COHOMOLOGICAL AND CATEGORICAL CONCENTRATION

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ABSTRACT. Given a torus action on a compact space X, a fundamental result of Borel and Atiyah–Segal asserts that the equivariant cohomology of X is concentrated in the fixed locus X^T, up to inverting enough Chern classes. We prove an analogue for algebraic varieties over an arbitrary field. In fact, we deduce this from a categorification at the level of equivariant derived categories and even equivariant stable motivic homotopy categories, which also gives concentration at the level of Voevodsky motives and for homotopy K-theory.

<center>INTRODUCTION</center>

0.1. **Cohomological concentration.** The starting point for this paper is the following fundamental result in equivariant cohomology (see [Hsi, §3.2, Thm. III.1]):

Theorem 0.1 (Borel, Atiyah–Segal). *Let X be a compact topological space with an action of a compact Lie group G. Let Σ be a set of nontrivial rank one G-representations and let $Z \subseteq X$ be a G-invariant closed subspace containing every point $x \in X$ such that for every $\rho \in \Sigma$ the restriction $\rho|_{G_x}$ is nontrivial (where G_x denotes the stabilizer of the G-action at x). Then the inclusion $i : Z \hookrightarrow X$ induces an isomorphism*

$$i^* : H_G^*(X)[\Sigma^{-1}] \to H_G^*(Z)[\Sigma^{-1}]$$

where Σ acts via multiplication by the first Chern class[1].

Here $H_G^*(-)$ denotes G-equivariant singular cohomology, defined via the Borel construction. One of our goals in this paper is to prove a variant of Theorem 0.1 in algebraic geometry. Let k be a field and let $H_G^*(-)$ denote one of:

(i) G-equivariant Betti cohomology $H_G^*(X(\mathbf{C}); \mathbf{Z})$ (if $k = \mathbf{C}$),
(ii) G-equivariant étale cohomology $H_G^*(X_{\text{ét}}; \mathbf{Z}/n\mathbf{Z})$ (if n is invertible in k),
(iii) G-equivariant ℓ-adic cohomology $H_G^*(X_{\text{ét}}; \mathbf{Z}_\ell)$ (if ℓ is invertible in k).

Then we have (see Corollary 3.4):

Theorem A (Cohomological concentration). *Let X be an algebraic space of finite type over k with an action of an algebraic group G. Let Σ be a set of nontrivial rank one G-representations and let $Z \subseteq X$ be a G-invariant closed subspace containing every geometric point x of X for which the restrictions $\rho|_{G_x}$ are nontrivial for all $\rho \in \Sigma$. Then the inclusion $i : Z \hookrightarrow X$ induces an isomorphism*

$$i^* : H_G^*(X)[\Sigma^{-1}] \to H_G^*(Z)[\Sigma^{-1}] \tag{0.2}$$

where Σ acts via multiplication by the first Chern class.

In fact, we prove a much more general statement about the cohomology of algebraic stacks. Each of the cohomology theories above admits a canonical extension to algebraic stacks in such a way that the cohomology of a quotient stack $[X/G]$ is the equivariant cohomology $H_G^*(X)$ (see e.g. [KR2]), and we have (see Corollary 3.4):

Theorem B (Cohomological stacky concentration). *Let $\mathfrak{X}$ be an algebraic stack*[2] *of finite type over k. Let Σ be a set of line bundles on $\mathfrak{X}$ and let $\mathfrak{Z} \subseteq \mathfrak{X}$ be a closed substack containing every geometric point x of $\mathfrak{X}$ for which the*

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[1]See e.g. [Ati, Appendix] for the definition of $c_1(\rho)$.

[2]Throughout the introduction, all algebraic stacks are assumed to have affine stabilizer groups.

restrictions $\mathscr{L}|_{B\underline{\mathrm{Aut}}_{\mathcal{X}}(x)}$ *are nontrivial for all* $\mathscr{L} \in \Sigma$. *Then the inclusion* $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ *induces an isomorphism*

$$i^* : \mathrm{H}^*(\mathcal{X})[\Sigma^{-1}] \to \mathrm{H}^*(\mathcal{Z})[\Sigma^{-1}] \tag{0.3}$$

where Σ *acts via multiplication by the first Chern class.*

One recovers Theorem A by applying Theorem B to the inclusion $\mathcal{Z} = [Z/G] \hookrightarrow \mathcal{X} = [X/G]$. Note in fact that this also implies that we may take X itself to be an algebraic stack in Theorem A (see e.g. [Rom] for an introduction to group actions on stacks).

Theorem B is a cohomological variant of the stacky concentration theorem of [AKLPR, Thm. B], which was proven for Borel–Moore homology. The two statements are equivalent when $\mathcal{X}$ and $\mathcal{Z}$ are smooth, via Poincaré duality. The arguments of *op. cit.* can be dualized to prove a variant for cohomology with *compact support* $\mathrm{H}_c^*(-)$, asserting the invertibility of the canonical map

$$\mathrm{H}_c^*(\mathcal{X})[\Sigma^{-1}] \to \mathrm{H}_c^*(\mathcal{Z})[\Sigma^{-1}], \tag{0.4}$$

but not for cohomology itself (when $\mathcal{X}$ is not proper).

We expect cohomological concentration to be useful in geometric representation theory, where one considers operators in cohomology induced by correspondences of schemes or stacks (possibly with action of a reductive algebraic group). In that context the spaces involved are often singular and it is important to distinguish between cohomology and Borel–Moore homology. In fact, one may also want to study cohomology with coefficients in more general sheaves, such as the intersection complex, see e.g. [Ach, GKM, Jos, Kir, Bry, EM]. Another example appears in the emerging subject of cohomological Donaldson–Thomas theory, which studies a certain perverse sheaf ϕ_M associated with oriented (-1)-shifted symplectic spaces M, such as the moduli of compactly supported coherent sheaves on a Calabi–Yau threefold; 3d cohomological Hall algebras are (partly conjectural) associative algebra structures on cohomology with coefficients in ϕ_M, see [KS, KL, BBDJS, Kin, KK].

We discuss "with coefficients" and categorified versions of concentration below, which surprisingly turn out to be instrumental in our proofs of Theorems A and B.

