3D COHOMOLOGICAL HALL ALGEBRAS FOR LOCAL SURFACES

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Abstract. Let X be the canonical bundle of a smooth algebraic surface S. We construct the (sheaf-level) 3d or critical cohomological Hall algebra of X. This refines the 2d cohomological Hall algebra of S constructed by Kapranov–Vasserot, and may be regarded as an instance of Joyce's conjecture for Lagrangians in (-1)-shifted symplectic spaces, which we prove in the conormal case. The proof uses a new theory of derived microlocalization. *This is a research announcement; details of some proofs will appear in a forthcoming work.*

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Introduction

The aim of this paper is to introduce a derived-geometric generalization of microlocal sheaf theory à la Kashiwara–Schapira [KS] and apply it to geometric representation theory of the moduli stack of compactly supported coherent sheaves on a local surface (i.e., the canonical bundle of a smooth algebraic surface). In particular, we will construct a structure called the 3-*dimensional* or *critical cohomological Hall product* on the categorified Donaldson–Thomas invariants of local surfaces, confirming expectations of Kontsevich–Soibelman [KSo] and Joyce [JS]. We begin by motivating the study of the cohomological Hall algebra.

0.1. **2d cohomological Hall algebras.** Let S be a smooth algebraic surface. Denote by $\mathrm{Coh}_{\mathrm{cpt}}(S)$ the abelian category of compactly supported coherent sheaves on S and by M_S the moduli stack of objects in $\mathrm{Coh}_{\mathrm{cpt}}(S)$. The *2d cohomological Hall algebra* (*2d CoHA* for short) is an associative algebra structure introduced by Kapranov–Vasserot [KV] on the Borel–Moore homology $\mathrm{H}^{\mathrm{BM}}_*(M_S)$.

We briefly recall the definition. Let M_S^{ext} be the moduli stack of short exact sequences in $\mathrm{Coh}_{\mathrm{cpt}}(S)$ and consider the following correspondence:

$$\begin{array}{ccc} & M_S^{\mathrm{ext}} & \\ {}^{(\mathrm{ev}',\mathrm{ev}'')}\swarrow & & \searrow^{\mathrm{ev}} \\ M_S \times M_S & & M_S, \end{array} \tag{0.1}$$

where ev' sends $[0 \to E' \to E \to E'' \to 0]$ to E', and $\mathrm{ev}, \mathrm{ev}''$ are defined similarly. The morphism $(\mathrm{ev}', \mathrm{ev}'')$ is quasi-smooth and ev is proper on each connected component of M_S^{ext}. The 2d CoHA product

$$*_{2d}^{\mathrm{Hall}} \colon \mathrm{H}^{\mathrm{BM}}_*(M_S) \otimes \mathrm{H}^{\mathrm{BM}}_*(M_S) = \mathrm{H}^{\mathrm{BM}}(M_S \times M_S) \to \mathrm{H}^{\mathrm{BM}}_{*+2\cdot\mathrm{rel.dim}(\mathrm{ev}',\mathrm{ev}'')}(M_S) \tag{0.2}$$

is defined as the composite $\mathrm{ev}_* \circ (\mathrm{ev}', \mathrm{ev}'')^!$ where $(\mathrm{ev}', \mathrm{ev}'')^!$ is the virtual pull-back [Kha1] and ev_* is the proper push-forward.

The 2d CoHA has striking applications in geometric representation theory. For example, Davison–Hennecart–Schlegel Mejia [DHS2, §10.2] recently showed that it can be used to recover the Heisenberg algebra action on the homology of Hilbert schemes of $\mathbf{A}^2$ [Nak, Gro].

0.2. **3d cohomological Hall algebras.** String theory predicts the existence of a certain graded associative algebra called the algebra of *BPS states* associated with a smooth Calabi–Yau threefold [HM]. Kontsevich and Soibelman [KSo] have proposed a mathematical definition in terms of a certain cohomological Hall algebra on a categorification of the Donaldson–Thomas invariants.

However, such an algebra structure has yet to be constructed in general. We now explain the state of the art.

0.2.1. *3d CoHA for quivers with potentials.* Let Q be a quiver (i.e., an oriented finite graph). A potential W of a quiver is a formal finite sum of oriented cycles of Q. A quiver with potential (Q,W) can be thought of as a local model for a Calabi–Yau threefold (see e.g. [Tod2, Thm. 1.1] for a precise statement). For example, one can construct from (Q,W) a 3-Calabi–Yau dg-algebra $\Gamma(Q,W)$ called the Ginzburg dg-algebra [Gin].

Let M_Q be the moduli stack of representations of Q. The potential W defines a function $f_W\colon M_Q \to \mathbf{A}^1$ by taking the trace, whose critical locus coincides with the moduli stack of representations over the Jacobi algebra $\mathrm{Jac}(Q,W) = \mathrm{H}^0(\Gamma(Q,W))$. Kontsevich–Soibelman defined the *cohomological Donaldson–Thomas invariant* (or *CoDT invariant* for short) for (Q,W) to be the vanishing cycle cohomology

$$\mathrm{H}^*(M_Q, \phi_{f_W}(\mathbf{Q}_{M_Q}))[\dim M_Q].$$

They also considered an algebra structure on the CoDT invariant called the *critical cohomological Hall algebra* (or *critical CoHA* for short). By virtual pull-back along ev, we have a sheaf-level CoHA product for Q

$$(\mathrm{ev}', \mathrm{ev}'')^* \mathbf{Q}_{M_Q \times M_Q}[2\mathrm{rel.dim}\ \mathrm{ev}] \to \mathrm{ev}^! \mathbf{Q}_{M_Q}, \tag{0.3}$$

where ev, ev′ and ev″ denote the evaluation maps as in Subsect. 0.1. Then the critical CoHA product is defined by applying the vanishing cycles functor with respect to the function $f_W \circ \mathrm{ev} = (f_W \boxplus f_W) \circ (\mathrm{ev}', \mathrm{ev}'')$.

Critical CoHAs for quivers with potentials play an essential role in the interplay between representation theory and Donaldson–Thomas (DT) theory. For example, Davison and Meinhardt [DM] used the critical CoHA product to construct a categorification of the wall-crossing identity of DT invariants for quivers with potentials. They also explained in the same paper that a PBW-type statement for the critical CoHA may be interpreted as a categorification of an integrality statement for the DT invariants.

0.2.2. *3d CoHA for Calabi–Yau threefolds.* Let X be a Calabi–Yau threefold and M_X be the derived moduli stack of compactly supported coherent sheaves on X. Fixing an orientation o, i.e, a choice of a line bundle $\mathcal{L} \in \mathrm{Pic}(M_X^{\mathrm{red}})$ and an isomorphism $o\colon \mathcal{L}^{\otimes 2} \simeq \det(\mathbf{L}_{M_X}|_{M_X^{\mathrm{red}}})$, we have the *Donaldson–Thomas perverse sheaf* (or *DT perverse sheaf* for short)

$$\phi_{M_X} = \phi_{M_X, o} \in \mathrm{Perv}(M_X).$$

defined by Joyce and his collaborators [BBBJ, BBDJS] and independently by Kiem–Li [KL]. When there is no ambiguity about the choice of orientation, we write $\phi_{M_X} = \phi_{M_X,o}$ for simplicity. The cohomology of the DT perverse sheaf

$$\mathrm{H}^*(M_X, \phi_{M_X})$$

is called the *CoDT invariant* for X.

Let us briefly recall the construction of the DT perverse sheaf. Using the (-1)-shifted symplectic structure [PTVV] on M_X, it is shown in [BBBJ] that the moduli stack M_X can be written smooth-locally as the derived critical locus of a function on a smooth scheme (see [BBBJ, Cor. 2.11]). The DT perverse sheaf is then defined by gluing the complexes of vanishing cycles for these locally defined functions (up to certain twists to guarantee compatibility on overlaps).

For a choice of orientation o compatible with direct sum (see e.g., [Kin3, Example 5.7]), it is expected that $\mathrm{H}^*(M_X, \phi_{M_X})$ carries the structure of an algebra called the *critical CoHA*. More generally, it is expected that there exists a canonical map

$$(\mathrm{ev}', \mathrm{ev}'')^*(\phi_{M_X} \boxtimes \phi_{M_X})[\mathrm{vdim}\, M_X^{\mathrm{ext}}] \to \mathrm{ev}^! \phi_{M_X} \tag{0.4}$$

satisfying an associativity property. Here ev, ev′ and ev″ denote the evaluation map as in Subsect. 0.1. We call this map the *sheaf-level critical CoHA product*.

At the moment, the construction of the critical CoHA product for a general Calabi–Yau threefold is an open problem. Given the definition of the DT perverse sheaves by gluing locally defined perverse sheaves, the obvious approach to defining a sheaf-level critical CoHA product is to glue morphisms locally defined as in (0.3). However, since the latter involves complexes that do not live in the perverse heart, performing such a gluing would require constructing an infinite system of homotopy coherence data in the derived ∞-category. From our perspective, the main difficulty is therefore the lack of a *global* construction of the DT perverse sheaf.

Because of the absence of the critical CoHA for Calabi–Yau threefolds, the CoDT theory for Calabi–Yau threefolds is still in its infancy compared to the case of quivers with potentials. Once the critical CoHA and its sheaf-level upgrade is constructed, it would be possible to globalize the work of Davison–Meinhardt [DM] and apply it to categorify some celebrated wall-crossing formulae such as the DT/PT correspondence [Bri, Tod1].

0.3. **Dimensional reduction for CoDT invariants.**

Let S be a smooth algebraic surface and $X = \mathrm{Tot}_S(\omega_S)$ the total space of the canonical bundle. In this case there exists a canonical choice of orientation for M_X and we have a *dimensional reduction* isomorphism [Dav1, Kin1]:

$$\mathrm{H}^*(M_X, \phi_{M_X}) \simeq \mathrm{H}^{\mathrm{BM}}_{-*+\mathrm{vdim}\, M_S}(M_S). \tag{0.5}$$

This theorem enables us to apply CoDT theory to the study of moduli stacks of objects in 2-Calabi–Yau categories. See e.g. [Dav2, DHS1, DHS2, KKo] for some applications in this direction[1].

One can construct an algebra structure on $\mathrm{H}^*(M_X, \phi_{M_X})$ by combining the isomorphism (0.5) and using the 2d CoHA we have recalled in §0.1. However, it is not satisfactory for applications for several reasons. One reason is, that when we consider enumerative invariants, we often pick an ample divisor H on X and work with the moduli stack of H-semistable objects $M_X^{H\text{-ss}} \subseteq M_X$. When K_S is positive, the projection map $M_X \to M_S$ does

[1]The works [Dav2, DHS1, DHS2] only use the *local* dimensional reduction theorem of [Dav1] along with results from the CoDT theory of quivers with potentials. The work [KKo] partly develops CoDT theory for local curves and applies it to the study of the topology of the moduli space of Higgs bundles by using *global* dimensional reduction [Kin1].

not preserve H-semistability in general, and hence dimensional reduction does not apply. Similarly, dimensional reduction *cannot* be applied directly to study the categorification of closed subscheme invariants and stable pair invariants [PT] for local surfaces.

Therefore, a sheaf-level critical CoHA product would be necessary for applications to enumerative geometry of local surfaces.

0.4. **Main result.** Our main result is the construction of sheaf-level critical CoHA products for local surfaces. We take a smooth surface S and set $X \coloneqq \mathrm{Tot}_S(\omega_S)$. We consider the following correspondence

$$M_X \times M_X \xleftarrow{(\mathrm{ev}',\mathrm{ev}'')} M_X^{\mathrm{ext}} \xrightarrow{\mathrm{ev}} M_X. \tag{0.6}$$

similarly to (0.6). Note that the map ev is proper on the connected component of the source.

Theorem A. *There exists a canonical map*

$$\nu\colon (\mathrm{ev}', \mathrm{ev}'')^*(\phi_{M_X} \boxtimes \phi_{M_X}) \to \mathrm{ev}^! \phi_{M_X}[-\mathrm{vdim}\, M_X^{\mathrm{ext}}]$$

which induces a map

$$*_{3d}^{\mathrm{Hall}}\colon \mathrm{H}^*(M_X, \phi_{M_X})^{\otimes 2} \to \mathrm{H}^{*-\mathrm{vdim}\, M_X^{\mathrm{ext}}}(M_X, \phi_{M_X})$$

on hypercohomology. It satisfies the property that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^*(M_X, \phi_{M_X})^{\otimes 2} & \xrightarrow{*_{3d}^{\mathrm{Hall}}} & \mathrm{H}^{*-\mathrm{vdim}\, M_X^{\mathrm{ext}}}(M_X, \phi_{M_X}) \\ \simeq \downarrow (0.5) & & \simeq \downarrow (0.5) \\ \mathrm{H}^{\mathrm{BM}}_{-*+\mathrm{vdim}\, M_S}(M_S)^{\otimes 2} & \xrightarrow{*_{2d}^{\mathrm{Hall}}} & \mathrm{H}^{\mathrm{BM}}_{-*+\mathrm{vdim}\, M_X^{\mathrm{ext}}+\mathrm{vdim}\, M_S}(M_S). \end{array}$$

Here $*_{2d}^{\mathrm{Hall}}$ is the 2d CoHA product recalled in (0.2).

As we will see below in Theorem D, we prove a much more general statement which may be regarded as an instance of the Joyce conjecture [JS, Conj. 1.1].

By adopting an argument of Toda [Tod3, §4], one can construct a right action of the CoHA for zero-dimensional sheaves on the cohomological closed subscheme invariants and a left action on the cohomological stable pair invariants. We also note that the construction of the sheaf-level critical CoHA plays an important role in forthcoming work of Davison and the second author [DK] on the construction of a bialgebra upgrade of 2d CoHA for Calabi–Yau surfaces.

0.5. **Derived microlocal sheaf theory.** The proof of Theorem A involves a global, or rather *microlocal*, definition of the DT perverse sheaf ϕ_{M_X} (for

X a local surface). To do this, we introduce a generalization of microlocal sheaf theory [KS] to the setting of derived algebraic geometry.[2]

Let $f\colon Y_1 \to Y_2$ be a morphism of lhfp[3] derived Artin stacks of relative virtual dimension d. We let

$$N^*_{Y_1/Y_2} = \mathbf{V}(\mathbf{L}^\vee_{Y_1/Y_2}[1])$$

be the conormal bundle and let $\pi_{Y_1/Y_2}\colon N^*_{Y_1/Y_2} \to Y_1$ be the projection. The following theorem generalizes the microlocalization functor for a regular embedding defined in [KS] to arbitrary lhfp morphisms between derived Artin stacks:

Theorem B. *There exists a functor $\mu_{Y_1/Y_2}\colon \mathbf{D}_{\mathrm{c}}(Y_2) \to \mathbf{D}_{\mathrm{c}}(N^*_{Y_1/Y_2})$ on constructible derived categories with the following property:*

$$\pi_{Y_1/Y_2,*} \circ \mu_{Y_1/Y_2} \simeq f^!, \quad \pi_{Y_1/Y_2,!} \circ \mu_{Y_1/Y_2} \simeq f^*[2d].$$

In particular, we have a canonical isomorphism

$$\mathrm{H}^*(N^*_{Y_1/Y_2}, \mu_{Y_1/Y_2}(\mathbf{Q}_{Y_2})) \simeq \mathrm{H}^*(Y_1, f^!\mathbf{Q}_{Y_2}). \tag{0.7}$$

See Theorem 3.22 for further properties of the microlocalization functor. As in the classical case, it is defined as the Fourier–Sato dual of a specialization functor.

Now let Y be quasi-smooth and 1-Artin. Microlocalizing the constant sheaf along the projection $Y \to \mathrm{pt}$ produces a canonical sheaf on $N^*_{Y/\mathrm{pt}}$ which is perverse up to a shift. On the other hand, $N^*_{Y/\mathrm{pt}}$ admits a canonical (-1)-shifted symplectic structure (see [Cal]) and a canonical orientation, so we also have the DT perverse sheaf $\phi_{N^*_{Y/\mathrm{pt}}}$. We have (see Theorem 4.2):

Theorem C. *For a quasi-smooth derived 1-Artin stack Y, we have*

$$\phi_{N^*_{Y/\mathrm{pt}}} \simeq \mu_{Y/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[-2\,\mathrm{vdim}\,Y]. \tag{0.8}$$

0.6. **Proof of Theorem A.** Let S be a smooth algebraic surface and set $X = \mathrm{Tot}_S(\omega_S)$. We explain how derived microlocalization is used to construct the 3d CoHA product for X.

There exists a canonical isomorphism of (-1)-shifted symplectic derived stacks $M_X \simeq N^*_{M_S/\mathrm{pt}}$, under which Theorem C yields an identification

$$\phi_{M_X} \simeq \mu_{M_S/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[-\,\mathrm{vdim}\,M_S]. \tag{0.9}$$

Then Theorem B recovers the dimensional reduction theorem (0.5).

Derived microlocalization allows us to prove the following generalization of Theorem A. Consider a correspondence of lhfp derived Artin stacks of the

[2]This is based on mostly unpublished work of the first author on a derived microlocalization functor for arbitrary topological weaves in the sense of [Kha2], using a derived generalization of the homogeneous Fourier–Laumon transform [Kha4]. For our applications here, it is important to use a derived Fourier–*Sato* transform instead.

[3]locally homotopically of finite presentation, see Subsect. 0.7

form

$$\begin{array}{ccc} & Y & \\ {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} \\ Y_1 & & Y_2. \end{array} \tag{0.10}$$

This gives rise to the *conormal correspondence*

$$\begin{array}{ccc} & N^*_{Y/Y_1\times Y_2}[-1] & \\ {\scriptstyle \tilde{f}_1}\swarrow & & \searrow{\scriptstyle \tilde{f}_2} \\ N^*_{Y_1/\mathrm{pt}} & & N^*_{Y_2/\mathrm{pt}}, \end{array} \tag{0.11}$$

where $N^*_{Y/Y_1\times Y_2}[-1] = \mathbf{V}(\mathbf{L}^\vee_{Y/Y_1\times Y_2}[2])$. One can show that the correspondence (0.6) is identified with the conormal correspondence associated with (0.1). Therefore Theorem A follows from (0.9) and the following theorem (see Theorems 4.18 and 4.32):

Theorem D. *There exists a canonical map*

$$\nu\colon \tilde{f}_1^* \mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\operatorname{vdim} f_1] \to \tilde{f}_2^! \mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}).$$

If f_1 is quasi-smooth and f_2 is proper, the map $\tilde{f}_2$ is proper and the map induced on hypercohomology

$$\mathrm{H}^{*+2\operatorname{vdim} f_1}(N^*_{Y_1/\mathrm{pt}}, \mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})) \to \mathrm{H}^*(N^*_{Y_2/\mathrm{pt}}, \mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}))$$

is identified with the composite

$$\mathrm{H}^{\mathrm{BM}}_{-*-2\operatorname{vdim} f_1}(Y_1) \xrightarrow{f_1^!} \mathrm{H}^{\mathrm{BM}}_{-*}(Y) \xrightarrow{f_{2,*}} \mathrm{H}^{\mathrm{BM}}_{-*}(Y_2)$$

under the isomorphism (0.7).

