array}
\tag{9.1}
$$

where p_ν and q_ν are the projections, and u and v_ν are the quotient maps.

Theorem 9.2. *For every $\mathcal{F} \in \mathbf{D}^{\lhd}([X/G])$, the unit maps induce a canonical isomorphism*

$$\mathcal{F} \to \varprojlim_\nu q_{\nu,*} q_\nu^*(\mathcal{F})$$

in $\mathbf{D}^{\lhd}([X/G])$.

Proof. For a fixed $\mathcal{F} \in \mathbf{D}^{\lhd}([X/G])$, the presheaf

$$F : {}^{\tau}\mathrm{LisStk}_{[X/G]}^{\mathrm{op}} \to \mathbf{D}^{\lhd}([X/G])$$

sending $(T, t : T \to [X/G]) \mapsto t_* t^*(\mathcal{F})$ is, by construction, lisse-extended from its restriction to $\mathrm{Lis}_{[X/G]}$. Thus the claim follows from Proposition 2.5 applied to $F|_{\mathrm{Lis}_{[X/G]}}$. $\square$

Consider the right Kan extension of $\mathbf{D}^\triangleleft$ to ind-objects, so that

$$\mathbf{D}^\triangleleft(X \underset{S}{\times} \{U_\nu\}_\nu) \simeq \varprojlim_\nu \mathbf{D}^\triangleleft(X \underset{S}{\times} U_\nu), \quad \mathbf{D}^\triangleleft(X \overset{G}{\underset{S}{\times}} \{U_\nu\}_\nu) \simeq \varprojlim_\nu \mathbf{D}^\triangleleft(X \overset{G}{\underset{S}{\times}} U_\nu)$$

where the transition functors are $*$-inverse image. We have the induced functors

$$p^* = (p_\nu^*)_\nu : \mathbf{D}^\triangleleft(X) \to \mathbf{D}^\triangleleft(X \underset{S}{\times} \{U_\nu\}_\nu),$$

$$q^* = (q_\nu^*)_\nu : \mathbf{D}^\triangleleft([X/G]) \to \mathbf{D}^\triangleleft(X \overset{G}{\underset{S}{\times}} \{U_\nu\}_\nu). \tag{9.3}$$

Corollary 9.4. *The functor (9.3) is fully faithful.*

Proof. The functor q^* admits as right adjoint $(\mathcal{F}_\nu) \mapsto \varprojlim_\nu q_{\nu,*}\mathcal{F}_\nu$, so fully faithfulness amounts to invertibility of the unit map

$$\mathcal{F} \to \varprojlim_\nu q_{\nu,*} q_\nu^*(\mathcal{F})$$

for all $\mathcal{F} \in \mathbf{D}([X/G])$, which is the assertion of Theorem 9.2. $\qquad\square$

We say that the group scheme G is *Nisnevich-special* if the quotient morphism $S \twoheadrightarrow [S/G] = BG$ admits Nisnevich-local sections, i.e., if every étale G-torsor is Nisnevich-locally trivial. For example, this includes special group schemes in the sense of Serre such as $\mathrm{GL}_{n,S}$.

Corollary 9.5. *If $\tau = $ ét or G is Nisnevich-special, then the squares (9.1) induce a cartesian square of ∞-categories*

$$\begin{array}{ccc}
\mathbf{D}^\triangleleft([X/G]) & \xrightarrow{q^*} & \mathbf{D}^\triangleleft(X \times_S^G \{U_\nu\}_\nu) \\
\downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\
\mathbf{D}^\triangleleft(X) & \xrightarrow{p^*} & \mathbf{D}^\triangleleft(X \times_S \{U_\nu\}_\nu).
\end{array}$$

Remark 9.6. If $\tau = $ Nis and G is not Nisnevich-special, then one still has a cartesian square of ∞-categories

$$\begin{array}{ccc}
\mathbf{D}^\triangleleft([X/G]) & \xrightarrow{q^*} & \mathbf{D}^\triangleleft(X \times_S^G \{U_\nu\}_\nu) \\
\downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\
\mathbf{D}^\triangleleft(Y) & \xrightarrow{p^*} & \mathbf{D}^\triangleleft(Y' \times_S^G \{U_\nu\}_\nu),
\end{array}$$

where $u : Y \twoheadrightarrow [X/G]$ is any smooth morphism with Nisnevich-local sections and $Y' \to Y$ is the G-torsor classified by $Y \to [X/G] \to BG$, since u^* is conservative in this case.

Lemma 9.7. *With notation as above, suppose that X is a scheme. Then the cartesian square*

$$\begin{array}{ccc}
X \times_S \{U_\nu\}_\nu & \xrightarrow{p} & X \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\
X \times_S^G \{U_\nu\}_\nu & \xrightarrow{q} & [X/G]
\end{array}$$

satisfies the smooth base change formula. That is, the natural transformation

$$u^* q_* \xrightarrow{\text{unit}} p_* p^* u^* q_* \simeq p_* v^* q^* q_* \xrightarrow{\text{counit}} p_* v^*$$

is invertible.