0.2. **Categorical concentration.** To prove Theorem B, we will demonstrate the following categorification of [AKLPR, Thm. B]. For a finite type algebraic space X over k, let $\mathbf{D}(X)$ denote one of the following:

(i) the derived ∞-category of sheaves of abelian groups on the topological space $X(\mathbf{C})$ (if $k = \mathbf{C}$),

(ii) the derived ∞-category of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules on the small étale site $X_{\text{ét}}$ (if n is invertible in k),

(iii) the derived ∞-category of ℓ-adic sheaves on the small étale site $X_{\text{ét}}$ (if ℓ is invertible in k).

Each of these admits a canonical extension to algebraic stacks, such that the Ext groups in $\mathbf{D}(\mathcal{X})$ calculate the cohomology groups $\mathrm{H}^*(\mathcal{X})$. We define the categorical localization $\mathbf{D}(\mathcal{X})[\Sigma^{-1}]$, roughly speaking, by formally inverting for every line bundle $\mathscr{L} \in \Sigma$ the first Chern class $c_1(\mathscr{L})$, regarded as a morphism $\Lambda_{\mathcal{X}} \to \Lambda_{\mathcal{X}}(1)[2]$ (where $\Lambda_{\mathcal{X}}$ denotes the constant sheaf on $\mathcal{X}$). Similarly, we define $\mathbf{D}(\mathcal{Z})[\Sigma^{-1}]$ for the substack $\mathcal{Z} \subseteq \mathcal{X}$ by inverting the morphisms $c_1(\mathscr{L}|_{\mathcal{Z}}) : \Lambda_{\mathcal{Z}} \to \Lambda_{\mathcal{Z}}(1)[2]$.[3]

Theorem C (Categorical stacky concentration)**.** *Let $\mathcal{X}$, $\mathcal{Z}$ and Σ be as in Theorem B. Then the inclusion $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ induces an equivalence*

$$i_* : \mathbf{D}(\mathcal{Z})[\Sigma^{-1}] \to \mathbf{D}(\mathcal{X})[\Sigma^{-1}]. \qquad (0.5)$$

In particular, for every $\mathscr{F} \in \mathbf{D}(\mathcal{X})$ the canonical morphism

$$\mathscr{F} \to i_* i^* \mathscr{F} \qquad (0.6)$$

is an isomorphism up to Σ-localization.

From this statement one deduces moreover that Theorems A and B hold with coefficients. For example, we have:

Corollary D. *Let $\mathcal{X}$, $\mathcal{Z}$ and Σ be as in Theorem B. Then for every $\mathscr{F} \in \mathbf{D}(\mathcal{X})$, the canonical map*

$$\mathrm{H}^*(\mathcal{X}; \mathscr{F})[\Sigma^{-1}] \to \mathrm{H}^*(\mathcal{X}; i_* i^* \mathscr{F})[\Sigma^{-1}] \simeq \mathrm{H}^*(\mathcal{Z}; i^* \mathscr{F})[\Sigma^{-1}] \qquad (0.7)$$

is invertible.

The idea to categorify comes from the fact that for any closed immersion $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ there is a Verdier sequence of stable ∞-categories

$$\mathbf{D}(\mathcal{Z}) \xrightarrow{i_*} \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X} \smallsetminus \mathcal{Z})$$

categorifying the localization triangle on Borel–Moore chains. We will show that this is preserved under Σ-localization. Therefore, to prove Theorem C it is possible to mimic the strategy of the proof of [AKLPR, Thm. B], reducing to the Σ-acyclicity of $\mathbf{D}(\mathcal{X} \smallsetminus \mathcal{Z})$. This is done by categorifying every step of the proof of *loc. cit.* As we alluded to earlier, this kind of localization argument does not work for cohomology itself but only for compactly supported cohomology.

0.3. **Motivic concentration.** Recently, the formalism of Grothendieck's six operations has been extended to various stable ∞-categories of motives over algebraic stacks (we refer the reader to [Voe] for a first introduction to stable motivic homotopy theory). Specifically, given a finite type algebraic stack $\mathcal{X}$ over k, with quasi-compact and separated diagonal, we may consider:

(i) the stable ∞-category $\mathrm{DM}(\mathcal{X})$ of relative Voevodsky motives over $\mathcal{X}$,

(ii) the stable ∞-category $\mathrm{D}_{\mathrm{MGL}}(\mathcal{X})$ of MGL-motives over $\mathcal{X}$, i.e., modules over the algebraic cobordism spectrum $\mathrm{MGL}_{\mathcal{X}}$ in $\mathrm{SH}(\mathcal{X})$.

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[3]See Subsect. 1.3 for the precise definitions.

(iii) (if $\mathfrak{X}$ is scalloped) the stable ∞-category $\mathrm{D}_{\mathrm{KGL}}$ of *genuine* KGL-motives over $\mathfrak{X}$, i.e., modules over the algebraic K-theory spectrum $\mathrm{KGL}_{\mathfrak{X}}$ in the genuine stable motivic homotopy category $\mathrm{SH}_{\mathrm{gen}}(\mathfrak{X})$.

See [Kha1, App. A], [Cho], [Kha4, §4] for the first two cases and [KR1, §2] for the case of genuine KGL-modules. Note that for quotient stacks $\mathfrak{X} = [X/G]$, the objects of (i) and (ii) are coefficients for *Borel-equivariant* motivic cohomology and algebraic cobordism, respectively (see [KR2]), whereas genuine KGL-cohomology recovers the homotopy K-theory of G-equivariant vector bundles on X (see [KR1, §12]). An interesting feature of our proof is that, since it only requires a suitable six functor formalism, we are able to treat both types of equivariance uniformly.

We find in particular that the results stated so far hold for motivic cohomology and motivic Borel–Moore homology. They even hold before taking hypercohomology, i.e., already for the cohomological and Borel–Moore motives

$$\mathrm{M}(\mathfrak{X}) \coloneqq f_* f^*(\mathbf{Z}), \ \mathrm{M}^{\mathrm{BM}}(\mathfrak{X}) \coloneqq f_* f^!(\mathbf{Z}) \ \in \ \mathrm{DM}(\mathrm{Spec}(k)) \qquad (0.8)$$

where $f : \mathfrak{X} \to \mathrm{Spec}(k)$ is the projection (and similarly in MGL-modules, see Variant 3.8). This kind of motivic concentration theorem makes it possible to study *motivic stable envelopes*, i.e., lifts of the theory of stable envelopes in cohomology (see [MO]) to Voevodsky motives, which we plan to investigate in future work.