Under the identification (0.8) between the DT perverse sheaf and the absolute microlocalization, the above theorem may be considered as an instance of the Joyce conjecture [JS, Conj. 1.1] in the conormal case. See Corollary 4.20.

0.7. **Conventions and notation.** We work over the field $\mathbf{C}$ of complex numbers. All (derived) schemes and stacks are implicitly defined over $\mathbf{C}$. We write $\mathrm{pt} = \mathrm{Spec}(\mathbf{C})$ and let $\mathbf{A}^1$ denote the affine line over $\mathbf{C}$ and $\mathbf{G}_m$ the complement of the zero section.

0.7.1. *Artin stacks.* A *stack* is an étale hypersheaf of ∞-groupoids[4] on the category of schemes.

A stack X is 0-*Artin* if its diagonal is a monomorphism representable by schemes, and there exists a scheme U with a morphism $U \to X$ which is étale surjective (i.e., whose base changes $U \times_X V \to V$ are étale surjective for every scheme V over X).

A stack is n-*Artin* for $n > 0$ if it has $(n-1)$-representable diagonal and admits a smooth and surjective morphism from a scheme. A stack is *Artin* if it is n-Artin for some n. See [Toë, §3.1] for details.

[4] our stacks are implicitly "higher"

Replacing the category of schemes above by the ∞-category of derived schemes, we obtain the notion of derived Artin stacks. We refer to [Toë, §5.2] for details.

We say that a morphism of derived Artin stacks is *lhfp* if it is locally homotopically of finite presentation, or equivalently if its relative cotangent complex is perfect and the induced morphism on classical truncations is locally of finite presentation (see e.g. [Kha3, Thm. 8.7.6]).

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1. Sheaves on stacks

1.1. Sheaves on Artin stacks.

1.1.1. *Sheaves.* Let $\mathcal{S}$ denote the category of locally of finite type schemes. Given $X \in \mathcal{S}$ we denote by $\mathbf{D}(X)$ the stable presentable ∞-category of sheaves on the topological space $X(\mathbf{C})$ with values in the derived ∞-category of $\mathbf{Q}$-vector spaces.

Proposition 1.1. *The presheaf $\mathbf{D}^*: \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ determined by the assignment*

$$X \mapsto \mathbf{D}(X), \quad f \mapsto f^* \tag{1.2}$$

satisfies descent for the étale topology. In particular, for every smooth surjection $p: U \twoheadrightarrow X$, the Čech descent diagram

$$\mathbf{D}(X) \to \mathbf{D}(U) \rightrightarrows \mathbf{D}(U \underset{X}{\times} U) \substack{\to\\\to\\\to} \mathbf{D}(U \underset{X}{\times} U \underset{X}{\times} U) \substack{\to\\\to\\\to\\\to} \cdots \tag{1.3}$$

exhibits $\mathbf{D}(X)$ as the limit.

Proof. If $(U_\alpha \to X)_\alpha$ is a jointly surjective family of étale morphisms of schemes, then $(U_\alpha(\mathbf{C}) \to X(\mathbf{C}))_\alpha$ is a jointly surjective family of local homeomorphisms. For the second claim, note that a smooth surjection $p: U \twoheadrightarrow X$ admits étale-local sections, hence generates a covering in the étale topology. □

1.1.2. *Sheaves on Artin stacks.*

Construction 1.4. Let $\mathcal{S}^+$ denote the ∞-category of locally of finite type Artin stacks. Consider the right Kan extension of (1.2) along the inclusion $\mathcal{S} \hookrightarrow \mathcal{S}^+$. By Proposition 1.1, the result is the unique étale sheaf $\mathbf{D}^*: (\mathcal{S}^+)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ extending (1.2). One can show that for $X \in \mathcal{S}^+$ it is the stable presentable ∞-category given by the limit

$$\mathbf{D}(X) \simeq \varprojlim_{(T,t)} \mathbf{D}(T)$$

over the ∞-category of pairs (T,t) where $T \in \mathcal{S}$ and $t: T \to X$ is a *smooth* morphism. Alternatively, if $p: U \twoheadrightarrow X$ is a smooth surjection from a scheme U then we may describe $\mathbf{D}(X)$ as the limit of the Čech diagram as in (1.3).

Remark 1.5. If X is a *derived* scheme, then the topological space $X(\mathbf{C})$ only depends on the classical truncation X_{cl}. Therefore, we may as well define $\mathbf{D}(X) \coloneqq \mathbf{D}(X_{\mathrm{cl}})$ for any derived Artin stack X.

1.1.3. *Six operations.*

Theorem 1.6. *We have the following operations on the ∞-categories $\mathbf{D}(X)$:*

(i) *For every locally of finite type derived Artin stack X, an adjoint pair of bifunctors*

$$\otimes : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X),$$
$$\underline{\mathrm{Hom}} : \mathbf{D}(X)^{\mathrm{op}} \times \mathbf{D}(X) \to \mathbf{D}(X).$$

(ii) *For every morphism $f: X \to Y$, an adjoint pair*

$$f^*: \mathbf{D}(Y) \to \mathbf{D}(X), \quad f_*: \mathbf{D}(X) \to \mathbf{D}(Y).$$

(iii) *For every morphism $f: X \to Y$, an adjoint pair*

$$f_!: \mathbf{D}(X) \to \mathbf{D}(Y), \quad f^!: \mathbf{D}(Y) \to \mathbf{D}(X).$$

These operations are subject to the following compatibilities:

(SO1) Base change formula: *For every cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

there are canonical isomorphisms

$$q^* f_! \simeq g_! p^*, \quad p_* g^! \simeq f^! q_*.$$

(SO2) Projection formula: *For every morphism $f: X \to Y$ there are canonical isomorphisms*

$$f_!(-) \otimes (-) \simeq f_!(- \otimes f^*(-)),$$
$$\underline{\mathrm{Hom}}_Y(f_!(-), -) \simeq f_* \underline{\mathrm{Hom}}_X(-, f^!(-)),$$
$$f^! \underline{\mathrm{Hom}}_Y(-,-) \simeq \underline{\mathrm{Hom}}_X(f^*(-), f^!(-)).$$

(SO3) Forgetting supports: *If $f: X \to Y$ is separated (has proper diagonal), there is a canonical morphism*

$$\mathrm{fsupp}_f: f_! \to f_*, \tag{1.7}$$

which is invertible when f is proper representable.

(SO4) Gysin: *If $f: X \to Y$ is quasi-smooth of relative virtual dimension d, there is a canonical morphism*

$$\mathrm{gys}_f: f^*[2d] \to f^!, \tag{1.8}$$

which is invertible when f is smooth (Poincaré duality).

(SO5) Localization: *If $i\colon Z \hookrightarrow X$ is a closed immersion with complementary open immersion $j\colon U \hookrightarrow X$, then there are canonical exact triangles*

$$j_!j^* \to \mathrm{id} \to i_!i^*$$
$$i_*i^! \to \mathrm{id} \to j_*j^!.$$

This theorem is essentially proven in [LZ][5], aside from the Gysin transformation (SO4). We briefly sketch the construction of the operations. We have f^* and $\otimes$ by construction, hence also f_* and $\underline{\mathrm{Hom}}$ by adjunction. Consider the presheaf $\mathbf{D}^!\colon \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ given by

$$X \mapsto \mathbf{D}(X), \quad f \mapsto f^!, \tag{1.9}$$

which can indeed be promoted to a functor of ∞-categories by the work of [LZ] or [GR]. The key observation is that its right Kan extension to $\mathcal{S}^+$ is, on objects, equivalent to $\mathbf{D}^*\colon (\mathcal{S}^+)^{\mathrm{op}} \to \mathrm{Cat}_\infty$. This is because the limits can be taken over smooth morphisms, for which we have the Poincaré duality isomorphisms $f^! \simeq f^*[2d]$. In this way we also get the operation $f^!$, as well as its left adjoint $f_!$ by formal reasons.

The Gysin transformation of (SO4) is constructed in [Kha1, §3.1, Rem. 3.8]. We recall that, for $f\colon X \to Y$ quasi-smooth of relative virtual dimension d, evaluating $\mathrm{gys}_f\colon f^*[2d] \to f^!$ on the constant sheaf gives rise to a canonical morphism

$$[X/Y]^{\mathrm{vir}} \coloneqq [f]^{\mathrm{vir}}\colon \mathbf{Q}_X[2d] \to f^!(\mathbf{Q}_Y) \tag{1.10}$$

called the relative virtual fundamental class in [Kha1]. When $Y = \mathrm{pt}$ this amounts to a morphism $\mathbf{Q}_X[2d] \to f^!(\mathbf{Q}) = \omega_X$, i.e., a Borel–Moore homology class

$$[X]^{\mathrm{vir}} \in \mathrm{H}^{\mathrm{BM}}_{2d}(X;\mathbf{Q}).$$

When X is Deligne–Mumford, this recovers the virtual fundamental class of [BF].

1.2. **Constructible complexes.** Recall that for a locally finite type (derived) $\mathbf{C}$-scheme X, a sheaf $\mathcal{F}$ of $\mathbf{Q}$-vector spaces on $X(\mathbf{C})$ is called *constructible* if, for some stratification $X = \coprod_\alpha X_\alpha$ by locally closed subschemes, each restriction $\mathcal{F}|_{X_\alpha}$ is locally constant of finite rank. A complex $\mathcal{F} \in \mathbf{D}(X)$ is called *constructible* if it has bounded and constructible cohomologies. See e.g. [Ach, Chap. 2].

Definition 1.11. Let X be a derived Artin stack. A complex $\mathcal{F} \in \mathbf{D}(X)$ is *constructible* if for every smooth morphism $t\colon T \to X$ where T is a quasi-compact scheme, $t^*(\mathcal{F}) \in \mathbf{D}(T)$ is constructible. This is equivalent to the existence of a single smooth surjection $p\colon U \twoheadrightarrow X$ where U is a scheme such that $p^*(\mathcal{F}) \in \mathbf{D}(U)$ is constructible.

We denote by $\mathbf{D}_{\mathrm{c}}(X) \subseteq \mathbf{D}(X)$ the full (stable) subcategory spanned by constructible complexes. For schemes it is well-known that the six operations preserve constructibility. For stacks one has the following:

[5] Although they consider the derived ∞-category of étale sheaves with torsion coefficients, their construction applies much more generally. This is explained for example in [Kha1, App. A].

Theorem 1.12.

(i) *For every morphism $f\colon X \to Y$ of derived Artin stacks, the functors f^* and $f^!$ preserve constructible complexes.*

(ii) *For every quasi-compact quasi-separated representable morphism $f\colon X \to Y$ of derived Artin stacks, the functors f_* and $f_!$ preserve constructible complexes.*

(iii) *For every derived Artin stack X, the functors $(-) \otimes (-)$ and $\underline{\mathrm{Hom}}(-,-)$ preserve constructibility in each argument.*

1.2.1. *Verdier duality.* Given a derived Artin stack X, we denote by

$$\omega_X \coloneqq a_X^!(\mathbf{Q})$$

the dualizing complex, where $a_X\colon X \to \mathrm{pt}$ is the projection. We set

$$\mathbb{D}_X \coloneqq \underline{\mathrm{Hom}}(-,\omega_X)\colon \mathbf{D}(X)^{\mathrm{op}} \to \mathbf{D}(X).$$

The following assertions reduce easily to the well-known case of schemes:

Theorem 1.13.

(i) *The canonical morphism*

$$\mathcal{F} \to \mathbb{D}_X\mathbb{D}_X(\mathcal{F})$$

is invertible for every constructible complex $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$.

(ii) *There are canonical isomorphisms*

$$\mathbb{D}_X(\mathcal{F} \otimes \mathbb{D}_X(\mathcal{G})) \simeq \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})$$

for all constructible complexes $\mathcal{F},\mathcal{G} \in \mathbf{D}_{\mathrm{c}}(X)$.

(iii) *For every morphism $f\colon X \to Y$, there are canonical isomorphisms*

$$\mathbb{D}_X(f^*\mathcal{G}) \simeq f^!(\mathbb{D}_Y\mathcal{G}),$$
$$\mathbb{D}_Y(f_*\mathcal{F}) \simeq f_!(\mathbb{D}_X\mathcal{F}),$$

for all constructible complexes $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$, $\mathcal{G} \in \mathbf{D}_{\mathrm{c}}(Y)$.

1.3. **Perverse sheaves.** Given a (derived) scheme X, we write

$$({}^p\mathbf{D}^{\leqslant 0}(X), {}^p\mathbf{D}^{\geqslant 0}(X))$$

for the perverse t-structure on the stable ∞-category $\mathbf{D}_{\mathrm{c}}(X)$. We refer to [BBDG] or [Ach, Chap. 3] for a textbook account.

Proposition 1.14. *Let X be a derived Artin stack. There exists a unique t-structure on the stable ∞-category $\mathbf{D}(X)$ such that $\mathcal{F} \in \mathbf{D}(X)$ belongs to ${}^p\mathbf{D}^{\leqslant 0}(X)$, resp. ${}^p\mathbf{D}^{\geqslant 0}(X)$, if and only if for every derived scheme T and every smooth morphism $t\colon T \to X$ of relative dimension d, $t^*(\mathcal{F})[d]$ belongs to ${}^p\mathbf{D}^{\leqslant 0}(T)$, resp. ${}^p\mathbf{D}^{\geqslant 0}(X)$.*

Proof. It is enough to prove that for any object $\mathscr{F} \in D_{\mathrm{c}}(X)$, we can find a fiber sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ such that $\mathscr{F}' \in {}^p\mathbf{D}^{\leqslant 0}(X)$ and $\mathscr{F}'' \in {}^p\mathbf{D}^{\geqslant 1}(X)$, since other axioms of t-structure can be checked locally. By definition, we have equivalences

$$\mathbf{D}(X) \simeq \varprojlim_t \mathbf{D}(T), \qquad {}^p\mathbf{D}^{\leqslant 0}(X) \simeq \varprojlim_t {}^p\mathbf{D}^{\leqslant 0}(T)$$

where the limits are taken over pairs $(T, t{:}\, T \to X)$ with T a derived scheme and t smooth of relative dimension d_t, and the transition functors are $t^*[d_t]$. Since the perverse truncation functor ${}^p\tau^{\leqslant 0}{:}\, \mathbf{D}_{\mathrm{c}}(T) \to {}^p\mathbf{D}^{\leqslant 0}(T)$ commutes with transition functors, it descends to a functor

$$ {}^p\tau^{\leqslant 0}{:}\, \mathbf{D}_{\mathrm{c}}(X) \to {}^p\mathbf{D}^{\leqslant 0}(X). $$

which makes ${}^p\mathbf{D}^{\leqslant 0}(X)$ a reflective subcategory of $\mathbf{D}_{\mathrm{c}}(X)$. For each $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(X)$, it follows from the construction that

$$ \mathrm{cofib}({}^p\tau^{\leqslant 0}(\mathscr{F}) \to \mathscr{F}) \in \mathbf{D}^{\geqslant 1}(X), $$

so we are done. □

Definition 1.15. Let X be a derived Artin stack. The t-structure on the stable ∞-category $\mathbf{D}(X)$ defined in Proposition 1.14 is called the *perverse t-structure*. A *perverse sheaf* is a constructible complex that belongs to the heart of this t-structure, which we denote $\mathrm{Perv}(X)$.

The following statements follow easily from the case of schemes:

Proposition 1.16. *Let X be a derived Artin stack.*

(i) *The Verdier duality functor $\mathbb{D}{:}\, \mathbf{D}_{\mathrm{c}}(X)^{\mathrm{op}} \to \mathbf{D}_{\mathrm{c}}(X)$ is perverse t-exact.*

(ii) *If $f{:}\, X \to Y$ is smooth of relative dimension d, then $f^*[d] \simeq f^![-d]$ is perverse t-exact.*

(iii) *If $f{:}\, X \to Y$ is proper representable with fibres of dimension $\leqslant d$, then $f_! \simeq f_*$ is of perverse amplitude $[-d, d]$.*

1.4. **Monodromic complexes.** Let X be a derived Artin stack with an action of $\mathbf{G}_m$.

Definition 1.17. We say that $\mathcal{F} \in \mathbf{D}(X)$ is *monodromic*[6] if for every point $x \in X$, the complex $\mathrm{act}_x^*(\mathcal{F})$ is locally constant, where act_x denotes the restricted action map

$$ \mathrm{act}_x{:}\, \mathbf{G}_m \times \{x\} \to \mathbf{G}_m \times X \xrightarrow{\mathrm{act}} X. $$

We let $\mathbf{D}_{\mathrm{mon}}(X) \subseteq \mathbf{D}(X)$ denote the full subcategory consisting of monodromic complexes.

The following is well-known in the case of a separated morphism of schemes; see [DG, Thm. C.5.3] for an argument that works in this generality.

Proposition 1.18 (Contraction lemma)**.** *Let $p{:}\, X \to S$ be a morphism of derived Artin stacks and $s{:}\, S \to X$ a section. Suppose there exists an $\mathbf{A}^1$-homotopy $h{:}\, \mathbf{A}^1 \times X \to X$ between id_X and $s \circ p$, so that the two composites*

$$ X \xrightarrow{i_0} X \times \mathbf{A}^1 \xrightarrow{h} X, $$

$$ X \xrightarrow{i_1} X \times \mathbf{A}^1 \xrightarrow{h} X $$

[6] Arguing as in [KS, Proposition 3.7.2], one can show that this is equivalent to the existence of an isomorphism $\widetilde{\mathrm{act}}^*(\mathcal{F}) \simeq \widetilde{\mathrm{pr}}_2^*(\mathcal{F})$, where $\widetilde{\mathbf{G}}_m \to \mathbf{G}_m$ is the universal cover and $\widetilde{\mathrm{act}}, \widetilde{\mathrm{pr}}_2{:}\, \widetilde{\mathbf{G}}_m \times X \to X$ denote the action and the projection maps, respectively. This requires the extension of $\mathbf{D}(-)$ to complex-analytic stacks, which we avoid here for simplicity.

are identified with id_X *and* $s \circ p$*, respectively. Then the canonical morphisms*

$$p_* \xrightarrow{\text{unit}} p_* s_* s^* \simeq s^*,$$
$$s^! \simeq p_! s_! s^! \xrightarrow{\text{counit}} p_!$$

are invertible on monodromic complexes.