Proof. This is the limit over n of the natural transformations

$$u^* q_{\nu,*} \to p_{\nu,*} v_\nu^*,$$

associated to the squares (9.1). Using descent for the Čech nerve of $u : X \twoheadrightarrow [X/G]$ and its base change v_ν, which we denote $X_\bullet$ and $Y_\bullet$ respectively, [GR, Vol. I, Pt. I, Chap. 1, 2.6.4] implies that this is map is in turn the limit of the corresponding natural transformations for all the squares

$$
\begin{array}{ccc}
Y_{m+1} & \xrightarrow{p_m} & X_{m+1} \\
\downarrow{\scriptstyle d^i} & & \downarrow{\scriptstyle d^i} \\
Y_m & \xrightarrow{q_m} & X_m
\end{array}
$$

where the horizontal arrows are base changed from q and p and the vertical arrows d^i are the face maps (for $0 \leqslant i \leqslant m$). By the smooth base change formula for schemes (ii), these are invertible for all m and all i. $\qquad\square$

Lemma 9.8. *Suppose given a commutative square of ∞-categories*

$$
\begin{array}{ccc}
\mathscr{C} & \xrightarrow{q^*} & \mathscr{C}' \\
\downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\
\mathscr{D} & \xrightarrow{p^*} & \mathscr{D}'
\end{array}
$$

where p^* and q^* are fully faithful with respective right adjoints p_* and q_*, the base change transformation

$$u^* q_* \xrightarrow{\text{unit}} p_* p^* u^* q_* \simeq p_* v^* q^* q_* \xrightarrow{\text{counit}} p_* v^*$$

is invertible, and v^* is conservative. Then the essential image of q^* is spanned by objects $c' \in \mathscr{C}'$ for which $v^*(c')$ belongs to the essential image of p^*.

Proof. Note that an object $c' \in \mathscr{C}'$ belongs to the essential image of q^* if and only if the counit $q^* q_*(c') \to c'$ is invertible. Indeed, the condition is clearly sufficient. Conversely, suppose $c' \simeq q^*(c)$ for an object $c \in \mathscr{C}$. By the adjunction identities, the composite

$$q^*(c) \xrightarrow{\text{unit}} q^* q_* q^*(c) \xrightarrow{\text{counit}} q^*(c)$$

is the identity. Since q^* is fully faithful, the first arrow is invertible. It follows that the second arrow is also invertible.

Now since v^* is conservative, invertibility of the counit $q^* q_*(c') \to c'$ is equivalent to invertibility of

$$\text{counit} : p^* p_* v^*(c') \simeq p^* u^* q_*(c') \simeq v^* q^* q_*(c') \xrightarrow{\text{counit}} v^*(c')$$

where we have used the base change isomorphism. As in the first paragraph, since p^* is fully faithful this is equivalent to the condition that $v^*(c')$ belongs to the essential image of p^*. $\qquad\square$

Proof of Corollary 9.5. Given $(T, t) \in \mathrm{Lis}_{[X/G]}$, we may form the base change of the squares (9.1) along $t : T \to [X/G]$ to get

$$
\begin{array}{ccc}
T' \times_S \{U_\nu\}_\nu & \xrightarrow{\ p_T\ } & T' \\
\downarrow{\scriptstyle v_T} & & \downarrow{\scriptstyle u_T} \\
T \times_{BG} \{U_\nu/G\}_\nu & \xrightarrow{\ q_T\ } & T
\end{array}
$$

where $T' \to T \simeq [T'/G]$ is the G-torsor classified by $T \to [X/G] \to BG$. By definition of $\mathbf{D}^{\triangleleft}$, the square in question is the limit over (T, t) of the squares

$$
\begin{array}{ccc}
\mathbf{D}^{\triangleleft}(T) & \xrightarrow{\ q_T^*\ } & \mathbf{D}^{\triangleleft}(T \times_{BG} \{U_\nu/G\}_\nu) \\
\downarrow{\scriptstyle u_T^*} & & \downarrow{\scriptstyle v^*} \\
\mathbf{D}^{\triangleleft}(T') & \xrightarrow{\ p^*\ } & \mathbf{D}^{\triangleleft}(T' \times_S \{U_\nu\}_\nu).
\end{array}
$$

We may therefore replace X by T' and thereby assume that X is a scheme.

By Corollary 9.4 the upper horizontal arrow is fully faithful. The same holds for the lower horizontal arrow (note that $\{U_\nu\}_\nu$ also serves as a Borel construction for the trivial group). This implies that the square is cartesian on mapping spaces. Essential surjectivity of the functor

$$
\mathbf{D}^{\triangleleft}([X/G]) \to \mathbf{D}^{\triangleleft}(X) \underset{\mathbf{D}^{\triangleleft}(X \times_S \{U_\nu\}_\nu)}{\times} \mathbf{D}^{\triangleleft}(X \overset{G}{\underset{S}{\times}} \{U_\nu\}_\nu)
$$

then follows from Lemma 9.8 in view of the base change formula $u^* q_* \simeq p_* v^*$ (Lemma 9.7) and the conservativity of v^* (since u and hence v admits τ-local sections). $\qquad\square$

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