For genuine KGL-cohomology we recover an analogue of Thomason's concentration theorem [Tho3, Thms. 2.1, 2.2] for homotopy K-theory (see [Hoy, §4] and [KR1, §10.1] for the definition of equivariant homotopy K-theory):

Corollary E (Homotopy K-theory). *Let X be a tame Deligne–Mumford stack of finite type over k with an action of an affine algebraic group G of multiplicative type. Let Σ be a set of nontrivial rank one G-representations and let $Z \subseteq X$ be a G-invariant closed substack containing every geometric point x of X for which the restrictions $\rho|_{G_x}$ are nontrivial for all $\rho \in \Sigma$. Then the inclusion $i : Z \hookrightarrow X$ induces an isomorphism*

$$i^* : \mathrm{KH}^G(X)[\Sigma^{-1}] \to \mathrm{KH}^G(Z)[\Sigma^{-1}] \qquad (0.9)$$

where $\rho \in \Sigma$ acts via multiplication by $\lambda_{-1}(\rho) = 1 - [\rho]$.

In case X is a quasi-projective scheme with linearizable action, we may take G to be any linearly reductive affine algebraic group. Using the Borel–Moore variant, which recovers equivariant G-theory (= algebraic K-theory of G-equivariant coherent sheaves, see [Hoy, Rem. 5.7]), we also recover Thomason's concentration [Tho3, Thm. 2.2]. In [KPR] we will give a direct proof of stacky concentration in G-theory (for possibly non-scalloped stacks).

Note that Corollary E cannot hold for equivariant algebraic K-theory $\mathrm{K}^G(-)$ itself (i.e., without imposing homotopy invariance) because it is not nil-invariant on singular stacks. However, if X is an algebraic space and Z is the homotopy fixed point space X^{hG}, then we do expect the isomorphism (0.9) to lift to $\mathrm{K}^G(X)[\Sigma^{-1}] \to \mathrm{K}^G(Z)[\Sigma^{-1}]$.

0.4. **Localization formulas.** One standard consequence of Theorem 0.1 in torus-equivariant cohomology is the localization formula of Atiyah–Bott and Berline–Vergne [AB, BV] for smooth compact manifolds. In our context, a straightforward consequence of Theorem A is that for any smooth algebraic space X of finite type, and any action of a split algebraic torus T with fixed locus $X^T \subseteq X$, the inverse of the isomorphism (Theorem A)

$$i^* : \mathrm{H}_T^*(X)[\Sigma^{-1}] \to \mathrm{H}_T^*(X^T)[\Sigma^{-1}]$$

admits the formula

$$(i^*)^{-1} = i_! \left(- \cap e(\mathscr{N}_{X^T/X})^{-1} \right) \tag{0.10}$$

where $i_! : \mathrm{H}^*(X^T) \to \mathrm{H}^*(X)$ is the Gysin map. By Poincaré duality, this can also be deduced from the localization formula in Borel–Moore homology proven in [AKLPR, Thm. D] (note that as X is smooth, so is X^T).

In the "virtual" case, where X is quasi-smooth, we have the following new result:

Corollary F (Cohomological virtual localization formula)**.** *Let X be a quasi-smooth derived algebraic space of finite type over k with T-action. Then the fixed locus $X^T \subseteq X$ is quasi-smooth (see [AKLPR, Prop. A.23, Cor. A.35]), and the isomorphism $i^* : \mathrm{H}_T^*(X)[\Sigma^{-1}] \to \mathrm{H}_T^*(X^T)[\Sigma^{-1}]$ of Theorem A has inverse given by*

$$(i^*)^{-1} = i_! \left(- \cap e(\mathscr{N}_{X^T/X})^{-1} \right). \tag{0.11}$$

See Corollary 7.8 for a general statement formulated in the context of a closed immersion $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ as in Theorem C[4]. Modulo Theorem A, the nontrivial part of Corollary F is to actually construct the Gysin map $i_!$ and the inverse Euler class $e(\mathscr{N}_{X^T/X})^{-1}$ in this situation, where the closed immersion i is typically *not* quasi-smooth, with conormal complex $\mathscr{N}_{X^T/X}$ of Tor-amplitude $[0, 1]$. We do this by categorifying the constructions used to prove the virtual localization formula in Borel–Moore homology [AKLPR, Thm. D]. In particular, we prove a categorification of Corollary F which computes the inverse of the unit map $\mathrm{id} \to i_* i^*$ in the Σ-localized category (see Theorem 6.1).

The first version of the virtual localization formula was proven in Chow homology by Graber–Pandharipande [GP] and has had numerous applications in enumerative geometry, e.g. in Gromov–Witten and Donaldson–Thomas theory, where invariants are defined via virtual fundamental classes [Kon]. The generalization proven in [AKLPR, Thm. D] is necessary for applications involving more complicated moduli spaces/stacks that do not admit global embeddings into smooth spaces/stacks or global resolutions of their cotangent complex (see e.g. [Joy, Rem. 2.20], [SS, Prop. 3.7], [AKLPR, Thm. 7.13]). Note that the version proven here also recovers the K-theoretic virtual localization formula of [CFK, §5], again with no hypotheses on global embeddability or resolutions. This generality will be even more important

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[4]For example, when X is Deligne–Mumford we can apply it to $i : [X^{rhT}/T] \to [X/T]$ where X^{rhT} is the reparametrized homotopy fixed point stack of [AKLPR, §A.6].

for applications to Gromov–Witten invariants in non-archimedean analytic geometry [PY1, PY2], where global embeddings and resolutions are rare (we believe our arguments carry over to the topological and analytic categories without much modification).

0.5. **Related work.**

0.5.1. *Singular cohomology.* In the context of equivariant singular cohomology of complex algebraic varieties, Theorem A is a special case of the classical Theorem 0.1[5]. We are not aware of any previous results like Theorems B or C.