1.5. **Nearby and vanishing cycles.** We let $\pi \colon \widetilde{\mathbf{G}}_m \to \mathbf{A}^1$ denote the natural projection map from the universal cover of $\mathbf{G}_m$ and set[7]

$$K_\psi := \pi_! \mathbf{Q}_{\widetilde{\mathbf{G}}_m}, \quad K_\phi := \mathrm{cofib}(\pi_! \mathbf{Q}_{\widetilde{\mathbf{G}}_m} \xrightarrow{\text{counit}} \mathbf{Q}_{\mathbf{A}^1}) \quad \text{in } \mathbf{D}(\mathbf{A}^1).$$

Given a derived Artin stack X and a morphism $t \colon X \to \mathbf{A}^1$, we let $i_0 \colon X_0 \hookrightarrow X$ denote the inclusion of the zero locus:

$$\begin{array}{ccc} X_0 & \xhookrightarrow{i_0} & X \\ \downarrow & & \downarrow t \\ \{0\} & \hookrightarrow & \mathbf{A}^1. \end{array}$$

The functors of *nearby* and *vanishing cycles* along t are defined respectively by

$$\psi_t := i_0^* \circ \mathbf{R}\underline{\mathrm{Hom}}(t^* K_\psi, -) \colon \mathbf{D}(X) \to \mathbf{D}(X_0),$$
$$\phi_t := i_0^* \circ \mathbf{R}\underline{\mathrm{Hom}}(t^* K_\phi, -) \colon \mathbf{D}(X) \to \mathbf{D}(X_0).$$

Note also that there is a canonical isomorphism $\psi_t \circ i_{0,*} \simeq 0$; equivalently, the canonical morphisms $\psi_t \circ j_{0,!} j_0^* \to \psi_t$ and $\psi_t \circ j_{0,!} \to \psi_t \circ j_{0,*}$ are invertible, where j_0 is the inclusion of the complement of X_0.

Theorem 1.19. *Let* X *be a derived Artin stack and* $t \colon X \to \mathbf{A}^1$ *a morphism. We have:*

(NC1) Monodromicity: *Suppose* X *admits a* $\mathbf{G}_m$*-action for which* t *is equivariant (with respect to the scaling action on* $\mathbf{A}^1$*), and regard* X_0 *with the induced action. Then the functors* ψ_t *and* ϕ_t *preserve monodromic complexes.*

(NC2) Triangles: *There are canonical exact triangles*

$$\phi_t \to i_0^* \to \psi_t, \tag{1.20}$$
$$\psi_t \to i_0^! \to \phi_t. \tag{1.21}$$

(NC3) Proper base change: *Given a morphism* $f \colon X' \to X$*, form the cartesian square*

$$\begin{array}{ccc} X'_0 & \longrightarrow & X' \\ \downarrow f_0 & & \downarrow f \\ X_0 & \longrightarrow & X \end{array}$$

[7]This is an abuse of notation: π is just the universal cover $\widetilde{\mathbf{C}^*} \to \mathbf{C}^* \hookrightarrow \mathbf{C}$, as a morphism of topological spaces, and $\pi_!$ is the compactly supported direct image functor $\mathbf{D}(\widetilde{\mathbf{C}^*}) \to \mathbf{D}(\mathbf{C})$.

and let $t' = t \circ f: X' \to \mathbf{A}^1$. Then there are canonical natural transformations

$$\mathrm{Ex}_{\psi,*} : \psi_t f_* \to f_{0,*}\psi_{t'}, \tag{1.22}$$

$$\mathrm{Ex}_{!,\psi} : f_{0,!}\psi_{t'} \to \psi_t f_!, \tag{1.23}$$

which are invertible if f is proper representable. Similarly for ϕ_t.

(NC4) Smooth base change: *With notation as in (NC3), there are canonical natural transformations*

$$\mathrm{Ex}^{*,\psi} : f_0^*\psi_t \to \psi_{t'} f^*, \tag{1.24}$$

$$\mathrm{Ex}^{\psi,!} : \psi_{t'} f^! \to f_0^!\psi_t, \tag{1.25}$$

which are invertible if f is smooth. Similarly for ϕ_t.

(NC5) Constructibility: *The functors ψ_t and ϕ_t preserve constructible complexes.*

(NC6) Perversity: *The functors $\psi_t[-1]$ and ϕ_t are perverse t-exact; in particular, they preserve perverse sheaves.*

(NC7) Duality: *For every constructible complex $\mathcal{F} \in \mathbf{D}(X)$, there are canonical natural isomorphisms*

$$\phi_t(\mathbb{D}\mathcal{F}) \to \mathbb{D}\phi_t(\mathcal{F}), \tag{1.26}$$

$$\psi_t(\mathbb{D}\mathcal{F})[-1] \to \mathbb{D}\psi_t(\mathcal{F})[-1]. \tag{1.27}$$

(NC8) Normalization: *Let $u: X \times \mathbf{A}^1 \to \mathbf{A}^1$ denote the projection. Then there are canonical isomorphisms*

$$\psi_u \circ p^* \simeq \psi_u \circ j_! \circ q^* \simeq \mathrm{id} \tag{1.28}$$

where $j: X \times \mathbf{G}_m \hookrightarrow X \times \mathbf{A}^1$ is the inclusion and $p: X \times \mathbf{A}^1 \to X$ and $q: X \times \mathbf{G}_m \to X$ are the projections.

1.5.1. *Monodromicity (NC1).* The proof is the same as in the case of schemes, see [Ver, Prop. 7.1].

1.5.2. *Triangles (NC2).* As in [KS, §8.6], there are canonical exact triangles

$$\Delta_1: \mathbf{Q}_{\mathbf{A}^1} \to K_\phi \to K_\psi[1],$$

$$\Delta_2: K_\psi[1] \to K_\phi \to \mathbf{Q}_0.$$

The exact triangles (1.20) and (1.21) are obtained by applying $i_0^*\mathbf{R}\underline{\mathrm{Hom}}(\Delta_1, -)$ and $i_0^*\mathbf{R}\underline{\mathrm{Hom}}(\Delta_2, -)$ respectively.

1.5.3. *Proper base change (NC3) and smooth base change (NC4).* They are direct consequences of exchange properties between six-operations.

1.5.4. *Constructibility (NC5).* Since constructibility is local for smooth covers, this follows from the case of schemes.

1.5.5. *Perversity (NC6).* The perverse t-structure is local for smooth covers, this follows from the case of schemes.

1.5.6. *Duality (NC7).* The proof is the same as in the case of schemes: see [Mas].

1.5.7. *Normalization* (NC8). The first isomorphism in (1.28) comes from $\psi_t \circ j_! j^* \simeq \psi_t$. Let $i_0 : X \hookrightarrow X \times \mathbf{A}^1$ denote the zero section. By the contraction lemma (Proposition 1.18) we have

$$\begin{aligned}\psi_f \circ p^*(\mathcal{F}) &\simeq i_0^* \underline{\mathrm{Hom}}(u^* K_\psi, p^* \mathcal{F}) \\ &\simeq p_* \underline{\mathrm{Hom}}(u^* K_\psi, p^* \mathcal{F}) \\ &\simeq \underline{\mathrm{Hom}}(p_! u^* K_\psi[2], \mathcal{F}),\end{aligned}$$

where the second isomorphism is the standard identity $p_* \underline{\mathrm{Hom}}(-, p^!(-)) \simeq \underline{\mathrm{Hom}}(p_!(-), -)$, adjoint to the projection formula, combined with $p^! \simeq p^*[2]$ (Poincaré duality). By the base change formula we compute

$$p_! u^* K_\psi[2] \simeq a_X^* a_{\mathbf{A}^1,!} K_\psi[2] \simeq a_X^* a_{\widetilde{\mathbf{G}}_m,!}(\mathbf{Q}_{\widetilde{\mathbf{G}}_m})[2],$$

where $a_Y : Y \to \mathrm{pt}$ denotes the projection for any Y. Since $\widetilde{\mathbf{G}}_m$ is contractible, we have $a_{\widetilde{\mathbf{G}}_m,!}(\mathbf{Q}) \simeq a_{\widetilde{\mathbf{G}}_m,!} a^!_{\widetilde{\mathbf{G}}_m}(\mathbf{Q})[-2] \simeq \mathbf{Q}[-2]$ by homotopy invariance. We get the canonical isomorphism $\psi_f \circ p^*(\mathcal{F}) \simeq \underline{\mathrm{Hom}}(\mathbf{Q}_X, \mathcal{F}) \simeq \mathcal{F}$ as claimed.

2. The Fourier–Sato transform

2.1. **Derived vector bundles.** Let X be a derived Artin stack over $\mathbf{C}$. Given a perfect complex $\mathcal{E} \in \mathrm{Perf}(X)$, we denote by $\mathbf{V}(\mathcal{E})$ the stack of cosections of $\mathcal{E}$, or equivalently sections of $\mathcal{E}^\vee$. That is, given a derived scheme T over X, the T-points of $\mathbf{V}(\mathcal{E})$ over X are morphisms $\mathcal{E}|_T \to \mathcal{O}_T$ in $\mathrm{Perf}(T)$. This agrees with Grothendieck's convention for vector bundles.

The derived stack $\mathbf{V}(\mathcal{E})$ is Artin, in fact affine if $\mathcal{E}$ is connective and n-Artin if $\mathcal{E}$ is $(-n)$-connective for $n > 0$. It is also lhfp and quasi-compact quasi-separated.

Definition 2.1. For a fixed base X, the assignment $\mathcal{E} \mapsto \mathbf{V}(\mathcal{E})$ determines a fully faithful contravariant functor from $\mathrm{Perf}(X)$ to the ∞-category of derived stacks over X with $\mathbf{G}_m$-action. The stable ∞-category $\mathrm{DVect}(X)$ of *derived vector bundles* over X is its essential image.

Example 2.2. Let X be an lhfp derived Artin stack. The *n-shifted tangent* and *n-shifted cotangent bundles* of X are the derived vector bundles

$$T_X[n] \coloneqq \mathbf{V}(\mathbf{L}_X[-n]), \quad T^*_X[n] \coloneqq \mathbf{V}(\mathbf{L}^\vee_X[-n]),$$

respectively. Similarly, given an lhfp morphism $f : X \to Y$, the relative n-shifted tangent and cotangent bundles are the derived vector bundles

$$T_{X/Y}[n] \coloneqq \mathbf{V}(\mathbf{L}_{X/Y}[-n]), \quad T^*_{X/Y}[n] \coloneqq \mathbf{V}(\mathbf{L}^\vee_{X/Y}[-n]),$$

over X. The *n-shifted normal* and *n-shifted conormal bundles* are

$$\begin{aligned}N_{X/Y}[n] &\coloneqq T_{X/Y}[n+1] \coloneqq \mathbf{V}(\mathbf{L}_{X/Y}[-n-1]), \\ N^*_{X/Y}[n] &\coloneqq T^*_{X/Y}[n-1] \coloneqq \mathbf{V}(\mathbf{L}^\vee_{X/Y}[-n+1]),\end{aligned}$$

respectively.

To avoid confusion with the dual convention using the assignment $\mathcal{E} \mapsto \mathbf{V}(\mathcal{E}^\vee)$, we will work with derived vector bundles directly and avoid referring to the corresponding perfect complexes.

Definition 2.3. We say $E = \mathbf{V}(\mathcal{E}) \in \mathrm{DVect}(X)$ is of *amplitude* $\leqslant n$, resp. $\geqslant n$, if $\mathcal{E}$ is of Tor-amplitude $\geqslant -n$, resp. $\leqslant -n$ (using homological grading). Similarly, E is of amplitude $[a,b]$, for integers $a \leqslant b$, if $\mathcal{E}$ is of Tor-amplitude $[-b,-a]$.

Notation 2.4. Given a derived vector bundle E over a derived Artin stack X, we denote by $\pi_E: E \to X$ the projection and $0_E: X \to E$ the zero section. The morphism π_E is affine if and only if it is representable, if and only if 0_E is a closed immersion, if and only if E is of amplitude $\leqslant 0$. The morphism π_E is smooth if and only if E is of amplitude $\geqslant 0$, if and only if $\pi_{E^\vee}$ is affine.

Corollary 2.5. *For every derived Artin stack X and every derived vector bundle E over X, the natural transformations*

$$\pi_{E,*} \xrightarrow{\text{unit}} \pi_{E,*}0_{E,*}0_E^* \simeq 0_E^*$$
$$0_E^! \simeq \pi_{E,!}0_{E,!}0_E^! \xrightarrow{\text{counit}} \pi_{E,!}$$

are invertible on monodromic complexes. In particular, the functors π_E^* and $0_{E,!}$ are fully faithful on monodromic complexes.

Proof. The main assertion is a special case of the contraction lemma (Proposition 1.18). For the second part, note that for every monodromic $\mathcal{F} \in \mathbf{D}(X)$ the composite

$$\mathcal{F} \xrightarrow{\text{unit}_{\pi_E}} \pi_{E,*}\pi_E^*(\mathcal{F}) \xrightarrow{\text{unit}_{0_E}} 0_E^*\pi_E^*(\mathcal{F}) \simeq \mathcal{F}$$

is identity and the second arrow is invertible by the first claim. This shows that unit: $\mathrm{id} \to \pi_{E,*}\pi_E^*$ is invertible on monodromic complexes. Similarly, the unit $\mathrm{id} \to 0_E^! 0_{E,!}$ is identified on monodromic complexes with the tautological isomorphism $\mathrm{id} \simeq \pi_{E,!}0_{E,!}$. □

2.2. **The Fourier–Sato transform.** Let E be a derived vector bundle over an lhfp derived Artin stack X. We let $\pi_E: E \to X$ denote the projection and

$$\mathrm{ev}_E: E \times_X E^\vee \to \mathbf{A}^1, \quad \mathrm{pr}_1: E \times_X E^\vee \to E, \quad \mathrm{pr}_2: E \times_X E^\vee \to E^\vee$$

the pairing function, and respective projections. We consider the closed half-space $\mathbf{A}^1_{\leqslant 0} \coloneqq \{z \in \mathbf{A}^1 \mid \mathfrak{R}(z) \leqslant 0\}$ and set $\mathcal{P}_\phi \coloneqq \mathrm{ev}_E^* \mathbf{Q}_{\mathbf{A}^1_{\leqslant 0}}$.

Definition 2.6. The *Fourier–Sato transform* is the functor $\mathrm{FS}_E: \mathbf{D}_{\mathrm{mon}}(E) \to \mathbf{D}(E^\vee)$ defined by the formula

$$\mathrm{FS}_E(\mathcal{F}) = \mathrm{pr}_{2,!}(\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathcal{P}_\phi)$$

for a monodromic complex $\mathcal{F} \in \mathbf{D}_{\mathrm{mon}}(E)$.

Theorem 2.7. *Let X be a derived Artin stack and $E \in \mathrm{DVect}(X)$. We have:*

- (FS1) Monodromicity: *For every monodromic complex $\mathcal{F} \in \mathbf{D}_{\mathrm{mon}}(E)$, the complex $\mathrm{FS}_E(\mathcal{F})$ is monodromic. In particular, FS_E determines a functor $\mathrm{FS}_E: \mathbf{D}_{\mathrm{mon}}(E) \to \mathbf{D}_{\mathrm{mon}}(E^\vee)$.*
- (FS2) Involutivity: *For every monodromic complex $\mathcal{F} \in \mathbf{D}(E)$, there is a canonical natural isomorphism*

$$\mathrm{invol}_E: \mathrm{FS}_{E^\vee}\mathrm{FS}_E(\mathcal{F}) \simeq a_E^*(\mathcal{F})[-2r]$$

where $r = \mathrm{rk}(E)$ and $a_E: E \to E$ is the antipodal map.

(FS3) Base change: *For every morphism $f\colon X' \to X$, FS_E commutes with the four operations f^*, f_*, $f_!$, and $f^!$. More precisely, there are canonical isomorphisms*

$$f^*_{E^\vee} \circ \mathrm{FS}_E \simeq \mathrm{FS}_{E'} \circ f^*_E, \tag{2.8}$$

$$f_{E^\vee,*} \circ \mathrm{FS}_{E'} \simeq \mathrm{FS}_E \circ f_{E,*}, \tag{2.9}$$

$$f_{E^\vee,!} \circ \mathrm{FS}_{E'} \simeq \mathrm{FS}_E \circ f_{E,!} \tag{2.10}$$

$$f^!_{E^\vee} \circ \mathrm{FS}_E \simeq \mathrm{FS}_{E'} \circ f^!_E \tag{2.11}$$

where $f_E\colon E' \to E$ and $f_{E^\vee}\colon E'^\vee \to E^\vee$ are the base changes of f.

(FS4) Functoriality: *For every morphism of derived vector bundles $\phi\colon E' \to E$ over X, there are canonical isomorphisms*

$$\mathrm{Ex}^{*,\mathrm{FS}}: \quad \phi^{\vee,*} \circ \mathrm{FS}_{E'} \to \quad \mathrm{FS}_E \circ \phi_! \tag{2.12}$$

$$\mathrm{Ex}^{\mathrm{FS},!}: \quad \mathrm{FS}_{E'} \circ \phi^! \to \quad \phi^\vee_* \circ \mathrm{FS}_E \tag{2.13}$$

$$\mathrm{Ex}^{!,\mathrm{FS}}: \phi^{\vee,!} \circ \mathrm{FS}_{E'}[2r'] \to \mathrm{FS}_E \circ \phi_*[2r] \tag{2.14}$$

$$\mathrm{Ex}^{\mathrm{FS},*}: \mathrm{FS}_{E'} \circ \phi^*[2r'] \to \phi^\vee_! \circ \mathrm{FS}_E[2r] \tag{2.15}$$

where $r = \mathrm{rk}(E)$ and $r' = \mathrm{rk}(E')$.

(FS5) Constructibility: *The functor $\mathrm{FS}_E\colon \mathbf{D}_{\mathrm{mon}}(E) \to \mathbf{D}_{\mathrm{mon}}(E^\vee)$ is constructible.*

(FS6) Perversity: *The functor $\mathrm{FS}_E[r]\colon \mathbf{D}_{\mathrm{mon}}(E) \to \mathbf{D}_{\mathrm{mon}}(E^\vee)$ is perverse t-exact where $r = \mathrm{rk}(E)$; in particular, it preserves perverse sheaves.*

(FS7) Duality: *For every derived vector bundle E over X and every monodromic constructible complex $\mathcal{F} \in \mathbf{D}(E)$, there is a canonical natural isomorphism*

$$\mathrm{FS}_E(\mathbb{D}\mathcal{F}) \to \mathbb{D}(\mathrm{FS}_E(\mathcal{F}))[-2r]$$

where $r = \mathrm{rk}(E)$.