Concentration for cohomology with coefficients in sheaves $\mathscr{F} \in \mathbf{D}_G(X)$, as in Corollary D, have been considered in the literature when $G = T$ is a split torus: in [GKM, Thm. 6.2] for sheaves of $\mathbf{R}$-vector spaces, and more generally but assuming various niceness conditions[6], for sheaves of modules over a factorial ring in [FW, Thm. 3.2]. Outside of these cases, the cohomological concentration isomorphism

$$\mathrm{H}_G^*(X; \mathscr{F})[\Sigma^{-1}] \to \mathrm{H}_G^*(X^G; \mathscr{F})[\Sigma^{-1}],$$

seems to require the new methods we introduce here, even in the classical situation of equivariant singular cohomology and X non-stacky. Note that for *compactly supported* cohomology, the isomorphism

$$\mathrm{H}_{T,c}^*(X; \mathscr{F})[\Sigma^{-1}] \to \mathrm{H}_{T,c}^*(X; i_* i^* \mathscr{F})[\Sigma^{-1}]$$

is much easier to prove and appears in [EM, Thm. B.2].

0.5.2. *Étale/ℓ-adic cohomology.* We did not find the statement of Theorem A for étale cohomology with torsion or adic coefficients anywhere in the literature. See however [IZ, Cor. 7.4] for an analogue of Quillen's concentration theorem for actions of elementary abelian ℓ-groups [Qui, Thm. 4.2].

0.5.3. *Chow groups and motives.* Edidin and Graham [EG2] proved concentration theorems for the equivariant Chow groups [EG1] over an arbitrary base field k. Chow groups form a Borel–Moore homology-type theory: in fact, the (equivariant) motivic Borel–Moore homology of a finite type k-scheme is isomorphic to the (equivariant) higher Chow groups, at least up to inverting $p = \mathrm{char}(k)$ when $p > 0$ (see [KR2, Cor. 6.4]).

Here we prove concentration for motivic *co*homology, which may be thought of as a (higher) Chow cohomology theory for possibly singular schemes. From that perspective, the only previous result in this direction we are aware of is a concentration theorem for *operational* Chow cohomology with

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[5]at least when X is proper; see [Qui, Thm. 4.4] for a generalization of Theorem 0.1 to the non-compact case.

[6]specifically, [FW] requires the following assumptions: X is normal and admits a cover by T-invariant affine opens each containing an attractive fixed point, and T acts with only finitely many 0- and 1-dimensional orbits and such that the closure of each 1-dimensional orbit is smooth. We thank an anonymous referee for pointing out this reference to us.

rational coefficients proven by Gonzales [Gon]. However, operational Chow cohomology (introduced by Fulton [Ful]) is only a coarse approximation of motivic cohomology (e.g. the former has no cycle class map [Tot, §8]).

After this work appeared, Haution proved in [Hau] a categorical concentration statement for the (genuine-equivariant) motivic stable homotopy category: for X an algebraic space of finite type over k, with action of a multiplicative type algebraic group G[7], there is an equivalence

$$\mathrm{SH}^G(X^G)[\Sigma^{-1}] \to \mathrm{SH}^G(X)[\Sigma^{-1}],$$

where Σ is defined similarly as in Theorem C but using the Euler classes of all G-representations with no trivial summand. The methods, quite different from ours, are specific to groups G of multiplicative type (or at least linearly reductive).

0.6. **Conventions and notation.** We fix an implicit noetherian base ring k. The term "stack" will mean *derived algebraic stack locally of finite type over k* throughout the paper, where *algebraic* is a synonym for *1-Artin* (see e.g. [Toë, §5.1]). Similarly, "schemes" and "algebraic spaces" will be derived and locally of finite type over k. We write Stk for the ∞-category of stacks.

0.7. **Acknowledgments.** We are grateful to Hyeonjun Park for helpful discussions and suggestions.

1. Categorical setup

1.1. **Weaves.** Throughout the paper we will work in the context of an *oriented topological weave*, which is a precise axiomatization of a sheaf theory with the six functor formalism. We give an informal account below and refer to [Kha4] for the precise definitions.

We fix a full subcategory $\mathbf{S} \subseteq$ Stk which is closed under finite coproducts and finite limits. Roughly speaking, a weave $\mathbf{D}$ on $\mathbf{S}$ [8] amounts to a collection of ∞-categories $\mathbf{D}(X)$ for every $X \in \mathbf{S}$, equipped with the six operations:

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[7]Haution assumes that X is a quasi-projective scheme with linearizable G-action, but this assumption can be dropped by using the theory in [KR1] (since $[X/G]$ is nicely scalloped by Theorem 2.14 there).

[8]More precisely: a weave on $(\mathbf{S}, \mathbf{S}^{\mathrm{repr}})$ where $\mathbf{S}^{\mathrm{repr}}$ is the subcategory of *representable* morphisms; that is, we only require the !-operations to be defined for representable morphisms (which are necessarily locally of finite type, since all objects of Stk are locally of finite type over k).

(1a) *Tensor:* $(-) \otimes (-) : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X)$,

(1b) *Internal Hom:* $\underline{\mathrm{Hom}}(-,-) : \mathbf{D}(X)^{\mathrm{op}} \times \mathbf{D}(X) \to \mathbf{D}(X)$,

for every $X \in \mathbf{S}$;

(2a) $*$-*Inverse image:* $f^* : \mathbf{D}(Y) \to \mathbf{D}(X)$,

(2b) $*$-*Direct image:* $f_* : \mathbf{D}(X) \to \mathbf{D}(Y)$,

for every morphism $f : X \to Y$;

(3a) !-*Direct image:* $f_! : \mathbf{D}(X) \to \mathbf{D}(Y)$,

(3b) !-*Inverse image:* $f^! : \mathbf{D}(Y) \to \mathbf{D}(X)$,

for every representable morphism $f : X \to Y$.

The operation $\otimes$ is left adjoint to $\underline{\mathrm{Hom}}$ (as bifunctors), f^* is left adjoint to f_*, and $f_!$ is left adjoint to $f^!$. These operations are subject to various standard compatibilities and coherences: for example, $\otimes$ and $\underline{\mathrm{Hom}}$ define a closed symmetric monoidal structure on $\mathbf{D}(X)$ (we write $\mathbf{1}_X \in \mathbf{D}(X)$ for the unit object); each of the operations f^*, f_*, $f_!$, $f^!$ is compatible with composition up to coherent homotopy; and !-direct image is compatible with $*$-inverse image (base change formula) and with $\otimes$ (projection formula) up to coherent homotopy. Moreover, for every proper representable morphism f there is a canonical isomorphism $f_! \simeq f_*$ and for every smooth representable morphism there is a canonical isomorphism $f^! \simeq f^*(-) \otimes f^!(\mathbf{1}_Y)$ where $f^!(\mathbf{1}_Y)$ is $\otimes$-invertible ("Poincaré duality").

The condition that a weave $\mathbf{D}$ is *topological* means that it satisfies the following:

(a) *Localization:* The ∞-category $\mathbf{D}(\varnothing)$ is trivial (consists of zero objects). For any $X \in \mathbf{S}$ and any closed-open decomposition $i : Z \to X$, $j : X \smallsetminus Z \to X$, there is a canonical exact triangle of functors

$$j_! j^* \to \mathrm{id} \to i_* i^*. \tag{1.1}$$

(b) *Homotopy invariance:* For any $X \in \mathbf{S}$ and affine bundle $\pi : Y \to X$ (i.e., a torsor under a vector bundle on X), the unit morphism $\mathrm{id} \to \pi_* \pi^*$ is invertible.

We also assume that $\mathbf{D}$ satisfies Nisnevich descent (if all objects of $\mathbf{S}$ are quasi-compact, this is automatic). These axioms imply that $\mathbf{D}(X)$ is stable for every X, and the localization axiom implies that the diagram

$$\mathbf{D}(Z) \xrightarrow{i_*} \mathbf{D}(X) \xrightarrow{j^*} \mathbf{D}(U)$$

defines a (split) Verdier sequence of stable ∞-categories in the sense of [Cal+8, App. A].

For a topological weave $\mathbf{D}$ we can write, for every smooth representable morphism $f : X \to Y$, $f^!(\mathbf{1}_Y) \simeq \mathbf{1}_X \langle \Omega_f \rangle$ where $\langle \Omega_f \rangle$ denotes the "Thom twist" by the relative cotangent sheaf Ω_f. The choice of *orientation* for $\mathbf{D}$ gives an identification

$$(-)\langle \Omega_f \rangle \simeq (-)\langle d \rangle := (-)(d)[2d]$$

where $d = \mathrm{rk}(\Omega_f)$ is the relative dimension of f, and (d), resp. $[2d]$, indicates a Tate twist, resp. shift. Thus for an oriented topological weave, Poincaré duality reads $f^! \simeq f^*(-)\langle d\rangle$ for f smooth representable of relative dimension d.

For a locally free sheaf $\mathscr{E}$ on $X \in \mathbf{S}$ of rank r, the top Chern class may be regarded as a morphism $c_r(\mathscr{E}) : \mathbf{1}_X \to \mathbf{1}_X\langle r\rangle$. For any object $\mathscr{F} \in \mathbf{D}(X)$ this yields, by tensoring, a morphism

$$c_r(\mathscr{E}) : \mathscr{F} \to \mathscr{F}\langle r\rangle \tag{1.2}$$

which in cohomology corresponds to cap product with $c_r(\mathscr{E})$. See [Kha1, Constr. A.23].

Definition 1.3. Given an oriented topological weave $\mathbf{D}$ on $\mathbf{S}$, we consider the following condition:

(PB1) Let $X \in \mathbf{S}$ and $\mathscr{E}$ a locally free sheaf on X of rank $r + 1$, $r \geqslant 0$, with associated projective bundle $p : \mathbf{P}(\mathscr{E}) \to X$. Then the natural transformations

$$\mathrm{id} \xrightarrow{\text{unit}} p_* p^* \xrightarrow{c_1(\mathscr{O}(-1))^{\circ i}} p_* p^*\langle i\rangle \tag{1.4}$$

induce a canonical isomorphism

$$\bigoplus_{0 \leqslant i \leqslant r} \mathrm{id}\langle -i\rangle \to p_* p^*. \tag{1.5}$$

(PB2) Let $X \in \mathbf{S}$ and $\mathscr{E}$ a locally free sheaf on X of rank $r + 1$, $r \geqslant 0$, with associated projective bundle $p : \mathbf{P}(\mathscr{E}) \to X$. Then the natural transformation

$$\text{unit} : \mathrm{id}_{\mathbf{D}(X)} \to p_* p^*$$

admits a retract.

By the projection formula for $p_* = p_!$, both conditions can be checked after evaluation on the unit object $\mathbf{1}_X$. The first condition is the projective bundle formula, and the second is the assertion that $p^* : \mathrm{C}^\bullet(X) \to \mathrm{C}^\bullet(\mathbf{P}(\mathscr{E}))$ admits a retract. (PB1) implies (PB2): the projective bundle formula implies $p_!(p^*(-) \cap c_1(\mathscr{O}(-1))^{\cup r}) \simeq \mathrm{id}_{\mathrm{C}^\bullet(X)}$, where $p_! : \mathrm{C}^\bullet(\mathbf{P}(\mathscr{E}))\langle r\rangle \to \mathrm{C}^\bullet(X)$ denotes the Gysin map in cohomology (notation as in Subsect. 3); see the discussion in [Dég, Rem. 3.2.3].

Remark 1.6. For $\mathscr{E} \simeq \mathscr{O}_X^{\oplus r+1}$ it is easy to show (PB1) using the localization triangle, homotopy invariance, and induction on r. Thus (PB1) holds for any $\mathscr{E}$ when X is a scheme or Zariski-locally quasi-separated algebraic space, as we can reduce to the free case using Nisnevich descent. Note also that (PB2) holds for any $X \in \mathbf{S}$ as soon as $\mathscr{E}$ admits a surjection $\mathscr{E} \twoheadrightarrow \mathscr{L}$ with $\mathscr{L}$ a line bundle, since we get a section s of $p : \mathbf{P}(\mathscr{E}) \to X$ giving rise to a retraction of $\mathrm{id} \to p_* p^*$.

In general we will need to impose (PB2) as an additional axiom, although we expect it to hold for any oriented topological weave; in the next section, we will see that it holds for all the examples we consider.

1.2. **Examples of weaves.**

1.2.1. *Lisse-extended weaves.* For $\mathbf{S} \subseteq \mathrm{Stk}$ the full subcategory of *schemes*, there are many examples of oriented topological weaves on $\mathbf{S}$. In fact, the notion is equivalent to that of a Voevodsky formalism on $\mathcal{S}$ (see [Kha4, §3.5]), and as such admits the following examples:

(a) *Betti sheaves:* If k is a $\mathbf{C}$-algebra, we may take $X \mapsto \mathbf{D}(X)$ sending X to the derived ∞-category $\mathrm{D}(X(\mathbf{C}), \mathbf{Z})$ of sheaves of abelian groups on the topological space $X(\mathbf{C})$.