Remark 2.16. See also [Kha4] for a variant of the derived Fourier–Sato transform, which generalizes Laumon's homogeneous Fourier transform.

2.3. **Proof of Theorem 2.7.** The proofs of all claims except involutivity (FS2) are either standard, or straightforward consequences of involutivity. We will only prove (FS2) here and defer the remaining proofs to [KK].

2.3.1. *Additive vs. multiplicative.* Our goal is to compute the composite $\mathrm{FS}_{E^\vee} \circ \mathrm{FS}_E\colon \mathbf{D}_{\mathrm{mon}}(E) \to \mathbf{D}_{\mathrm{mon}}(E)$. We first note that $\mathrm{FS}_{E^\vee} \circ \mathrm{FS}_E$ can be described as an integral transform with respect to a "multiplicative" kernel $\mathcal{P}''_{\mathrm{mult}} \in \mathbf{D}(E \times_X E^\vee \times_X E)$. Indeed, define

$$\mathcal{P}''_{\mathrm{mult}} \coloneqq \mathrm{Ev}^*(\mathbf{Q}_{\mathbf{A}^1_{\leqslant 0} \times \mathbf{A}^1_{\leqslant 0}})$$

where

$$\mathrm{Ev}\colon E \times_X E^\vee \times_X E \to \mathbf{A}^1 \times \mathbf{A}^1$$

is the morphism $\mathrm{Ev} = (\mathrm{ev}_E \circ \mathrm{pr}_{12}, \mathrm{ev}_{E^\vee} \circ \mathrm{pr}_{23})$. Then we have

$$\mathcal{P}''_{\mathrm{mult}} \simeq \mathrm{pr}^*_{12}\mathcal{P}_\phi \otimes \mathrm{pr}^*_{23}\mathcal{P}'_\phi$$

where $\mathcal{P}'_\phi \coloneqq \mathrm{ev}^*_{E^\vee}\mathbf{Q}_{\mathbf{A}^1_{\leqslant 0}}$ and pr_{ij} denotes the projection from $E \times_X \times E^\vee \times_X E$ to the i-th and j-th components. By the proper base change theorem, the integral transform with respect to the right-hand side is precisely the composite $\mathrm{FS}_{E^\vee} \circ \mathrm{FS}_E$. That is,

$$\mathrm{FS}_{E^\vee} \circ \mathrm{FS}_E(\mathcal{F}) \simeq \mathrm{FS}''_{\mathrm{mult}}(\mathcal{F}) \coloneqq \mathrm{pr}_{3,!}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{P}''_{\mathrm{mult}}).$$

Now considering instead the subspace $\mathbf{A}^2_{\Delta^-\leqslant 0} \coloneqq \{(z,w) \in \mathbf{A}^2 \mid \mathfrak{R}(z+w) \leqslant 0\}$ we have also the "additive" kernel

$$\mathcal{P}''_{\mathrm{add}} \coloneqq \mathrm{Ev}^*(\mathbf{Q}_{\mathbf{A}^2_{\Delta^-\leqslant 0}}).$$

We denote by $\mathrm{FS}''_{\mathrm{add}} \colon \mathbf{D}_{\mathrm{mon}}(E) \to \mathbf{D}_{\mathrm{mon}}(E)$ the associated integral transform

$$\mathcal{F} \mapsto \mathrm{pr}_{3,!}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{P}''_{\mathrm{add}}).$$

We have by restriction a natural morphism of kernels $\mathcal{P}''_{\mathrm{add}} \to \mathcal{P}''_{\mathrm{mult}}$, whence a natural transformation of integral transforms

$$\sigma \colon \mathrm{FS}''_{\mathrm{add}} \to \mathrm{FS}''_{\mathrm{mult}} \simeq \mathrm{FS}_{E^\vee} \circ \mathrm{FS}_E.$$

Lemma 2.17. *The natural transformation σ is invertible.*

Proof. Fix a monodromic complex $\mathcal{F} \in \mathbf{D}_{\mathrm{mon}}(E)$ and a point $v \in E$. It is enough to prove the following restriction map is invertible:

$$R\Gamma_c(\mathrm{pr}_3^{-1}(v), (\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{P}''_{\mathrm{add}})|_{\mathrm{pr}_3^{-1}(v)}) \to R\Gamma_c(\mathrm{pr}_3^{-1}(v), (\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{P}''_{\mathrm{mult}})|_{\mathrm{pr}_3^{-1}(v)})$$

By considering the projection to $\mathbf{A}^2$ by Ev, it suffices to check invertibility of the following map:

$$R\Gamma_c(\mathbf{A}^2, \mathcal{G}_v \otimes \mathbf{Q}_{\mathbf{A}^2_{\Delta^-\leqslant 0}}) \to R\Gamma_c(\mathbf{A}^2, \mathcal{G}_v \otimes \mathbf{Q}_{\mathbf{A}^1_{\leqslant 0} \times \mathbf{A}^1_{\leqslant 0}}).$$

where $\mathcal{G}_v \coloneqq \mathrm{Ev}_!(\mathrm{pr}_1^*\mathcal{F} \otimes \mathbf{Q}_{\mathrm{pr}_3^{-1}(v)})$. By further pushing along $\pi = (\mathfrak{R},\mathfrak{R}) \colon \mathbf{A}^2 \to \mathbf{R}^2$, we reduce to checking invertibility of the map

$$R\Gamma_c(\mathbf{R}^2, \mathcal{G}_v^{\mathfrak{R}} \otimes \mathbf{Q}_{\mathbf{R}^2_{\Delta^-\leqslant 0}}) \to R\Gamma_c(\mathbf{R}^2, \mathcal{G}_v^{\mathfrak{R}} \otimes \mathbf{Q}_{\mathbf{R}_{\leqslant 0} \times \mathbf{R}_{\leqslant 0}})$$

where we set $\mathcal{G}_v^{\mathfrak{R}} \coloneqq \pi_!\mathcal{G}_v$ and $\mathbf{R}^2_{\Delta^-\leqslant 0} \coloneqq \{(a,b) \in \mathbf{R}^2 \mid a+b \leqslant 0\}$. By construction, $\mathcal{G}_v^{\mathfrak{R}}$ is $\mathbf{R}^+$-equivariant with respect to the $\mathbf{R}^+$-action on the first coordinate. Hence the claim follows from Lemma 2.18 below. □

Lemma 2.18. *For any $\mathbf{R}^+$-equivariant complex $\mathcal{H}$ on $\mathbf{R}^2$, the natural map*

$$\mathrm{pr}_{2,!}(\mathcal{H} \otimes \mathbf{Q}_{\mathbf{R}^2_{\Delta^-\leqslant 0}}) \to \mathrm{pr}_{2,!}(\mathcal{H} \otimes \mathbf{Q}_{\mathbf{R}_{\leqslant 0} \times \mathbf{R}_{\leqslant 0}})$$

is invertible.

Proof. It will suffice to show that the map in question is an isomorphism on stalks at all points $a \in \mathbf{R}$.

Assume first that $a > 0$. Then it is enough to prove the vanishing

$$R\Gamma_c(\mathbf{R}_{\leqslant -a}, \mathcal{H}|_{\{a\} \times \mathbf{R}_{\leqslant -a}}) = 0. \tag{2.19}$$

Since $\mathcal{H}|_{\{a\} \times \mathbf{R}_{<0}}$ is $\mathbf{R}^+$-equivariant, there exists an object $M_a \in \mathbf{D}(\mathrm{pt})$ such that we have an equivalence $\mathcal{H}|_{\{a\} \times \mathbf{R}_{<0}} \simeq (\{a\} \times \mathbf{R}_{<0} \to \mathrm{pt})^* M_a$. Therefore the vanishing (2.19) follows from

$$R\Gamma_c(\mathbf{R}_{\leqslant -a}, \mathbf{Q}_{\mathbf{R}_{\leqslant -a}}) = 0.$$

Assume now that $a \leqslant 0$. Then it is enough to prove that the following restriction map is an equivalence

$$R\Gamma_c(\mathbf{R}_{\leqslant -a}, \mathcal{H}|_{\{a\}\times \mathbf{R}_{\leqslant -a}}) \to R\Gamma_c(\mathbf{R}_{\leqslant 0}, \mathcal{H}|_{\{a\}\times \mathbf{R}_{\leqslant 0}}),$$

which is equivalent to the vanishing

$$R\Gamma_c((0,-a], \mathcal{H}|_{\{a\}\times(0,-a]}) = 0.$$

Similarly to the case $a < 0$, this follows from the vanishing

$$R\Gamma_c((0,-a], \mathbf{Q}_{(0,-a]}) = 0.$$

We conclude. □

2.3.2. *Fourier–Sato of the constant sheaf.* Set $\mathscr{L}^E \coloneqq (0_E)^! \mathrm{FS}_{E^\vee}(\mathbf{Q}_{E^\vee})$.

Lemma 2.20. *There exists a natural isomorphism*

$$\alpha_E \colon \mathscr{L}^E \simeq \mathbf{Q}_X[-2\operatorname{rk} E].$$

Proof. When E has a global resolution $E^\bullet$ by a complex of vector bundles, we can construct a natural isomorphism $\eta_{E^\bullet} \colon (0_E)^! \mathrm{FS}_{E^\vee}(\mathbf{Q}_{E^\vee}) \simeq \mathbf{Q}_X[-2\operatorname{rk} E]$ by reducing to the case of classical vector bundles; see the proof of [FYZ, Lemma A.14]. One can show that

$$\alpha_{E^\bullet} \coloneqq (-1)^{\binom{\operatorname{rk} E}{2} + \binom{\operatorname{rk} E^0}{2} + \sum_{i<0} \operatorname{rk} E^i} \cdot \eta_{E^\bullet}$$

does not depend on the choice of the global resolution, using [Kin2, Prop. 2.3]; details will be given in [KK]. In general, we may choose a smooth affine cover of the base over which E admits a resolution, and glue the above isomorphisms (the gluing can be reduced to the heart of the t-structure, so only requires checking a cocycle condition for these isomorphisms). □

Lemma 2.21. *The following map is an isomorphism:*

$$(0_E)_! \mathscr{L}^E = (0_E)_!(0_E)^! \mathrm{FS}_{E^\vee}(\mathbf{Q}_{E^\vee}) \to \mathrm{FS}_{E^\vee}(\mathbf{Q}_{E^\vee}).$$

Proof. The proof is identical to the proof of [FYZ, Lemma A.12] or [Kha4, Prop. 1.29], so we omit the details here. □

2.3.3. *Conclusion of proof of* (FS2). By Lemma 2.17 and Lemma 2.20, it is enough to show the equivalence

$$\mathrm{FS}''_{\mathrm{add}} \simeq a_E^*(- \otimes \mathrm{pr}_E^* \mathscr{L}^E). \tag{2.22}$$

Consider the following cartesian diagram

$$\begin{array}{ccc} E \times_X E^\vee \times_X E & \xrightarrow{\widetilde{\mathrm{add}}_E} & E \times_X E^\vee \\ \Big\downarrow{\scriptstyle \widetilde{\mathrm{pr}}_{13}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_E} \\ E \times E & \xrightarrow{\mathrm{add}_E} & E \end{array}$$

where $\widetilde{\mathrm{pr}}_{13}$ denotes the projection to the first and third components, add_E denotes the addition map, and $\widetilde{\mathrm{add}}_E \coloneqq (\mathrm{pr}_1 + \mathrm{pr}_3, \mathrm{pr}_2)$. Note that we haved a natural isomorphism

$$\mathcal{P}''_{\mathrm{add}} \simeq \widetilde{\mathrm{add}}_E^{\,*} \mathcal{P}_\phi.$$

Therefore for $\mathcal{F} \in \mathbf{D}_{\mathrm{mon}}(E)$, we have a natural isomorphisms

$$\begin{aligned}\mathrm{FS}''_{\mathrm{add}}(\mathcal{F}) &\simeq \widetilde{\mathrm{pr}}_{3,!}(\widetilde{\mathrm{pr}}_1^*\mathcal{F} \otimes \widetilde{\mathrm{add}_E}^*\mathcal{P}_\phi) \\ &\simeq \mathrm{pr}_{2,!}\widetilde{\mathrm{pr}}_{13,!}(\widetilde{\mathrm{pr}}_{13}^*\mathrm{pr}_1^*\mathcal{F} \otimes \widetilde{\mathrm{add}_E}^*\mathcal{P}_\phi) \\ &\simeq \mathrm{pr}_{2,!}(\mathrm{pr}_1^*\mathcal{F} \otimes \widetilde{\mathrm{pr}}_{13,!}\widetilde{\mathrm{add}_E}^*\mathcal{P}_\phi) \\ &\simeq \mathrm{pr}_{2,!}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{add}_E^*\mathrm{pr}_{E,!}\mathcal{P}_\phi).\end{aligned}$$

Here $\widetilde{\mathrm{pr}}_i$ denotes the i-th projection from $E \times_X E^\vee \times_X E$ and pr_i denotes the i-th projection from $E \times_X E$. Now we claim an isomorphism

$$\mathrm{add}_E^*\mathrm{pr}_{E,!}\mathcal{P}_\phi \simeq \Delta_!^-\pi_E^*\mathscr{L}^E \tag{2.23}$$

where we set $\Delta^- := (\mathrm{id}_E, -\mathrm{id}_E)\colon E \to E \times_X E$ and $\pi_E\colon E \to X$ denotes the projection. Assuming this, the isomorphism (2.22) follows from the following isomorphisms

$$\begin{aligned}\mathrm{FS}''_{\mathrm{add}}(\mathcal{F}) &\simeq \mathrm{pr}_{2,!}(\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{add}_E^*\mathrm{pr}_{E,!}\mathcal{P}_\phi) \\ &\simeq \mathrm{pr}_{2,!}(\mathrm{pr}_1^*\mathcal{F} \otimes \Delta_!^-\mathrm{pr}_E^*\mathscr{L}^E) \\ &\simeq \mathrm{pr}_{2,!}\Delta_!^-(\mathcal{F} \otimes \mathrm{pr}_E^*\mathscr{L}^E) \simeq a_E^*(\mathcal{F} \otimes \mathrm{pr}_E^*\mathscr{L}^E).\end{aligned}$$

Now we prove the isomorphism (2.23). Consider the following cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{\Delta^-} & E \times_X E \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \mathrm{add}_E} \\ X & \xrightarrow{0_E} & E. \end{array}$$

Using the proper base change theorem, the isomorphism (2.23) is reduced to proving

$$\mathrm{pr}_{E,!}\mathcal{P}_\phi \simeq 0_{E,!}\mathscr{L}^E.$$

which follows from Lemma 2.21 and Lemma 2.20.

2.4. **An adjunction identity.** The involutivity property (FS2) shows that the functor $a_E^*\mathrm{FS}_{E^\vee}[2\,\mathrm{rk}\,E]$ provides a canonical inverse to FS_E; in particular, there is an adjunction $(\mathrm{FS}_E, a_E^*\mathrm{FS}_{E^\vee}[2r])$. However, the respective involutivity isomorphisms

$$\begin{aligned}\mathrm{invol}_E\colon \mathrm{FS}_{E^\vee}\mathrm{FS}_E(-) &\simeq a_E^*(-)[-2\,\mathrm{rk}\,E], \\ \mathrm{invol}_{E^\vee}\colon \mathrm{FS}_E\mathrm{FS}_{E^\vee}(-) &\simeq a_{E^\vee}^*(-)[-2\,\mathrm{rk}\,E]\end{aligned}$$

do not define a unit and counit for this adjunction on the nose, but only up to a sign. Indeed, we have the following triangle identity:

Proposition 2.24. *The following diagram commutes up to the sign* $(-1)^{\mathrm{rk}\,E}$*:*

$$\begin{array}{ccc} a_{E^\vee}^*\mathrm{FS}_E[-2\,\mathrm{rk}\,E] & \xrightarrow{\mathrm{FS}_E(\mathrm{invol}_E^{-1})} & \mathrm{FS}_E \circ \mathrm{FS}_{E^\vee} \circ \mathrm{FS}_E \\ & \searrow{=} & \downarrow{\scriptstyle \mathrm{invol}_{E^\vee}\mathrm{FS}_E} \\ & & a_{E^\vee}^*\mathrm{FS}_E[-2\,\mathrm{rk}\,E]. \end{array}$$

In particular, the tuple

$$(\mathrm{FS}_E, a_E^*\mathrm{FS}_{E^\vee}[2\operatorname{rk} E], a_E^*\mathrm{invol}_E^{-1}[2\operatorname{rk} E], (-1)^{\operatorname{rk} E} a_{E^\vee}^* \mathrm{invol}_{E^\vee})$$

defines an adjoint equivalence.

Proof. We define a map $\widetilde{\mathrm{Ev}}\colon E \times E^\vee \times E \times E^\vee \to \mathbf{A}^3$ by

$$\widetilde{\mathrm{Ev}}(v_0, w_0, v_1, w_1) \mapsto (\langle v_0, w_0\rangle, \langle v_1, w_0\rangle, \langle v_1, w_1\rangle).$$

The functor $\mathrm{FS}_E \circ \mathrm{FS}_{E^\vee} \circ \mathrm{FS}_E$ is the integral transform with respect to the kernel $\mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* \mathbf{Q}^{\boxtimes 3}_{\mathbf{A}^1_{\leqslant 0}}$. We claim the existence of an isomorphism

$$\mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* \mathbf{Q}^{\boxtimes 3}_{\mathbf{A}^1_{\leqslant 0}} \simeq (a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi \otimes \pi^*_{E \times_X E^\vee} \mathscr{L}^{E^\vee}. \tag{2.25}$$

To prove this, we first note that there exists an isomorphism:

$$\mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* \mathbf{Q}^{\boxtimes 3}_{\mathbf{A}^1_{\leqslant 0}} \simeq \mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* (\mathbf{Q}_{\mathbf{A}^2_{\Delta^- \leqslant 0}} \boxtimes \mathbf{Q}_{\mathbf{A}^1_{\leqslant 0}})$$

This can be proved in the same manner as Lemma 2.17. Now by arguing as the proof of (FS2), we obtain an isomorphism in $\mathbf{D}(E \times_X E \times_X E^\vee)$

$$\mathrm{pr}_{134,!}\widetilde{\mathrm{Ev}}^* (\mathbf{Q}_{\mathbf{A}^2_{\Delta^- \leqslant 0}} \boxtimes \mathbf{Q}_{\mathbf{A}^1_{\leqslant 0}}) \simeq \mathrm{pr}_{12}^*(\Delta^-_! \pi_E^* \mathscr{L}^E) \otimes \mathrm{pr}_{23}^* \mathcal{P}_\phi.$$

Therefore we obtain the isomorphism (2.25).

We can also construct a natural isomorphism

$$\mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* \mathbf{Q}^{\boxtimes 3}_{\mathbf{A}^1_{\leqslant 0}} \simeq (a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi \otimes \pi^*_{E \times_X E^\vee} \mathscr{L}^{E^\vee}. \tag{2.26}$$

by using the isomorphism

$$\mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* \mathbf{Q}^{\boxtimes 3}_{\mathbf{A}^1_{\leqslant 0}} \simeq \mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* (\mathbf{Q}_{\mathbf{A}^1_{\leqslant 0}} \boxtimes \mathbf{Q}_{\mathbf{A}^2_{\Delta^- \leqslant 0}}).$$

By construction, the isomorphism $\mathrm{FS}_{E^\vee}(\mathrm{invol}_E^{-1})$ corresponds to the isomorphism (2.25) and $\mathrm{invol}_E \mathrm{FS}_E$ corresponds to the isomorphism (2.26). Therefore it is enough to show that the automorphism of $(a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi$ constructed by (2.25), (2.26) and Lemma 2.20, is multiplication by $(-1)^{\operatorname{rk} E}$. Let $q\colon E \times_X E^\vee \to X$ be the projection. Then we have an isomorphism

$$q_* \underline{\mathrm{Hom}}((a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi, (a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi) \simeq \mathbf{Q}_X$$

which can be checked for example by taking a local resolution. In particular, any endomorphism of the object $(a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi$ is scalar multiplication by some locally constant function. Therefore we can reduce to the case where X is a point.

We first treat the case when E is a classical vector bundle over the point. We need to show that the following composite is multiplication by $(-1)^{\operatorname{rk} E}$:

$$\begin{aligned}(a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi[-2\operatorname{rk} E] &\simeq (a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi \otimes \pi^*_{E \times_X E^\vee} \mathscr{L}^E \\ &\simeq \mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^* \mathbf{Q}^{\boxtimes 3}_{\mathbf{A}^1_{\leqslant 0}} \\ &\simeq (a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi \otimes \pi^*_{E \times_X E^\vee} \mathscr{L}^{E^\vee} \\ &\simeq (a_E \times \mathrm{id}_{E^\vee})^* \mathcal{P}_\phi[-2\operatorname{rk} E].\end{aligned}$$

As we have already seen that this map is given by a scalar multiplication, it is enough to show that the map induced on the stalk at $(0,0)$ is multiplication by $(-1)^{\mathrm{rk}\,E}$. Note that we have an isomorphism

$$\mathrm{pr}_{14,!}\widetilde{\mathrm{Ev}}^*(\mathbf{Q}^{\boxtimes 3}_{\mathbf{A}^1_{\leqslant 0}})_{(0,0)} \simeq R\Gamma_c(P_\phi).$$

where $P_\phi \coloneqq \{\{w,v\} \in E^\vee \times E \mid \langle v,w\rangle \leqslant 0\}$. The inclusion $i_0\colon E^\vee \times \{0\} \hookrightarrow P_\phi$ induces the first isomorphism on the stalk and the inclusion $i_1\colon \{0\} \times E \hookrightarrow P_\phi$ induces the latter. We take a trivialization $E \simeq \mathbf{A}^r$ and define maps $F', F''\colon E \times [0,1] \hookrightarrow P_\phi$ by

$$F'\colon (z_1,\dots,z_r,t) \mapsto (z_1,\dots,z_r,-t\bar{z}_1,\dots,-t\bar{z}_r)$$
$$F''\colon (z_1,\dots,z_r,t) \mapsto ((1-t)z_1,\dots,(1-t)z_r,-\bar{z}_1,\dots,-\bar{z}_r).$$

These maps define a proper homotopy between i_0 and $-\bar{i}_1$. Therefore the trivializations $R\Gamma_c(P_\phi) \simeq \mathbf{Q}[r]$ defined by i_0 and i_1 differ by $(-1)^r$ where $r = \mathrm{rk}\,E$.

The general case can be reduced to the case of a classical vector bundle since we are working over a point. Note that we need a sign modification as in the proof of Lemma 2.20. □

2.5. **Forgetting supports vs. Gysin.** Let $\phi\colon E_1 \to E_2$ be a morphism between derived vector bundles. Assume that its fibre is of amplitude $\leqslant 0$. This implies that ϕ is separated and its dual $\phi^\vee\colon E_2^\vee \to E_1^\vee$ is quasi-smooth. Thus we have the natural transformations

$$\mathrm{fsupp}_\phi\colon \phi_! \to \phi_*, \quad \mathrm{gys}_{\phi^\vee}\colon \phi^{\vee,*}[-2d] \to \phi^{\vee,!},$$

see (1.7) and (1.8), where $d = \mathrm{rk}(E_1^\vee) - \mathrm{rk}(E_2^\vee) = \mathrm{rk}(E_1) - \mathrm{rk}(E_2)$.

For $\mathcal{F}_1 \in \mathbf{D}_{\mathrm{mon}}(E_1)$ we can consider the following diagram:

$$\begin{array}{ccc} \mathrm{FS}_{E_2}(\phi_!\mathcal{F}_1) & \xrightarrow{\mathrm{FS}_{E_2}(\mathrm{fsupp}_\phi)} & \mathrm{FS}_{E_2}(\phi_*\mathcal{F}_1) \\ {\scriptstyle (2.12)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle (2.14)}\downarrow{\scriptstyle \simeq} \\ \phi^{\vee,*}\mathrm{FS}_{E_1^\vee}(\mathcal{F}_1) & \xrightarrow{\mathrm{gys}_{\phi^\vee}} & \phi^{\vee,!}\mathrm{FS}_{E_1^\vee}(\mathcal{F})[2d] \end{array}$$

which we expect to commute. We have the following partial results:

Proposition 2.27. *If ϕ is a closed immersion, the above diagram commutes.*

Proof. This is established in the ℓ-adic setting in [FYZ, Prop. 6.8]. The same argument works in our setting. □

Proposition 2.28. *Let $\pi_E\colon E \to X$ be the projection of a derived vector bundle of amplitude $\leqslant 0$. For every $\mathcal{F} \in \mathbf{D}_{\mathrm{mon}}(E)$, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{FS}_E(\pi_{E,!}\mathcal{F}) & \xrightarrow{\mathrm{FS}_E(\mathrm{fsupp}_\pi)} & \mathrm{FS}_E(\pi_{E,*}\mathcal{F}) \\ {\scriptstyle (2.12)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle (2.14)}\downarrow{\scriptstyle \simeq} \\ 0^*_{E^\vee}\mathcal{F} & \xrightarrow{\mathrm{gys}_{0_{E^\vee}}} & 0^!_{E^\vee}\mathcal{F}[2\,\mathrm{rk}\,E]. \end{array}$$

This can be proven using Proposition 2.27. Details will be given in [KK].

3. Specialization and microlocalization

3.1. **Co/normal bundles.**

3.1.1. Let $f: X \to Y$ be an lhfp morphism of derived Artin stacks. The *normal bundle* $N_{X/Y} \coloneqq T_{X/Y}[1]$ is the 1-shifted tangent bundle, and the *conormal bundle* $N^*_{X/Y} \coloneqq T^*_{X/Y}[-1]$ is the (-1)-shifted cotangent bundle. We denote by

$$\tau_{X/Y}: N_{X/Y} \to X, \qquad \pi_{X/Y}: N^*_{X/Y} \to X$$

the projections. We denote both zero sections by $0_{X/Y}: X \to N_{X/Y}$, $0_{X/Y}: X \to N^*_{X/Y}$.

3.1.2. *Functoriality.* Given a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

where f and f' are lhfp, we denote by $Nq: N_{X'/Y'} \to N_{X/Y}$ the composite

$$Nq: N_{X'/Y'} \xrightarrow{dq} N_{X/Y} \underset{X}{\times} X' \xrightarrow{q_\tau} N_{X/Y} \tag{3.1}$$

where dq is the canonical morphism of derived vector bundles over X' with fibre $N_{X'/X \times_Y Y'}$, and q_τ is the base change of p.

On conormals we have the correspondence

$$N^*_{X'/Y'} \xleftarrow{dq^\vee} N^*_{X/Y} \underset{X}{\times} X' \xrightarrow{q_\pi} N^*_{X/Y} \tag{3.2}$$

where $dq^\vee$ is the canonical morphism of derived vector bundles over X' with cofibre $N^*_{X'/X \times_Y Y'}$, and q_π is the base change of p.

3.1.3. *Normal deformation.* According to [HKR] there is a $\mathbf{G}_m$-equivariant deformation diagram:

$$\begin{array}{ccccc} X & \xrightarrow{0} & X \times \mathbf{A}^1 & \longleftarrow & X \times \mathbf{G}_m \\ \downarrow{\scriptstyle 0_{X/Y}} & & \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f} \\ N_{X/Y} & \xrightarrow{i_D} & D_{X/Y} & \xleftarrow{j_D} & Y \times \mathbf{G}_m \\ \downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \mathrm{pt} & \xrightarrow{0} & \mathbf{A}^1 & \longleftarrow & \mathbf{G}_m \end{array} \tag{3.3}$$

where the vertical composites are the obvious projections, and each square is homotopy cartesian. By definition, $D_{X/Y}$ is the derived Weil restriction of $X \times \{0\} \to Y \times \{0\}$ along $Y \times \{0\} \to Y \times \mathbf{A}^1$, or equivalently the derived mapping stack

$$D_{X/Y} = \underline{\mathrm{Map}}_{Y \times \mathbf{A}^1}(Y \times \{0\}, X \times \mathbf{A}^1).$$

This is Artin, specifically $(n+1)$-Artin if X and Y are n-Artin, by the main result of [HKR][8]. Its T-points for a scheme T over Y are commutative squares

$$\begin{array}{ccc} D & \hookrightarrow & T \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $D \hookrightarrow T$ is a virtual Cartier divisor in the sense of [KR].

Given a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array} \tag{3.4}$$

where f and f' are lhfp, we have a $\mathbf{G}_m$-equivariant commutative diagram

$$\begin{array}{ccccc} N_{X'/Y'} & \xrightarrow{i'_D} & D_{X'/Y'} & \xleftarrow{j'_D} & Y' \times \mathbf{G}_m \\ \downarrow{\scriptstyle Nq} & & \downarrow{\scriptstyle Dq} & & \downarrow{\scriptstyle q\times \mathrm{id}} \\ N_{X/Y} & \xrightarrow{i_D} & D_{X/Y} & \xleftarrow{j_D} & Y \times \mathbf{G}_m \\ \downarrow & & \downarrow{\scriptstyle t} & & \downarrow \\ \mathrm{pt} & \xrightarrow{0} & \mathbf{A}^1 & \longleftarrow & \mathbf{G}_m \end{array} \tag{3.5}$$

which factorizes the lower rectangle (3.3) for the morphism $f' : X' \to Y'$. We recall the following fact from [HKR]:

Proposition 3.6.

(i) *Suppose q and Nq are lhfp of Tor-amplitude $\leqslant n$. Then $Dq : D_{X'/Y'} \to D_{X/Y}$ has the same property. In particular, if q and Nq are smooth (resp. quasi-smooth), then so is Dq.*

(ii) *Suppose q is proper and the square (3.4) is excessive: it is cartesian on classical truncations and the morphism $N_{X'/Y'} \to N_{X/Y} \times_X X'$ is a closed immersion. Then $Dq : D_{X'/Y'} \to D_{X/Y}$ is proper.*

3.2. **Specialization.**

Definition 3.7. Let $f : X \to Y$ be an lhfp morphism of derived Artin stacks. The functor of *specialization* along $f : X \to Y$ is defined by

$$\mathrm{sp}_{X/Y} = \psi_t \circ j_{D,!} \circ \mathrm{pr}_1^* : \mathbf{D}(Y) \to \mathbf{D}(N_{X/Y}),$$

using the normal deformation (3.3), where $\mathrm{pr}_1 : Y \times \mathbf{G}_m \to Y$ is the projection.

Note that we could have taken $j_{D,*}$ in the definition, as the canonical morphism $\psi_t \circ j_{D,!} \to \psi_t \circ j_{D,*}$ is invertible. Note also that $\mathrm{sp}_{X/Y}$ preserves constructible objects, since all functors involved in its definition do.

Theorem 3.8. *Let $f : X \to Y$ be an lhfp morphism of derived Artin stacks. Then we have:*

[8] If X and Y are 1-Artin and Y has affine diagonal, we can alternatively appeal to [HP, Thm. 5.1.1].

(SP0) Identity: *For* $f = \mathrm{id}_X$*, there is a canonical isomorphism* $\mathrm{sp}_{X/X} \simeq \mathrm{id}$.

(SP1) Monodromicity: *For every* $\mathcal{F} \in \mathbf{D}(Y)$*, the complex* $\mathrm{sp}_{X/Y}(\mathcal{F})$ *is monodromic. In other words,* $\mathrm{sp}_{X/Y}$ *determines a functor* $\mathbf{D}(Y) \to \mathbf{D}_{\mathrm{mon}}(N_{X/Y})$.

(SP2) Proper base change: *For any commutative square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where f *and* f' *are lhfp, there are canonical natural transformations*

$$\mathrm{Ex}_{\mathrm{sp},*} \colon \mathrm{sp}_{X/Y} \circ q_* \to Nq_* \circ \mathrm{sp}_{X'/Y'}, \tag{3.9}$$

$$\mathrm{Ex}_{!,\mathrm{sp}} \colon Nq_! \circ \mathrm{sp}_{X'/Y'} \to \mathrm{sp}_{X/Y} \circ q_!. \tag{3.10}$$

If q *is proper and the square is excessive (Proposition 3.6), then both* $\mathrm{Ex}_{\mathrm{sp},*}$ *and* $\mathrm{Ex}_{!,\mathrm{sp}}$ *are invertible.*

(SP3) Smooth base change: *For any commutative square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where f *and* f' *are lhfp, there is a canonical natural transformation*

$$\mathrm{Ex}^{*,\mathrm{sp}} \colon Nq^* \circ \mathrm{sp}_{X/Y} \to \mathrm{sp}_{X'/Y'} \circ q^*, \tag{3.11}$$

$$\mathrm{Ex}^{\mathrm{sp},!} \colon \mathrm{sp}_{X'/Y'} \circ q^! \to Nq^! \circ \mathrm{sp}_{X/Y}. \tag{3.12}$$

If q *and* Nq *are smooth, then both* $\mathrm{Ex}^{*,\mathrm{sp}}$ *and* $\mathrm{Ex}^{\mathrm{sp},!}$ *are invertible.*

(SP4) Perversity: *The functor* $\mathrm{sp}_{X/Y}$ *is perverse t-exact; in particular, it preserves perverse sheaves.*

(SP5) Duality: *For every constructible* $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(Y)$*, there is a canonical isomorphism*

$$\mathrm{sp}_{X/Y}(\mathbb{D}\mathcal{F}) \to \mathbb{D}(\mathrm{sp}_{X/Y}(\mathcal{F})).$$

(SP6) Restriction to zero: *Consider the canonical morphisms*

$$\mathrm{Ex}^{*,\mathrm{sp}} \colon 0^*_{X/Y} \circ \mathrm{sp}_{X/Y} \to \mathrm{sp}_{X/X} \circ f^* \simeq f^*, \tag{3.13}$$

$$\mathrm{Ex}^{\mathrm{sp},!} \colon f^! \simeq \mathrm{sp}_{X/X} \circ f^! \to 0^!_{X/Y} \circ \mathrm{sp}_{X/Y}. \tag{3.14}$$

Then (3.13) is invertible and (3.14) is invertible on constructible complexes.

Remark 3.15. The derived specialization functor exists in the context of any topological weave [Kha2] with a formalism of nearby/vanishing cycles. Proofs of the above properties in that context will appear in forthcoming work of the first-named author.

3.2.1. *Identity* (SP0). For $f = \mathrm{id}_X\colon X \to X$, the vertical arrows in the upper half of the diagram (3.3) are invertible. Thus we have

$$\mathrm{sp}_{X/X} \coloneqq \psi_{\mathrm{pr}_2} \circ j_! \circ \mathrm{pr}_1^* \simeq \mathrm{id}$$

by (NC8), where $j\colon X \times \mathbf{G}_m \hookrightarrow X \times \mathbf{A}^1$ is the inclusion, and $\mathrm{pr}_1\colon X \times \mathbf{G}_m \to X$ and $\mathrm{pr}_2\colon X \times \mathbf{A}^1 \to \mathbf{A}^1$ are the projections.

3.2.2. *Monodromicity* (SP1). Since $t\colon D_{X/Y} \to \mathbf{A}^1$ is $\mathbf{G}_m$-equivariant, this follows from (NC1).

3.2.3. *Proper base change* (SP2). We consider the diagram (3.5) and write $t' = t \circ Dq$ for the canonical $\mathbf{G}_m$-equivariant function on $D_{X'/Y'}$.

The natural transformation $\mathrm{Ex}_{\mathrm{sp},*}\colon \mathrm{sp}_{X/Y} \circ q_* \to Nq_* \circ \mathrm{sp}_{X'/Y'}$ (3.9) is induced by $\mathrm{Ex}_{\psi,*}\colon \psi_t Dq_* \to Nq_* \psi_{t'}$ as follows:

$$\psi_t j_{D,*} \mathrm{pr}_1^* q_* \simeq \psi_t j_{D,*} (q \times \mathrm{id})_* \mathrm{pr}_1^* \simeq \psi_t Dq_* j'_{D,*} \mathrm{pr}_1^* \xrightarrow{\mathrm{Ex}_{\psi,*}} Nq_* \psi_{t'} j'_{D,*} \mathrm{pr}_1^*$$

Similarly, $\mathrm{Ex}_{!,\mathrm{sp}}\colon Nq_! \circ \mathrm{sp}_{X'/Y'} \to \mathrm{sp}_{X/Y} \circ q_!$ (3.10) is induced by $\mathrm{Ex}_{!,\psi}\colon Nq_! \psi_{t,!} \to \psi_t Dq_!$ as follows:

$$Nq_! \circ \psi_{t'} j'_{D,!} \mathrm{pr}_1^* \xrightarrow{\mathrm{Ex}_{!,\psi}} \psi_t Dq_! j'_{D,!} \mathrm{pr}_1^* \simeq \psi_t j_{D,!} (q \times \mathrm{id})_! \mathrm{pr}_1^* \simeq \psi_t j_{D,!} \mathrm{pr}_1^* q_!.$$

For the second claim, the assumptions imply that $Dq\colon D_{X'/Y'} \to D_{X/Y}$ and $Nq\colon N_{X'/Y'} \to N_{X/Y}$ are proper (Proposition 3.6), so that $\mathrm{Ex}_{\psi,*}$ and $\mathrm{Ex}_{!,\psi}$ are both invertible.