(b) *Étale sheaves (finite coefficients):* If $n \in k^{\times}$, we may take $X \mapsto \mathbf{D}(X)$ sending X to the derived ∞-category $\mathrm{D}_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z})$ of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules on the small étale site of X.

(c) *Étale sheaves (ℓ-adic coefficients):* If $\ell \in k^{\times}$, we may take $X \mapsto \mathbf{D}(X)$ sending X to the ℓ-adic derived ∞-category $\mathrm{D}_{\text{ét}}(X, \mathbf{Z}_\ell)$ of sheaves on the small étale site of X, i.e., the limit $\mathrm{D}_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ over $n > 0$.

(d) *Motives:* Take $X \mapsto \mathbf{D}(X)$ sending X to the ∞-category $\mathrm{DM}(X) \coloneqq \mathrm{D}_{H\mathbf{Z}}(X)$ of modules over the integral motivic Eilenberg–MacLane spectrum $H\mathbf{Z}_X$ as in [Spi]. (See [CD2, Thm. 5.1] and [CD1, §14] for comparisons with other constructions of derived categories of motives.)

(e) *Cobordism motives:* Take $X \mapsto \mathbf{D}(X)$ sending X to the ∞-category $\mathrm{D}_{\mathrm{MGL}}(X)$ of modules over Voevodsky's algebraic cobordism spectrum MGL_X (see e.g. [EHKSY]).

Via the procedure of *lisse extension* (see [Kha4, §4]), every oriented topological weave $\mathbf{D}$ on schemes gives rise to a canonical oriented topological weave on (a large class of) stacks. More precisely, let $\mathbf{S}$ denote the ∞-category of stacks[9] with quasi-compact and separated diagonal. Given $X \in \mathbf{S}$ we consider the limit

$$\mathbf{D}(X) \coloneqq \varprojlim_{(T,t)} \mathbf{D}(T)$$

over the ∞-category Lis_X of pairs (T, t) where T is a scheme and $t : T \to X$ is a smooth morphism, where the transition functors are $*$-inverse image. This can be extended to a topological weave on $\mathbf{S}$ (see [Kha4, §4], [Cho]). In case $\mathbf{D}$ satisfies étale descent (e.g. for the Betti or étale weaves), we can take $\mathbf{S}$ to be the entire ∞-category of stacks, i.e., without the extra conditions on the diagonal (see [Kha1, App. A], [LZ1, LZ2] for the construction in that case). Moreover, any orientation of $\mathbf{D}$ on schemes also lisse-extends to an orientation on $\mathbf{S}$.

Proposition 1.7. *Let $\mathbf{D}$ be an oriented topological weave on $\mathbf{S}$. Suppose that $\mathbf{D}$ is lisse-extended, i.e., that for every $X \in \mathbf{S}$ the family of functors $t^* : \mathbf{D}(X) \to \mathbf{D}(T)$ is jointly conservative as t ranges over smooth morphisms $t : T \to X$ with T a scheme. Then $\mathbf{D}$ satisfies condition (PB1) of Definition 1.3.*

Proof. We reduce to the case where X is a scheme, which holds by Remark 1.6 (see also [AKLPR, Thm. 1.19]). □

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[9]i.e., derived algebraic stacks locally of finite type over k (see Subsect. 0.6)

1.2.2. *Genuine* KGL-*motives.* Let **S** denote the ∞-category of *nicely* or *linearly scalloped* stacks as in [KR1, §2.3, App. A]. The class of nicely scalloped stacks includes tame algebraic stacks in the sense of [AOV], as well as quotients thereof by multiplicative type group actions. The class of linearly scalloped stacks includes quotients $[X/G]$ where G is a linearly reductive affine group scheme acting linearly on a quasi-projective scheme X.

If X is nicely (resp. linearly) scalloped and $Y \to X$ is representable (resp. quasi-projective), then Y is nicely (resp. linearly) scalloped.

The main construction of [KR1] provides a topological weave $\mathrm{SH}^{\mathrm{gen}}$ on **S** sending X to the *genuine* stable motivic homotopy category $\mathrm{SH}^{\mathrm{gen}}(X)$. This restricts to SH on schemes, just like its lisse-extended version considered above, but does not agree for quotient stacks. We refer to [KR1, §12] for an extended discussion of this point.

Taking modules over the genuine algebraic cobordism spectrum [KR1, §10.2] gives the oriented topological weave $\mathrm{D}^{\mathrm{gen}}_{\mathrm{MGL}}$ of genuine MGL-motives. Unfortunately, we do not know whether (PB1) or (PB2) holds in general for this weave[10]. Therefore, we restrict our attention to modules over the genuine algebraic K-theory spectrum [KR1, §10.1], which gives an oriented topological weave $\mathrm{D}^{\mathrm{gen}}_{\mathrm{KGL}}$ of genuine KGL-motives.

Proposition 1.8. *The oriented topological weave* $\mathrm{D}^{\mathrm{gen}}_{\mathrm{KGL}}$ *satisfies condition (PB1) of Definition 1.3.*

Proof. In view of [KR1, Prop. 3.7 and Thm. 10.7] this amounts to the condition that for every nicely (resp. linearly) scalloped stack X, every locally free sheaf $\mathscr{E}$ on X of rank $r+1$, and every smooth representable (resp. quasi-projective) $Y \to X$, the maps of spectra

$$p_Y^*(-) \cup \lambda_{-1}(\mathscr{O}(-1)) : \mathrm{KH}(Y) \to \mathrm{KH}(\mathbf{P}(\mathscr{E}|_Y),$$

where $p_Y : \mathbf{P}_Y(\mathscr{E}|_Y) \to Y$, induce an isomorphism

$$\bigoplus_{0 \leqslant i \leqslant r} \mathrm{KH}(Y) \to \mathrm{KH}(\mathbf{P}(\mathscr{E}|_Y)).$$

This follows from the corresponding formula for (non-homotopy) algebraic K-theory, see e.g. [Kha2, Cor. 3.6]. $\qquad\square$

1.3. **Σ-localization.** From now on, **D** will be an oriented topological weave on **S** satisfying condition (PB2) of Definition 1.3.