3.2.4. *Smooth base change* (SP3). We consider the diagram (3.5) and write $t' = t \circ Dq$ for the canonical $\mathbf{G}_m$-equivariant function on $D_{X'/Y'}$.

The natural transformation $\mathrm{Ex}^{*,\mathrm{sp}}\colon Nq^* \circ \mathrm{sp}_{X/Y} \to \mathrm{sp}_{X'/Y'} \circ q^*$ (3.11) is induced by $\mathrm{Ex}_{*,\psi}\colon Nq^* \psi_t \to \psi_{t'} Dq^*$ as follows:

$$Nq^* \psi_t j_{D,!} \mathrm{pr}_1^* \xrightarrow{\mathrm{Ex}^{*,\psi}} \psi_{t'} Dq^* j_{D,!} \mathrm{pr}_1^* \simeq \psi_{t'} j'_{D,!} (q \times \mathrm{id})^* \mathrm{pr}_1^* \simeq \psi_{t'} j'_{D,!} \mathrm{pr}_1^* q^*.$$

Similarly, $\mathrm{Ex}^{\mathrm{sp},!}\colon \mathrm{sp}_{X'/Y'} \circ q^! \to Nq^! \circ \mathrm{sp}_{X/Y}$ (3.12) is induced by $\mathrm{Ex}_{\psi,!}\colon \psi_{t'} Dq^! \to Nq^! \psi_t$ as follows:

$$\psi_{t'} j'_{D,*} \mathrm{pr}_1^* q^! \simeq \psi_{t'} j'_{D,*} (q \times \mathrm{id})^! \mathrm{pr}_1^* \simeq \psi_{t'} Dq^! j_{D,*} \mathrm{pr}_1^* \xrightarrow{\mathrm{Ex}^{\psi,!}} Nq^! \psi_t j_{D,*} \mathrm{pr}_1^*.$$

The second claim follows from Proposition 3.6 because $\mathrm{Ex}^{*,\psi}$ and $\mathrm{Ex}^{\psi,!}$ are both invertible when Nq and Dq are smooth.

3.2.5. *Perversity* (SP4). Since the functor $j_{D,!}$ is right perverse t-exact and ψ_t is perverse t-exact by (NC6), we conclude that $\mathrm{sp}_{X/Y} = \psi_t \circ j_{D,!}$ is right perverse t-exact. Similarly, the formula $\mathrm{sp}_{X/Y} \simeq \psi_t \circ j_{D,*}$ implies the left t-exactness of $\mathrm{sp}_{X/Y}$.

3.2.6. *Duality* (SP5). We have the following isomorphisms:

$$\begin{aligned}\psi_t \circ j_{D,!} \circ \mathrm{pr}_1^* \circ \mathbb{D} &\to \psi_t \circ j_{D,!} \circ \mathbb{D} \circ \mathrm{pr}_1^! \\ &\to \psi_t \circ \mathbb{D} \circ j_{D,*} \circ \mathrm{pr}_1^! \\ &\to \mathbb{D} \circ \psi_t \circ j_{D,*} \circ \mathrm{pr}_1^![-2] \\ &\simeq \mathbb{D} \circ \psi_t \circ j_{D,*} \circ \mathrm{pr}_1^* \\ &\simeq \mathbb{D} \circ \psi_t \circ j_{D,!} \circ \mathrm{pr}_1^*.\end{aligned}$$

using the canonical isomorphisms $\psi \circ \mathbb{D} \to [1] \circ \mathbb{D} \circ \psi[-1] \simeq \mathbb{D} \circ \psi[-2]$, $\psi \circ j_{D,!} \simeq \psi \circ j_{D,*}$, and $\mathrm{pr}_1^*[2] \to \mathrm{pr}_1^!$.

3.2.7. *Restriction to zero* (SP6). The morphisms in question are the exchange transformations $\mathrm{Ex}^{*,\mathrm{sp}}$ and $\mathrm{Ex}^{\mathrm{sp},!}$ (see (SP3)) associated with the square

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow f \\ X & \xrightarrow{f} & Y. \end{array}$$

We first consider $\mathrm{Ex}^{*,\mathrm{sp}}$ (3.13). Note that the claim is local on X and Y. Indeed, suppose given a commutative square

$$\begin{array}{ccc} U & \xrightarrow{f_0} & V \\ \downarrow u & & \downarrow v \\ X & \xrightarrow{f} & Y \end{array} \tag{3.16}$$

where u and v are smooth. The induced map $N_{U/V} \to N_{X/Y}$ is the composite

$$N_{U/V} \to u^* N_{X/Y} \to N_{X/Y} \tag{3.17}$$

where the second arrow is a base change of u, hence is smooth. The first arrow is also smooth, as a torsor under $N_{U/X \times_Y V}$, which is a vector bundle stack since $U \to X \times_Y V$ is quasi-smooth (as both source and target are smooth over X). Thus the smooth base change formula (SP3) implies that the natural transformation $u^*\mathrm{Ex}^{*,\mathrm{sp}}\colon u^* 0^*_{X/Y} \mathrm{sp}_{X/Y} \to u^* f^*$ is identified with

$$\mathrm{Ex}^{*,\mathrm{sp}} v^*\colon 0^*_{U/V} \mathrm{sp}_{U/V} v^* \to f_0^* v^*.$$

Note that it is always possible to choose a square as in (3.16) where f_0 is a closed immersion:

Lemma 3.18. *Let $f\colon X \to Y$ be an lhfp morphism of derived Artin stacks. Then there exists a family of commutative squares*

$$\begin{array}{ccc} U_\alpha & \xrightarrow{f_\alpha} & V_\alpha \\ \downarrow u_\alpha & & \downarrow v_\alpha \\ X & \xrightarrow{f} & Y \end{array}$$

with $(u_\alpha)_\alpha$ and $(v_\alpha)_\alpha$ jointly surjective families of smooth morphisms and each f_α an lhfp closed immersion of affine derived schemes.

Proof. Choose a jointly surjective family $(V'_\beta \to Y)_\beta$ of smooth morphisms with V'_β affine. Replacing Y by V'_β (and X by $X \times_Y V'_\beta$) we may assume that Y is affine. Choose a jointly surjective family $(U_\alpha \to X)_\alpha$ with U_α affine. Each $U_\alpha \to X \to Y$ is lhfp, hence induces a finitely presented morphism of affine schemes on classical truncations. Choose a surjection $\pi_0\mathcal{O}_Y[T_1,\ldots,T_n] \twoheadrightarrow \pi_0\mathcal{O}_{U_\alpha}$. Lifting the images of T_i to points of $\mathcal{O}_{U_\alpha}$, we get a closed immersion $U_\alpha \hookrightarrow V_\alpha \coloneqq \mathbf{A}^{n_\alpha}_Y$ over Y. □

We are thus reduced to the case where $i \coloneqq f\colon X \hookrightarrow Y$ is a closed immersion. Using the localization triangle we may assume that $\mathcal{F} \simeq i_*i^*(\mathcal{F})$ or $\mathcal{F} \simeq j_!j^*(\mathcal{F})$ where $j\colon X \smallsetminus Y \hookrightarrow X$. For the first case, note that $\mathrm{Ex}^{*,\mathrm{sp}} i_*\colon 0^*\mathrm{sp}_{X/Y} i_* \to i^*i_*$ is identified by proper base change (SP2) with the isomorphism $0^*_{X/Y} 0_{X/Y,*} \to \mathrm{id}$.

Suppose $\mathcal{F} \simeq j_!j^*(\mathcal{F})$. Consider the derived blow-up square

$$\begin{array}{ccc} E & \xrightarrow{i_E} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{i} & Y \end{array} \tag{3.19}$$

defined as in [Hek] (see also [HKR]): the square is excessive in the sense of Proposition 3.6, i_E is a virtual Cartier divisor, q is proper and induces an isomorphism $Y' \smallsetminus E \to Y \smallsetminus X$. Up to the latter isomorphism we have $j_! \simeq q_! j_{E,!}$ where $j_E\colon Y' \smallsetminus E \hookrightarrow Y'$, so $\mathcal{F}$ is in the essential image of $q_!$. By proper base change (SP2) we may further replace i by i_E and assume i is a virtual Cartier divisor. Localizing further on Y, we may moreover assume that it is globally cut out as the derived zero locus of a function $s\colon Y \to \mathbf{A}^1$.

Since Y_{cl} is locally of finite presentation, we may assume $Y \simeq \mathrm{Spec}(A)$ where $\pi_0(A) = k[x_1,\ldots,x_n]/(f_1,\ldots,f_m)$. Let $a_1,\ldots,a_n \in A$ be points lifting $x_1,\ldots,x_n$ in $\pi_0(A)$. Together with $s \in A$, these determine a closed immersion $Y \hookrightarrow \mathbf{A}^n \times \mathbf{A}^1$ such that the square

$$\begin{array}{ccc} X & \xhookrightarrow{f} & Y \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle (x,s)} \\ \mathbf{A}^n & \xhookrightarrow{(\mathrm{id},0)} & \mathbf{A}^n \times \mathbf{A}^1 \end{array} \tag{3.20}$$

is homotopy cartesian. By proper base change we are thus reduced to the case of the inclusion $(\mathrm{id},0)\colon \mathbf{A}^n \hookrightarrow \mathbf{A}^n \times \mathbf{A}^1$ for any $n \geqslant 0$. Since this is a regular closed immersion between schemes, the derived specialization functor in this case agrees with the classical one from [Ver], hence the claim now follows from the property (SP5) proven there.

We have shown that (3.13) is invertible. By Verdier duality (SP5) we deduce that (3.14) is invertible on constructible objects.

3.3. **Microlocalization.**

Definition 3.21. Let $f\colon X \to Y$ be an lhfp morphism of derived Artin stacks. The functor of *microlocalization* along $f\colon X \to Y$ is defined by

$$\mu_{X/Y} = \mathrm{FS}_{N_{X/Y}} \circ \mathrm{sp}_{X/Y}\colon \mathbf{D}(Y) \to \mathbf{D}(N^*_{X/Y}),$$

i.e., the Fourier-Sato transform of the specialization.

Theorem 3.22. *Let $f\colon X \to Y$ be an lhfp morphism of derived Artin stacks. Then we have:*

(M1) Conicity: *For every $\mathcal{F} \in \mathbf{D}(Y)$, the complex $\mu_{X/Y}(\mathcal{F})$ is monodromic. In other words, $\mu_{X/Y}$ determines a functor $\mathbf{D}(Y) \to \mathbf{D}_{\mathrm{mon}}(N^*_{X/Y})$.*

(M2) Proper base change: *For any commutative square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where f and f' are lhfp of relative virtual dimension d and d', respectively, there are canonical natural transformations

$$\mathrm{Ex}_{\mu,*}: \quad \mu_{X/Y} \circ q_*[-2d] \to q_{\pi,*} \circ dq^{\vee,!} \circ \mu_{X'/Y'}[-2d'] \tag{3.23}$$

$$\mathrm{Ex}_{!,\mu}: q_{\pi,!} \circ dq^{\vee,*} \circ \mu_{X'/Y'} \to \mu_{X/Y} \circ q_! \tag{3.24}$$

where $dq^\vee\colon N^*_{X/Y} \times_X X' \to N^*_{X'/Y'}$ and $q_\pi\colon N^*_{X/Y} \times_X X' \to N^*_{X/Y}$ are as in Subsect. 3.1. If q is proper and the square is excessive (Proposition 3.6), then $\mathrm{Ex}_{\mu,*}$ is invertible.

(M3) Smooth base change: *For any commutative square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where f and f' are lhfp of relative virtual dimension d and d', respectively, there are canonical natural transformations

$$\mathrm{Ex}^{*,\mu}\colon dq^\vee_! \circ q^*_\pi \circ \mu_{X/Y}[-2d] \to \mu_{X'/Y'} \circ q^*[-2d'], \tag{3.25}$$

$$\mathrm{Ex}^{\mu,!}\colon \quad \mu_{X'/Y'} \circ q^! \to dq^\vee_* \circ q^!_\pi \circ \mu_{X/Y}. \tag{3.26}$$

If q and Nq are smooth, then $\mathrm{Ex}^{*,\mu}$ and $\mathrm{Ex}^{\mu,!}$ are both invertible.

(M4) Perversity: *The functor $\mu_{X/Y}[-d]$ is perverse t-exact; in particular, it preserves perverse sheaves. Here d denotes the relative virtual dimension of f.*

(M5) Duality: *For every constructible complex $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(Y)$, there is a canonical natural isomorphism*

$$\mu_{X/Y}(\mathbb{D}\mathcal{F}) \to \mathbb{D}(\mu_{X/Y}(\mathcal{F}))[2d].$$

(M6) Restriction to zero: *Consider the canonical morphisms*

$$\mathrm{Ex}^{*,\mu}\colon \pi_{X/Y,!} \circ \mu_{X/Y}[-2d] \to f^*, \tag{3.27}$$

$$\mathrm{Ex}^{\mu,!}\colon f^! \to \pi_{X/Y,*} \circ \mu_{X/Y}. \tag{3.28}$$

The map (3.27) is invertible and (3.28) is invertible on constructible complexes. In particular, there is a canonical isomorphism[9]

$$0^!_{X/Y} \circ \mu_{X/Y}[-2d] \to f^* \tag{3.29}$$

[9] by Corollary 2.5

and a canonical morphism

$$f^! \to 0^*_{X/Y} \circ \mu_{X/Y} \tag{3.30}$$

which is invertible on constructible complexes.

3.3.1. *Monodromicity (M1).* Combine (SP1) and (FS1).

3.3.2. *Proper base change (M2).* This follows by combining (SP2) with (FS3) and (FS4).

The natural transformation $\mathrm{Ex}_{!,\mu}\colon q_{\pi,!} \circ dq^{\vee,*} \circ \mu_{X'/Y'} \to \mu_{X/Y} \circ q_!$ (3.24) is defined, up to the identifications

$$\begin{aligned} q_{\pi,!} \circ dq^{\vee,*} \circ \mathrm{FS}_{N_{X'/Y'}} &\simeq \mathrm{FS}_{N_{X/Y}} \circ q_{\tau,!} \circ dq_! \\ &\simeq \mathrm{FS}_{N_{X/Y}} \circ Nq_! \end{aligned}$$

coming from $\mathrm{Ex}^{*,\mathrm{FS}}$ (2.12) and (2.10), as the exchange transformation

$$\mathrm{Ex}_{!,\mathrm{sp}}\colon \mathrm{FS}_{N_{X/Y}} \circ Nq_! \circ \mathrm{sp}_{X'/Y'} \to \mathrm{FS}_{N_{X/Y}} \circ \mathrm{sp}_{X/Y} \circ q_!$$

of (3.10).

Similarly, $\mathrm{Ex}_{\mu,*}\colon \mu_{X/Y} \circ q_*[-2d] \to q_{\pi,*} \circ dq^{\vee,!}\mu_{X'/Y'}[-2d']$ (3.23) is the exchange transformation

$$\mathrm{Ex}_{\mathrm{sp},*}\colon \mathrm{FS}_{N_{X/Y}} \circ \mathrm{sp}_{X/Y} \circ q_*[-2d] \to \mathrm{FS}_{N_{X/Y}} \circ Nq_* \circ \mathrm{sp}_{X'/Y'}[-2d]$$

up to the identifications

$$\begin{aligned} \mathrm{FS}_{N_{X/Y}} \circ Nq_*[-2d] &\simeq \mathrm{FS}_{N_{X/Y}} \circ q_{\tau,*} \circ dq_*[-2d] \\ &\simeq q_{\pi,*} \circ \mathrm{FS}_{N_{X/Y} \times_X X'} \circ dq_*[-2d] \\ &\simeq q_{\pi,*} \circ dq^{\vee,!} \circ \mathrm{FS}_{N_{X'/Y'}}[-2d']. \end{aligned}$$

coming from (2.9) and $\mathrm{Ex}^{!,\mathrm{FS}}$ (2.14).

3.3.3. *Smooth base change (M3).* This follows by combining (SP3) with (FS3) and (FS4).

The natural transformation $\mathrm{Ex}^{*,\mu}\colon dq^{\vee}_! \circ q^*_\pi \circ \mu_{X/Y}[-2d] \to \mu_{X'/Y'} \circ q^*[-2d']$ (3.25) is defined, up to the identifications

$$\begin{aligned} dq^{\vee}_! \circ q^*_\pi \circ \mathrm{FS}_{N_{X/Y}}[-2d] &\simeq dq^{\vee}_! \circ \mathrm{FS}_{N_{X/Y} \times_X X'} \circ q^*_\tau[-2d] \\ &\simeq \mathrm{FS}_{N_{X/Y}} \circ dq^* \circ q^*_\tau[-2d'] \\ &\simeq \mathrm{FS}_{N_{X/Y}} \circ Nq^*[-2d'] \end{aligned}$$

coming from $\mathrm{Ex}^{\mathrm{FS},*}$ (2.15) and (2.8), as the exchange transformation

$$\mathrm{Ex}^{*,\mathrm{sp}}\colon \mathrm{FS}_{N_{X'/Y'}} \circ Nq^* \circ \mathrm{sp}_{X/Y}[-2d'] \to \mathrm{FS}_{N_{X'/Y'}} \circ \mathrm{sp}_{X'/Y'} \circ q^*[-2d']$$

of (3.11).

Similarly, $\mathrm{Ex}^{\mu,!}\colon \mu_{X'/Y'} \circ q^! \to dq^{\vee}_* \circ q^!_\pi \circ \mu_{X/Y}$ (3.26) is the exchange transformation

$$\mathrm{Ex}^{\mathrm{sp},!}\colon \mathrm{FS}_{N_{X'/Y'}} \circ \mathrm{sp}_{X'/Y'} \circ q^! \to \mathrm{FS}_{N_{X'/Y'}} \circ Nq^! \circ \mathrm{sp}_{X/Y}$$

up to the identifications

$$\begin{aligned}\mathrm{FS}_{N_{X'/Y'}} \circ Nq^! &\simeq \mathrm{FS}_{N_{X'/Y'}} \circ dq^! \circ q_\tau^! \\ &\simeq dq_*^\vee \circ \mathrm{FS}_{N_{X/Y} \times_X X'} \circ q_\tau^! \\ &\simeq dq_*^\vee \circ q_\pi^! \circ \mathrm{FS}_{N_{X/Y} \times_X X'}\end{aligned}$$

coming from (2.11) and $\mathrm{Ex}^{\mathrm{FS},!}$ (2.14).

3.3.4. *Perversity (M4).* Combine (SP4) and (FS6).

3.3.5. *Duality (M5).* Combine (SP5) and (FS7).

3.3.6. *Restriction to zero (M6).* The morphisms are induced by $\mathrm{Ex}^{*,\mu}$ and $\mathrm{Ex}^{\mu,!}$ of (M3) with $q = f$ and $f' = \mathrm{id}_X$. The claim follows by combining (SP6) and (FS4).

3.4. **Virtual fundamental classes via the specialization functor.** Let $f: X \to Y$ be a quasi-smooth morphism between derived Artin stacks. Recall that the Gysin transformation (SO4) gives rise to a relative virtual fundamental class (1.10)

$$[f]^{\mathrm{vir}}: \mathbf{Q}_X[2\,\mathrm{vdim}\, f] \to f^!(\mathbf{Q}_Y),$$

which recovers the virtual fundamental class $[X]^{\mathrm{vir}}$ in the absolute case $Y = \mathrm{pt}$.

We describe an alternative construction of (1.10) in terms of the specialization sheaf $\mathrm{sp}_{X/Y}(\mathbf{Q}_Y) \in \mathbf{D}_{\mathrm{mon}}(N_{X/Y})$. Consider the canonical natural transformation

$$\mathrm{Ex}_\otimes^{*!}: 0_{X/Y}^*(-) \otimes 0_{X/Y}^!(\mathbf{Q}_Y) \to 0_{X/Y}^!(-)$$

adjoint to the projection formula (SO2). We have a canonical isomorphism

$$0_{X/Y}^! \mathbf{Q}_{N_{X/Y}} \simeq 0_{X/Y}^! \tau_{X/Y}^! \mathbf{Q}_X[2\,\mathrm{vdim}\, f] \simeq \mathbf{Q}_X[2\,\mathrm{vdim}\, f]$$

by Poincaré duality for $\tau_{X/Y}: N_{X/Y} \to X$. Indeed, $N_{X/Y}$ is of amplitude $\geqslant 0$, hence smooth over X, since f is quasi-smooth. The following will be proven in [KK]:

Theorem 3.31. *The following diagram commutes:*

$$\begin{array}{ccc} 0_{X/Y}^* \mathrm{sp}_{X/Y}(\mathbf{Q}_Y) \otimes 0_{X/Y}^! \mathbf{Q}_{N_{X/Y}} & \xrightarrow{\mathrm{Ex}_\otimes^{*!}} & 0_{X/Y}^! \mathrm{sp}_{X/Y}(\mathbf{Q}_Y) \\ \simeq \downarrow (3.13) & & \simeq \downarrow (3.14) \\ \mathbf{Q}_X[2\,\mathrm{vdim}\, f] & \xrightarrow{[f]^{\mathrm{vir}}} & f^! \mathbf{Q}_Y. \end{array}$$

4. Applications

In this section, we discuss some applications of the results in the previous sections. Among other things, we prove a conjecture of Joyce [JS, Conj. 1.1] in the shifted conormal case and use it to construct a 3-dimensional refinement of the CoHA product.

4.1. **DT perverse sheaves via microlocalization.** Given a derived 1-Artin stack X with a (-1)-shifted symplectic structure [PTVV] and an orientation in the sense of [BBBJ, Def. 3.6], Joyce and collaborators have constructed the *DT perverse sheaf*

$$\phi_X \in \mathrm{Perv}(X),$$

whose study forms the subject of cohomological Donaldson–Thomas theory. See [BBBJ, Cor. 4.9], as well as [Kin3] for a survey. Informally speaking, ϕ_X is constructed by gluing vanishing cycle complexes defined on Darboux charts.

We study the DT perverse sheaf in the following situation. For a derived Artin stack Y, the conormal bundle $N^*_{Y/\mathrm{pt}} = T^*_Y[-1]$ admits a canonical (-1)-shifted symplectic structure (see [Cal, Thm. 2.2]). Moreover, from the exact triangle

$$\pi^*\mathbf{L}_Y \to \mathbf{L}_{N^*_{Y/\mathrm{pt}}} \to \mathbf{L}_{N^*_{Y/\mathrm{pt}}/Y} \simeq \pi^*\mathbf{T}_Y[1]$$

where $\pi \coloneqq \pi_{Y/\mathrm{pt}}\colon N^*_{Y/\mathrm{pt}} \to Y$ is the projection, we deduce a natural isomorphism

$$\pi^* \det(\mathbf{L}_Y)^{\otimes 2} \simeq \det(\mathbf{L}_{N^*_{Y/\mathrm{pt}}}), \tag{4.1}$$

which gives an orientation for $N^*_{Y/\mathrm{pt}}$. When Y is quasi-smooth and 1-Artin, $N^*_{Y/\mathrm{pt}}$ is of amplitude $\leqslant 0$, hence affine over Y and in particular also 1-Artin. Thus we have the DT perverse sheaf

$$\phi_{N^*_{Y/\mathrm{pt}}} \in \mathrm{Perv}(N^*_{Y/\mathrm{pt}}).$$

It admits the following microlocal description:

Theorem 4.2. *Let Y be a quasi-smooth derived 1-Artin stack. Then there exists a natural isomorphism*

$$\phi_{N^*_{Y/\mathrm{pt}}} \simeq \mu_{Y/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[-\mathrm{vdim}\, Y].$$

Sketch of Proof. Assume first that Y is a quasi-smooth derived scheme with a global embedding $i\colon Y \hookrightarrow U$ into a smooth scheme U. Since U is smooth, the canonical morphism $\gamma_2\colon N^*_{Y/\mathrm{pt}} \to N^*_{Y/U}$ is a closed immersion. Then the smooth base change theorem for the microlocalization (M3) yields an isomorphism

$$(\gamma_2)_*\mu_{Y/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \simeq \mu_{Y/U}(\mathbf{Q}_U[\dim U]).$$

In [Sch1, Thm. 6.1] it is shown that there exists an isomorphism[10]

$$(\gamma_2)_*\phi_{N^*_{Y/\mathrm{pt}}} \simeq \mu_{Y/U}(\mathbf{Q}_U[\dim U - \mathrm{vdim}\, Y]).$$

Combining these isomorphisms and restricting along γ_2 yields the claimed isomorphism in this case.

For a general quasi-smooth derived Artin stack Y, we may adapt the arguments of [Kin1, §5] to construct the desired isomorphism of perverse sheaves by gluing the above isomorphism. We refer to [KK] for the details. □

[10] In the case of quasi-smooth closed immersions, such as $Y \hookrightarrow U$, the derived microlocalization functor was defined independently by K. Schefers in [Sch1, Def. 4.19].

As a consequence of Theorem 4.2 and the isomorphisms (3.27) and (3.28) of (M6), we recover the dimensional reduction theorem in cohomological Donaldson–Thomas theory (see [Kin1, Thm. 4.14]):

Corollary 4.3 (Dimensional reduction)**.** *There are canonical isomorphisms*

$$\rho\colon \pi_*(\phi_{N^*_{Y/\mathrm{pt}}}) \simeq \omega_Y[-\operatorname{vdim} Y],$$
$$\mathbb{D}\rho\colon \pi_!(\phi_{N^*_{Y/\mathrm{pt}}}) \simeq \mathbf{Q}_Y[\operatorname{vdim} Y]$$

in $\mathbf{D}(Y)$*.*

Remark 4.4. The morphism ρ (resp. $\mathbb{D}\rho$) differs from the one constructed in [Kin1, Thm. 4.14] by the sign $(-1)^{\binom{\operatorname{vdim} Y}{2}+1}$ (resp. -1). See [Kin2, Prop. 4.5] and the paragraph after (2.17) in [Kin1].

Combining Theorem 4.2 with Proposition 2.28 and Theorem 3.31, we obtain the following new construction of the virtual fundamental class (1.10), as conjectured in [Kin1, Conj. 5.4]:

Corollary 4.5. *The following diagram commutes:*

$$\begin{array}{ccc} \pi_!(\phi_{N^*_{Y/\mathrm{pt}}}) & \longrightarrow & \pi_*(\phi_{N^*_{Y/\mathrm{pt}}}) \\ \simeq\big\downarrow \mathbb{D}\rho & & \simeq\big\downarrow \rho \\ \mathbf{Q}_Y[\operatorname{vdim} Y] & \xrightarrow{\cdot [Y]^{\mathrm{vir}}} & \omega_Y[-\operatorname{vdim} Y]. \end{array}$$

4.2. **Quasi-smooth correspondences and 2d CoHAs.**

4.2.1. *Correspondences.* Suppose given a correspondence of derived Artin stacks

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f_1} & & \searrow^{f_2} & \\ Y_1 & & & & Y_2 \end{array} \tag{4.6}$$

where f_1 is quasi-smooth.

Via the Gysin transformation (SO4), such a correspondence gives rise to a *cohomological correspondence* (in the sense of [SGA5, Exp. III, §3]):

$$f_1^*\omega_{Y_1}[2\operatorname{vdim} f_1] \xrightarrow{\mathrm{gys}_{f_1}} f_1^!\omega_{Y_1} = \omega_X = f_2^!\omega_{Y_2}. \tag{4.7}$$

We call this the *Gysin correspondence* associated with (4.6).

Any such cohomological correspondence, or rather its right transpose $\omega_{Y_1}[2\operatorname{vdim} f_1] \to f_{1,*}f_2^!\omega_{Y_2}$, gives rise on derived global sections to a canonical morphism

$$R\Gamma(Y_1,\omega_{Y_1})[2\operatorname{vdim} f_1] \to R\Gamma(Y_1, f_{1,*}f_2^!\omega_{Y_2}) = R\Gamma(X, f_2^!\omega_{Y_2}).$$

If f_2 is moreover proper representable, there exists a canonical map from the last term

$$R\Gamma(X, f_2^!\omega_{Y_2}) = R\Gamma(Y_2, f_{2,*}f_2^!\omega_{Y_2}) = R\Gamma(Y_2, f_{2,!}f_2^!\omega_{Y_2}) \xrightarrow{\mathrm{counit}} R\Gamma(Y_2,\omega_{Y_2}),$$

so we get the canonical morphism

$$R\Gamma(Y_1,\omega_{Y_1})[2\operatorname{vdim} f_1] \to R\Gamma(Y_2,\omega_{Y_2}). \tag{4.8}$$

4.2.2. *2d CoHA.* We recall how the cohomological Hall algebra of a surface, as constructed by [KV], may be regarded as an instance of the above constructions.

Let S be a smooth algebraic surface and M_S be the derived moduli stack of compactly supported coherent sheaves on S. For a compactly supported cohomology class $\gamma \in \mathrm{H}^*_c(S)$, we let $M_{S,\gamma} \subseteq M_S$ be the open substack consisting of compactly supported sheaves whose Chern character coincides with γ. It is a union of connected components of M_S. For cohomology classes $\gamma', \gamma'' \in \mathrm{H}^*_c(S)$, we let $M^{\mathrm{ext}}_{S,\gamma',\gamma''}$ denote the moduli stack of short exact sequences of compactly supported coherent sheaves

$$0 \to E' \to E \to E'' \to 0$$

with $\mathrm{ch}(E') = \gamma'$ and $\mathrm{ch}(E'') = \gamma''$.

We may now consider the following correspondence of derived Artin stacks:

$$\begin{array}{ccc} & M^{\mathrm{ext}}_{S,\gamma',\gamma''} & \\ {\scriptstyle (\mathrm{ev}',\mathrm{ev}'')} \swarrow & & \searrow {\scriptstyle \mathrm{ev}} \\ M_{S,\gamma'} \times M_{S,\gamma''} & & M_{S,\gamma'+\gamma''}, \end{array}$$

where ev' sends $[0 \to E' \to E \to E'' \to 0]$ to E', and $\mathrm{ev}, \mathrm{ev}''$ are defined similarly. The morphism ev is proper representable and $(\mathrm{ev}', \mathrm{ev}'')$ is quasi-smooth (see [KV, Props. 4.2.3, 4.3.2], [PS, Prop. 3.10]).

The construction (4.8) in this case yields, after taking hypercohomology, the canonical morphism

$$\begin{aligned} \mathrm{H}^{\mathrm{BM}}_*(M_{S,\gamma'}) \otimes \mathrm{H}^{\mathrm{BM}}_*(M_{S,\gamma''}) &\simeq \mathrm{H}^{\mathrm{BM}}_*(M_{S,\gamma'} \times M_{S,\gamma''}) \\ &\to \mathrm{H}^{\mathrm{BM}}_{*+2\,\mathrm{vdim}(\mathrm{ev}',\mathrm{ev}'')}(M_{S,\gamma'+\gamma''}). \end{aligned} \tag{4.9}$$

Unravelling the definitions, we see that this is given by composing the virtual pull-back $(\mathrm{ev}', \mathrm{ev}'')^!$ with the proper push-forward ev_*:

$$\begin{aligned} \mathrm{H}^{\mathrm{BM}}_*(M_{S,\gamma'} \times M_{S,\gamma''}) &\xrightarrow{(\mathrm{ev}',\mathrm{ev}'')^!} \mathrm{H}^{\mathrm{BM}}_{*+2\,\mathrm{vdim}(\mathrm{ev}',\mathrm{ev}'')}(M^{\mathrm{ext}}_{S,\gamma',\gamma''}) \\ &\xrightarrow{(\mathrm{ev})_*} \mathrm{H}^{\mathrm{BM}}_{*+2\,\mathrm{vdim}(\mathrm{ev}',\mathrm{ev}'')}(M_{S,\gamma'+\gamma''}). \end{aligned} \tag{4.10}$$

It is shown in [KV, Thm. 4.4.2] that this defines the structure of an an associative algebra on $\bigoplus_\gamma \mathrm{H}^{\mathrm{BM}}_*(M_{S,\gamma})$.

4.3. **Conormal correspondences and 3d CoHAs.**

4.3.1. *The Joyce conjecture.* Let M be an oriented (-1)-shifted symplectic derived 1-Artin stack and $\tau\colon L \to M$ an oriented Lagrangian. The following conjecture was proposed by Joyce [JS, Conj. 1.1]:

Conjecture 4.11. *There exists a canonical morphism*

$$\nu\colon \mathbf{Q}_L[\mathrm{vdim}\, L] \to \tau^! \phi_M \tag{4.12}$$

in $\mathbf{D}(L)$.

We call (4.12) the *Lagrangian cycle* morphism. To make the conjecture precise, one should require some properties of this morphism. For example, on a Darboux chart it is supposed to coincide with the construction of [AB, Prop. 5.20]. We refer to [Kin3, §5] for a survey of the Joyce conjecture and further expected properties.

For our purposes it is convenient to reformulate Conjecture 4.11 as a cohomological correspondence, analogously to (4.7).

Conjecture 4.13. *Suppose given an oriented Lagrangian correspondence*

$$\begin{array}{ccc} & L & \\ {}^{\tau_1}\swarrow & & \searrow^{\tau_2} \\ M_1 & & M_2 \end{array} \tag{4.14}$$

Then there exists a canonical morphism

$$\hat{\nu}\colon \tau_1^* \phi_{M_1}[\operatorname{vdim} L] \to \tau_2^! \phi_{M_2} \tag{4.15}$$

in $\mathbf{D}(L)$.

We call (4.15) the *Lagrangian Gysin correspondence* associated with (4.14). Its existence follows directly from Conjecture 4.11 and the Verdier self-duality of the perverse sheaf ϕ_{M_1}. Conversely, it is clear that Conjecture 4.13 implies Conjecture 4.11.

4.3.2. *Conormal correspondences.* Given a correspondence of derived Artin stacks

$$\begin{array}{ccc} & X & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ Y_1 & & Y_2, \end{array} \tag{4.16}$$

we consider the *conormal correspondence*

$$\begin{array}{ccc} & N^*_{X/Y}[-1] & \\ {}^{\tilde{f}_1}\swarrow & & \searrow^{\tilde{f}_2} \\ N^*_{Y_1/\mathrm{pt}} & & N^*_{Y_2/\mathrm{pt}}, \end{array} \tag{4.17}$$

which is a Lagrangian correspondence by [Cal, Thm. 2.8]. One can moreover show that it admits a canonical orientation, so this fits into the situation of Conjecture 4.13. (We omit the construction of the orientation, since for our purposes here it will not play any role.)

Theorem 4.18. *For every correspondence of the form (4.16), there exists a canonical morphism*

$$\tilde{f}_1^* \mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2 \operatorname{vdim} f_1] \to \tilde{f}_2^! \mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}). \tag{4.19}$$

We will prove Theorem 4.18 in (4.3.4) below. Under the identification of the DT perverse sheaf $\phi_{N^*_{X/\mathrm{pt}}}$ with the microlocalization $\mu_{X/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})$ (Theorem 4.2), it gives a candidate for the Lagrangian Gysin correspondence of Conjecture 4.13 associated with the conormal correspondence (4.17).

Corollary 4.20. *For every correspondence of the form (4.16) where Y_1, Y_2, and X are quasi-smooth 1-Artin, there exists a canonical morphism*

$$\tilde{f}_1^* \phi_{N^*_{Y_1/\mathrm{pt}}}[2\,\mathrm{vdim}\, f_1] \to \tilde{f}_2^! \phi_{N^*_{Y_2/\mathrm{pt}}}. \tag{4.21}$$

4.3.3. *3d CoHA.* We now consider the total space of the canonical bundle $X \coloneqq \mathrm{Tot}_S(\omega_S)$. Consider the following correspondence:

$$\begin{array}{ccc} & M^{\mathrm{ext}}_{X,\gamma',\gamma''} & \\ {\scriptstyle (\tilde{\mathrm{ev}}', \tilde{\mathrm{ev}}'')} \swarrow & & \searrow {\scriptstyle \tilde{\mathrm{ev}}} \\ M_{X,\gamma'} \times M_{X,\gamma''} & & M_{X,\gamma'+\gamma''}. \end{array} \tag{4.22}$$

It is shown in [BD, Corollary 6.9] that this correspondence is naturally equipped with a structure of Lagrangian correspondence. We have the following propopsition:

Proposition 4.23. *The Lagrangian correspondence (4.22) is identified with the conormal Lagrangian correspondence:*

$$\begin{array}{ccc} & N^*_{M^{\mathrm{ext}}_{S,\gamma',\gamma''}/M_{S,\gamma'} \times M_{S,\gamma''}}[-1] & \\ \swarrow & & \searrow \\ N^*_{M_{S,\gamma'}/\mathrm{pt}} \times N^*_{M_{S,\gamma''}/\mathrm{pt}} & & N^*_{M_{X,\gamma'+\gamma''}/\mathrm{pt}}. \end{array}$$

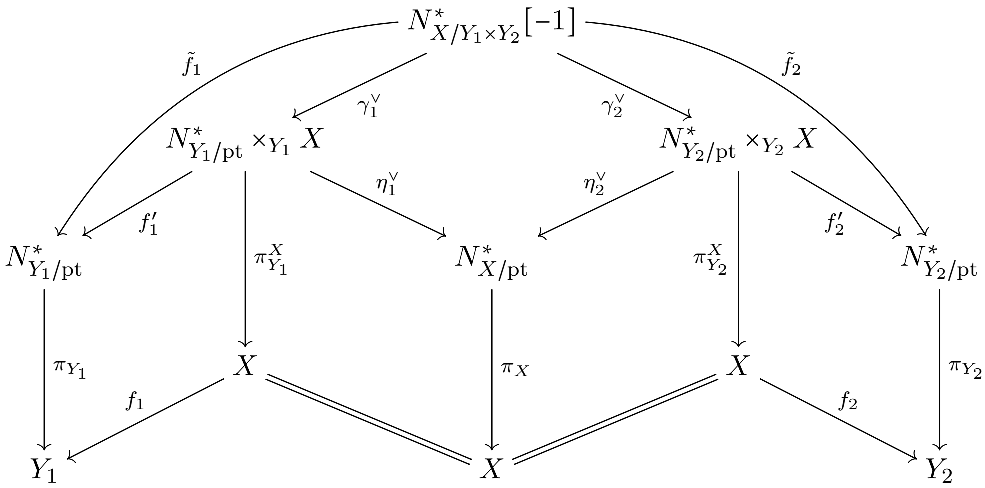
This statement can be proved using the theory of relative Calabi–Yau completion and using the work [BCS]. The detail will be given in [KK].