Given a set of line bundles $\Sigma \subseteq \mathrm{Pic}(X)$, we denote by $\mathscr{S}_\Sigma$ the (essentially small) set consisting of the morphisms (1.2)

$$c_1(\mathscr{L}) : g_!(\mathbf{1}_Z)[m]\langle n\rangle \to g_!(\mathbf{1}_Z)[m]\langle n+1\rangle$$

for all $\mathscr{L} \in \Sigma$, all representable morphisms of finite type $g : Z \to X$, and all integers $m, n \in \mathbf{Z}$.

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[10]To avoid confusion, let us reiterate that we do know this for lisse-extended cobordism motives.

We consider the localization of $\mathbf{D}(X)$ at $\mathscr{S}_\Sigma$ in the sense of [Cis, §7.1], or equivalently the Verdier quotient by the full (stable) subcategory spanned by the cofibres of the morphisms in $\mathscr{S}_\Sigma$ (see e.g. [Cal$^+$8, Prop. A.1.5]). This is a (large) stable ∞-category $\mathbf{D}(X)[\Sigma^{-1}]$ with an essentially surjective and exact functor

$$L_\Sigma : \mathbf{D}(X) \to \mathbf{D}(X)[\Sigma^{-1}] \tag{1.9}$$

which inverts $\mathscr{S}_\Sigma$ and satisfies the following universal property: for every stable ∞-category $\mathscr{C}$, restriction along (1.9) determines an equivalence

$$\mathrm{Fun}_{\mathrm{ex}}(\mathbf{D}(X)[\Sigma^{-1}], \mathscr{C}) \to \mathrm{Fun}_{\mathrm{ex},\Sigma}(\mathbf{D}(X), \mathscr{C}), \tag{1.10}$$

where the decoration "ex", resp. "ex, Σ", indicates that we consider the full subcategory of exact functors, resp. exact functors inverting $\mathscr{S}_\Sigma$. We say that a morphism in $\mathbf{D}(X)$ is a Σ-*local equivalence* if it is inverted by L_Σ.

Given any two functors F and G out of $\mathbf{D}(X)[\Sigma^{-1}]$, there is a canonical isomorphism of ∞-groupoids of natural transformations

$$\mathrm{Nat}(F, G) \to \mathrm{Nat}(F \circ L_\Sigma, G \circ L_\Sigma), \tag{1.11}$$

by taking mapping ∞-groupoids in (1.10).

We will require some very limited functoriality of the Σ-localization. Given a representable finite type morphism $f : Y \to X$, we also consider the localization of $\mathbf{D}(Y)$ with respect to the set $f^* \mathscr{S}_\Sigma$:

$$L_\Sigma : \mathbf{D}(Y) \to \mathbf{D}(Y)[\Sigma^{-1}].$$

Since f^* tautologically sends $\mathscr{S}_\Sigma$ to $f^* \mathscr{S}_\Sigma$, we obtain by universal property an essentially unique exact functor

$$f_\Sigma^* : \mathbf{D}(X)[\Sigma^{-1}] \to \mathbf{D}(Y)[\Sigma^{-1}]$$

such that $f_\Sigma^* \circ L_\Sigma \simeq L_\Sigma \circ f^*$.

Moreover, $f_!$ sends $f^* \mathscr{S}_\Sigma$ to $\mathscr{S}_\Sigma$. Indeed, given $\mathscr{L} \in \Sigma$ and a representable finite type morphism $g : Z \to X$, consider the endomorphism $c_1(\mathscr{L}) : g_!(\mathbf{1}_Z) \to g_!(\mathbf{1}_Z)\langle 1 \rangle$. The induced endomorphism

$$f_! f^*(c_1(\mathscr{L})) : f_! f^* g_!(\mathbf{1}_Z) \to f_! f^* g_!(\mathbf{1}_Z)\langle 1 \rangle$$

is identified under the base change formula $f_! f^* g_!(\mathbf{1}_Z) \simeq h_!(\mathbf{1}_W)$, where $h : W \to X$ is the projection of $W = Y \times_X Z$, with the endomorphism induced by the action of $c_1(\mathscr{L})$ on $h_!(\mathbf{1}_W)$. Thus there exists an essentially unique exact functor

$$f_!^\Sigma : \mathbf{D}(Y)[\Sigma^{-1}] \to \mathbf{D}(X)[\Sigma^{-1}]$$

such that $f_!^\Sigma \circ L_\Sigma \simeq L_\Sigma \circ f_!$.

If f is proper, then f_Σ^* is left adjoint to $f_*^\Sigma := f_!^\Sigma$. For example, to construct the unit transformation $\mathrm{id} \to f_*^\Sigma f_\Sigma^*$ when f is proper, use the isomorphism

$$\mathrm{Nat}(\mathrm{id}_{\mathbf{D}(X)[\Sigma^{-1}]}, f_*^\Sigma f_\Sigma^*) \to \mathrm{Nat}(L_\Sigma, f_*^\Sigma f_\Sigma^* L_\Sigma)$$

of (1.11). The desired natural transformation corresponds to the morphism $L_\Sigma \to L_\Sigma f_* f^* \simeq f_*^\Sigma f_\Sigma^* L_\Sigma$ induced by the unlocalized unit.

Similarly, for every $n \in \mathbf{Z}$ the operation $\langle n \rangle : \mathbf{D}(Y) \to \mathbf{D}(Y)$ preserves $f^* \mathscr{S}_\Sigma$, hence induces an operation on $\mathbf{D}(Y)[\Sigma^{-1}]$ which we still denote by $\langle n \rangle$.

If f is smooth of relative dimension d, then $f_!^\Sigma$ is left adjoint to $f_\Sigma^! := f_\Sigma^* \langle d \rangle$ by the same reasoning.