We let $\phi_{M_{X,\gamma}} \in \mathrm{Perv}(M_{X,\gamma})$ be the DT perverse sheaf [BBBJ, Corollary 4.9] which is identified w ith the canonical orientation (4.1) under the isomorphism $M_{X,\gamma} \simeq N^*_{M_{S,\gamma}/\mathrm{pt}}$. The Lagrangian Gysin correspondence (Corollary 4.20) gives in this case, in view of Proposition 4.23, a canonical morphism

$$((\tilde{\mathrm{ev}}')^* \phi_{M_{X,\gamma'}} \otimes (\tilde{\mathrm{ev}}'')^* \phi_{M_{X,\gamma''}})[2\,\mathrm{vdim}\, M^{\mathrm{ext}}_{S,\gamma',\gamma''}] \to \mathrm{ev}^! \phi_{M_{X,\gamma'+\gamma''}}. \tag{4.24}$$

This cohomological correspondence gives rise to the 3d CoHA product for X, see [Kin3, 5.3.3] and (4.4.2) below.

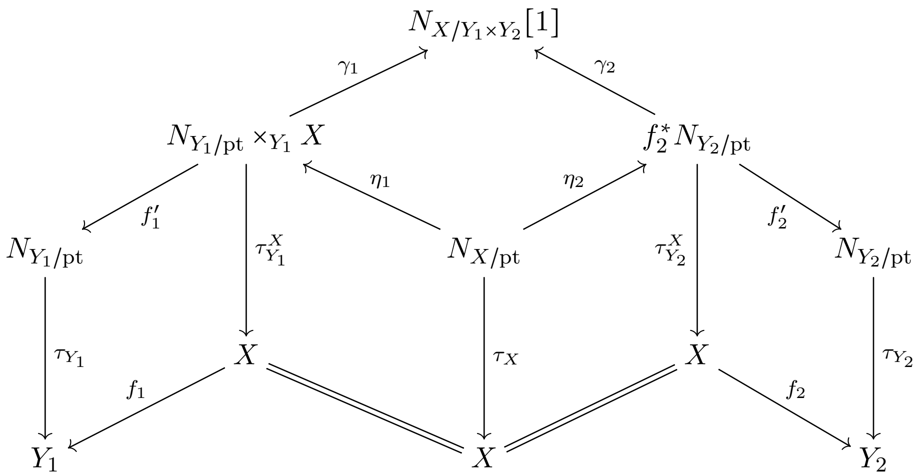
4.3.4. *Proof of Theorem 4.18.* Consider the following commutative diagram:



Note that the upper middle diamond is cartesian. Our goal is to construct a morphism of the form

$$(\gamma_1^\vee)^*(\tilde{f}_1')^*\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\, f_1] \to (\gamma_2^\vee)^!(\tilde{f}_2')^!\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}). \tag{4.25}$$

We will construct this by a Fourier–Sato transform. Dualizing the diagram above, we have a commutative diagram:



Under the Fourier–Sato transform, and using the isomorphisms (2.8) (2.11), (2.12) and (2.14), it is enough to construct a morphism

$$(\gamma_1)_!(f_1')^*\mathrm{sp}_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \to (\gamma_2)_*(f_2')^!\mathrm{sp}_{Y_2}(\mathbf{Q}_{\mathrm{pt}}) \tag{4.26}$$

Note that using the exchange properties of the specialization functor (3.11) and (3.12), we have a canonical morphism

$$(\eta_1)^*(f_1')^*\mathrm{sp}_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \to \mathrm{sp}_{X/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \to (\eta_2)^!(f_2')^!\mathrm{sp}_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}).$$

Its left transpose is a morphism

$$(\eta_2)_!(\eta_1)^*(f_1')^*\mathrm{sp}_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \to (f_2')^!\mathrm{sp}_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}).$$

By the base change formula (SO1) for the middle diamond of the above diagram, we obtain the canonical map

$$(\gamma_2)^*(\gamma_1)_!(f_1')^*\mathrm{sp}_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \to (f_2')^!\mathrm{sp}_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}).$$

By transposing $(\gamma_2)^*$ to the right, we obtain the desired map (4.26).

4.4. **2d vs. 3d CoHAs of a surface.**

4.4.1. *Comparison of Gysin correspondences.* Given a quasi-smooth correspondence (4.6), we explain the compatibility between the associated Gysin correspondence (4.7)

$$f_1^*\omega_{Y_1}[2\,\mathrm{vdim}\,f_1] \to f_2^!\omega_{Y_2}.$$

and the cohomological correspondence associated with the conormal correspondence (4.17), i.e., the morphism (4.19)

$$\tilde{f}_1^*\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\,f_1] \to \tilde{f}_2^!\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})$$

in $\mathbf{D}(N^*_{X/Y}[-1])$.

Consider the projection $\pi \coloneqq \pi_{N^*_{X/Y}[-1]}\colon N^*_{X/Y}[-1] \to X$. We claim there is a commutative diagram

$$\begin{array}{ccc} f_1^*\omega_{Y_1}[2\,\mathrm{vdim}\,f_1] & \xrightarrow{(4.7)} & f_2^!\omega_{Y_2} \\ \downarrow & & \uparrow \\ \pi_*\tilde{f}_1^*\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\,f_1] & \xrightarrow{\pi_*(4.19)} & \pi_*\tilde{f}_2^!\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \end{array} \tag{4.27}$$

where the vertical morphisms are to be defined below.

We adopt the notation from the proof of Theorem 4.18. The left-hand vertical morphism is the composite

$$\begin{aligned} f_1^*\omega_{Y_1}[2\,\mathrm{vdim}\,f_1] &\simeq f_1^*(\pi_{Y_1})_*\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\,f_1] \\ &\to (\pi^X_{Y_1})_*f_1'^*\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\,f_1] \\ &\to (\pi^X_{Y_1})_*(\gamma_1^\vee)_*(\gamma_1^\vee)^*f_1'^*\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\,f_1] \\ &\simeq \pi_*\tilde{f}_1^*\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\,f_1], \end{aligned} \tag{4.28}$$

where the first isomorphism is (M6).

The right-hand vertical morphism is the composite

$$\begin{aligned} \pi_*\tilde{f}_2^!\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) &= (\pi^X_{Y_2})_*(\gamma_2^\vee)_!(\gamma_2^\vee)^!(f_2')^!\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \\ &\to (\pi^X_{Y_2})_*(f_2')^!\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \\ &\simeq f_2^!(\pi_{N^*_{Y_2/\mathrm{pt}}})_*\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}) \\ &\simeq f_2^!\omega_{Y_2}. \end{aligned} \tag{4.29}$$

where we use the fact that γ_2 is a closed immersion (since f_1 is quasi-smooth).

Theorem 4.30. *The diagram (4.27) commutes.*

We defer the proof to [KK].

4.4.2. *Comparison of 2d and 3d CoHAs.* We assume now that in the correspondence (4.6), Y_1, Y_2 and X are all quasi-smooth. We set $\tilde{Y}_1 \coloneqq N^*_{Y_1/\mathrm{pt}}$, $\tilde{Y}_2 \coloneqq N^*_{Y_2/\mathrm{pt}}$ and $\tilde{X} \coloneqq N^*_{X/Y_1\times Y_2}[-1]$. Under the assumptions, one sees that the morphism $\tilde{f}_2\colon N^*_{X/Y}[-1] \to N^*_{Y_2/\mathrm{pt}}$ is proper.

In this situation the morphism (4.19) takes the form

$$\tilde{f}_1^* \phi_{\tilde{Y}_1}[2d - d_1 - d_2] \to \tilde{f}_2^! \phi_{\tilde{Y}_2} \tag{4.31}$$

as in Corollary 4.20, where we set $d_1 \coloneqq \operatorname{vdim} Y_1$, $d_2 \coloneqq \operatorname{vdim} Y_2$, $d \coloneqq \operatorname{vdim} X$.

Corollary 4.32. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{H}^*(\tilde{Y}_1, \phi_{\tilde{Y}_1}) & \xrightarrow[\mathrm{H}^*(\tilde{Y}_1,\rho)]{\simeq} & H^{\mathrm{BM}}_{-*+d_1}(Y_1) \\
\downarrow & & \downarrow{\scriptstyle f_1^!} \\
\mathrm{H}^*(\tilde{X}, \tilde{f}_1^* \phi_{\tilde{Y}_1}) & & \mathrm{H}^{\mathrm{BM}}_{-*-d_1+2d}(X) \\
\downarrow{\scriptstyle (4.31)} & & \downarrow{\scriptstyle f_{2,*}} \\
\mathrm{H}^{*-d_1-d_2+2d}(\tilde{X}, \tilde{f}_2^! \phi_{\tilde{Y}_2}) & & \\
\downarrow & & \\
\mathrm{H}^{*-d_1-d_2+2d}(\tilde{Y}_2, \phi_{\tilde{Y}_2}) & \xrightarrow[\mathrm{H}^*(\tilde{Y}_2,\rho)]{\simeq} & \mathrm{H}^{\mathrm{BM}}_{-*-d_1+2d}(Y_2)
\end{array}$$

where the horizontal arrows are the dimensional reduction isomorphisms (Corollary 4.3).

Proof. This is a direct consequence of the comparison of cohomological correspondences (Theorem 4.30). □

Now let S be a smooth algebraic surface and adopt again the notation of (4.2.2) and (4.3.3). In this situation, Corollary 4.32 shows that the morphism (4.24) gives a 3d refinement of the Kapranov–Vasserot CoHA product (4.9). More precisely, the following composite

$$\begin{aligned}
&\mathrm{H}^{\mathrm{BM}}_*(M_{S,\gamma'}) \otimes \mathrm{H}^{\mathrm{BM}}_*(M_{S,\gamma''}) \\
&\simeq \mathrm{H}^{-*+d'}(M_{X,\gamma'}, \phi_{M_{X,\gamma'}}) \otimes \mathrm{H}^{-*+d''}(M_{X,\gamma''}, \phi_{M_{X,\gamma''}}) \\
&\simeq \mathrm{H}^{-*+d'+d''}(M_{X,\gamma'} \times M_{X,\gamma''}, \phi_{M_{X,\gamma'}} \boxtimes \phi_{M_{X,\gamma''}}) \\
&\to \mathrm{H}^{-*+d'+d''}(M^{\mathrm{ext}}_{X,\gamma',\gamma''}, (\mathrm{ev}')^* \phi_{M_{X,\gamma'}} \otimes (\mathrm{ev}'')^* \phi_{M_{X,\gamma''}}) \\
&\xrightarrow{(4.24)} \mathrm{H}^{-*+d'+d''-2d^{\mathrm{ext}}}(M^{\mathrm{ext}}_{X,\gamma',\gamma''}, \mathrm{ev}^! \phi_{M_{X,\gamma'+\gamma''}}) \\
&\to \mathrm{H}^{-*+d'+d''-2d^{\mathrm{ext}}}(M_{X,\gamma'+\gamma''}, \phi_{M_{X,\gamma'+\gamma''}}) \\
&\simeq \mathrm{H}^{\mathrm{BM}}_{*+2\operatorname{vdim}(\mathrm{ev}',\mathrm{ev}'')}(M_{S,\gamma'+\gamma''})
\end{aligned}$$

coincides with the map (4.9). Here we set $d' \coloneqq \operatorname{vdim} M_{S,\gamma'}$, $d'' \coloneqq \operatorname{vdim} M_{S,\gamma''}$ and $d^{\mathrm{ext}} \coloneqq \operatorname{vdim} M^{\mathrm{ext}}_{S,\gamma',\gamma''}$.

4.5. **Microlocal virtual pull-back.** Let Y be a quasi-smooth derived Artin stack and $\Lambda \subseteq N^*_{Y/\mathrm{pt}}$ be a closed conic subset. Following [Sch1, Def. 1.10], we define

$$\mu_Y^{\Lambda} \coloneqq 0_Y^* i_\Lambda^! \mu_{Y/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}).$$

where $i_\Lambda \colon \Lambda \hookrightarrow N^*_{Y/\mathrm{pt}}$ denotes the inclusion map. We set

$$\mathrm{H}^*_\Lambda(Y) \coloneqq \mathrm{H}^{-*}(Y, \mu_Y^\Lambda)$$

and call it the *microlocal homology* of Y with prescribed singular support in Λ. This object can be seen as a decategorified version of the singular support for Ind-coherent sheaves introduced in [AG, Def. 4.1.4]. More precisely, the periodic cyclic homology $\mathrm{HP}_*(\mathrm{IndCoh}_\Lambda(X))$ of the category of ind-coherent sheaves with prescribed singular support in Λ is expected to coincide with the microlocal homology $\mathrm{H}^*_\Lambda(Y)$ after 2-periodization. See [Sch2, Thm. 9.1] for the precise statement and the proof in the case of derived global complete intersections.

By (M6) we have canonical isomorphisms

$$\mu_Y^{N^*_{Y/\mathrm{pt}}} \simeq \omega_Y, \quad \mu_Y^Y \simeq \mathbf{Q}_Y[2\,\mathrm{vdim}\,Y].$$

Therefore the microlocal homology interpolates the Borel–Moore homology and the cohomology. Note that if there is an inclusion of subsets $\Lambda_1 \subseteq \Lambda_2$, we have a natural map

$$\mu_Y^{\Lambda_1} \to \mu_Y^{\Lambda_2}.$$

We will apply Theorem 4.18 to study the functoriality of microlocal homology.

Assume that we are given a morphism of quasi-smooth derived Artin stacks $f \colon Y_1 \to Y_2$. We do not assume that f itself is quasi-smooth. Consider the following correspondence

$$\begin{array}{ccccc} & & f^* N^*_{Y_2/\mathrm{pt}} & & \\ & \overset{f'}{\swarrow} & & \overset{\eta^\vee}{\searrow} & \\ N^*_{Y_2/\mathrm{pt}} & & & & N^*_{Y_1/\mathrm{pt}} \end{array}$$

as in the proof of Theorem 4.18, which is naturally identified with the conormal Lagrangian correspondence for

$$\begin{array}{ccccc} & & Y_1 & & \\ & \overset{f}{\swarrow} & & \searrow\!\!\searrow & \\ Y_2 & & & & Y_1. \end{array}$$

Definition 4.33. Given closed subsets $\Lambda_1 \subseteq N^*_{Y_1}$ and $\Lambda_2 \subseteq N^*_{Y_2}$, we say that Λ_1 and Λ_2 are *f-admissible* if:

(i) There is an inclusion $(f')^{-1}(\Lambda_2) \subseteq (\eta^\vee)^{-1}(\Lambda_1)$.

(ii) The morphism $\eta^\vee$ restricts to a closed immersion

$$\eta^\vee|_{(f')^{-1}(\Lambda_2)} \colon (f')^{-1}(\Lambda_2) \to \Lambda_1.$$

These conditions are satisfied when f_1 is quasi-smooth and $\Lambda_1 = N^*_{Y_1}$. Note that Theorem 4.18 yields a canonical map

$$(f')^* \mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})[2\,\mathrm{vdim}\, f_1] \to (\eta^\vee)^! \mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}}).$$

By the condition (i) of f-admissibility, it induces a map

$$(f'|_{(f')^{-1}(\Lambda_2)})^* (\mu_{Y_2/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})|^!_{\Lambda_2})[2\,\mathrm{vdim}\, f_1] \to (\eta^\vee|_{(f')^{-1}(\Lambda_2)})^! (\mu_{Y_1/\mathrm{pt}}(\mathbf{Q}_{\mathrm{pt}})|^!_{\Lambda_1}).$$

The condition (ii) of f-admissibility implies that this induces a map on Borel–Moore homology

$$(f^!)^{\mathrm{micro}}_{\Lambda_1,\Lambda_2} \colon \mathrm{H}^{\Lambda_2}_*(Y_2) \to \mathrm{H}^{\Lambda_1}_{*+2\,\mathrm{vdim}\, f_1}(Y_1)$$

which we call the *microlocal virtual pull-back*.

By Corollary 4.32, when f is quasi-smooth we have the following identity:

$$f^! = (f^!)^{\mathrm{micro}}_{N^*_{Y_1/\mathrm{pt}}, N^*_{Y_2/\mathrm{pt}}},$$

so the microlocal virtual pull-back may be regarded as a generaliztion of the virtual pull-back to non-quasi-smooth morphisms (satisfying the admissibility condition).

By a similar argument, say Λ_1, Λ_2 are *f-coadmissible* if there is an inclusion $(\eta^\vee)^{-1}(\Lambda_1) \subseteq (f')^{-1}(\Lambda_2)$ and f' restricts to a proper morphism

$$f'_{(\eta^\vee)^{-1}(\Lambda_1)} \colon (\eta^\vee)^{-1}(\Lambda_1) \to \Lambda_2.$$

We can then define the *microlocal virtual push-forward*

$$(f_*)^{\mathrm{micro}}_{\Lambda_1,\Lambda_2} \colon \mathrm{H}^{\Lambda_1}_*(Y_1) \to \mathrm{H}^{\Lambda_2}_*(Y_2).$$

By Corollary 4.32, when f is proper we have the following identity:

$$f_* = (f_*)^{\mathrm{micro}}_{N^*_{Y_1/\mathrm{pt}}, N^*_{Y_2/\mathrm{pt}}}.$$

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