Lemma 1.12. *Let $i : Z \to X$ be a closed immersion and denote by $j : U \to X$ the complementary open immersion. Then there is a Verdier sequence of stable ∞-categories*

$$\mathbf{D}(Z)[\Sigma^{-1}] \xrightarrow{i_*^\Sigma} \mathbf{D}(X)[\Sigma^{-1}] \xrightarrow{j_\Sigma^*} \mathbf{D}(U)[\Sigma^{-1}].$$

Proof. Since $\mathbf{D}$ satisfies the localization property, the claim holds before Σ-localization. It is clear that $j_\Sigma^* \circ i_*^\Sigma \simeq 0$ since this can be checked after applying L_Σ on the right. Moreover, the localization triangle (1.1) gives rise, by applying L_Σ on the left, to an exact triangle

$$j_!^\Sigma j_\Sigma^* (L_\Sigma \mathscr{F}) \to L_\Sigma \mathscr{F} \to i_*^\Sigma i_\Sigma^* (L_\Sigma \mathscr{F})$$

for every $\mathscr{F} \in \mathbf{D}(X)$. The claim now follows formally.   □

Variant 1.13. If $\mathbf{D}$ is cocomplete (i.e., $\mathbf{D}(Y)$ admits small colimits for every $Y \in \mathbf{S}$), then we could instead consider the *cocontinuous* localization in the sense of [Cis, Rem. 7.7.10]. That is, for a representable finite type morphism $f : Y \to X$, there exists a cocomplete ∞-category $\mathbf{D}(Y)[\![\Sigma^{-1}]\!]$ equipped with a colimit-preserving localization functor

$$L_\Sigma : \mathbf{D}(Y) \to \mathbf{D}(Y)[\![\Sigma^{-1}]\!]$$

and with a universal property like (1.10) but for *colimit-preserving* functors instead of arbitrary exact functors. For example, if $\mathbf{D}(Y)$ is presentable then L_Σ admits a right adjoint (by the adjoint functor theorem) which is automatically fully faithful and identifies $\mathbf{D}(Y)[\![\Sigma^{-1}]\!]$ with the full subcategory of $f^* \mathscr{S}_\Sigma$-local objects in $\mathbf{D}(Y)$ (i.e., the localization in the sense of [Lur1, §5.2.7]).

Our proof of concentration (Theorem 2.5) will go through *mutatis mutandis* for $\mathbf{D}(-)[\![\Sigma^{-1}]\!]$ in place of $\mathbf{D}(-)[\Sigma^{-1}]$ when $\mathbf{D}$ is cocomplete. However, we need to consider the "plain" localization $\mathbf{D}(-)[\Sigma^{-1}]$ in order to derive cohomological concentration (Corollary 3.4)[11].

Remark 1.14. We note one pleasant advantage of the cocontinuous localization of Variant 1.13. If $\mathbf{D}$ is presentable and satisfies continuity (i.e., $R \mapsto \mathbf{D}(\mathrm{Spec}(R))$ preserves filtered colimits, see [Kha3, Def. 2.16]), then in the cocontinuous variant of categorical concentration mentioned above, one can drop the quasi-compactness hypothesis and thus consider *locally* of finite type stacks. In contrast, this fails for the "plain" localization and for the cohomological localization, cf. [AKLPR, §2.7].

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[11]Indeed, the functor (3.1) typically does not preserve colimits when X is a stack. In fact, $X \mapsto \mathrm{C}^\bullet(X)$ preserves colimits if and only if $\mathbf{1}_X$ is a compact object of $\mathbf{D}(X)$, an assertion which fails already for the simplest relevant example $X = B\mathbf{G}_m$.

2. Categorical concentration

2.1. **Acyclicity.**

Theorem 2.1 (Categorical acyclicity). *Let $X \in \mathbf{S}$ with affine stabilizers, and $\Sigma \subseteq \mathrm{Pic}(X)$ a subset. Let $U \subseteq X$ be a quasi-compact open satisfying the following condition:*

(L) *For every geometric point x of U, there exists a line bundle $\mathscr{L}(x) \in \Sigma$ whose restriction $\mathscr{L}(x)|_{B\underline{\mathrm{Aut}}_X(x)}$ is trivial.*

Then we have $\mathbf{D}(U)[\Sigma^{-1}] = 0$.

Definition 2.2. If an open $U \subseteq X$ satisfies condition (L), we say simply that U is Σ-*admissible*.

Remark 2.3. In fact, the proof of Theorem 2.1 will show that there exists a single Σ-local equivalence s such that $\mathbf{D}(U)[s^{-1}] = 0$.

Proof of Theorem 2.1. Replacing X by U, Σ by $j^*\Sigma$, and j by the identity, we may assume $X = U$ is Σ-admissible and quasi-compact.

Let us write
$$\mathscr{F}\langle * \rangle := \bigoplus_{n \in \mathbf{Z}} \mathscr{F}\langle n \rangle$$
for any $\mathscr{F} \in \mathbf{D}(X)$. We claim that there exists a Σ-local equivalence $\mathbf{1}_X\langle * \rangle \to \mathbf{1}_X\langle * \rangle$ which is null-homotopic. In particular, for every $\mathscr{F} \in \mathbf{D}(X)$ the induced endomorphism of $\mathbf{1}_X\langle * \rangle \otimes \mathscr{F} \simeq \mathscr{F}\langle * \rangle$ is a null-homotopic Σ-local equivalence, hence $L_\Sigma(\mathscr{F}\langle * \rangle) \simeq 0$. Since $L_\Sigma(\mathscr{F})$ is a direct summand of $L_\Sigma(\mathscr{F}\langle * \rangle)$ (by functoriality), the claim will follow.

Suppose we are given a Σ-local equivalence $s : \mathbf{1}_X\langle * \rangle \to \mathbf{1}_X\langle * \rangle$. We begin with the following observations:

(a) If $\pi : X' \to X$ is an affine bundle, then π^* is fully faithful by homotopy invariance. Thus if $\pi^*(s)$ is null-homotopic, then s is itself null-homotopic.

(b) If $\pi : X' \to X$ is a projective bundle, then s is a retract of $\pi^*(s)$ by (PB2) (see Definition 1.3). Thus if $\pi^*(s)$ is null-homotopic, then s is itself null-homotopic.

(c) If $X' \subseteq X$ is a nonempty open such that $s|_{X'}$ is null-homotopic over X', then there exists some Σ-local equivalence $t : \mathbf{1}_X\langle * \rangle \to \mathbf{1}_X\langle * \rangle$ which is null-homotopic over X. Indeed, by noetherian induction on the complement this reduces to the following observation. Suppose given Σ-local equivalences $s', s'' : \mathbf{1}_X\langle * \rangle \to \mathbf{1}_X\langle * \rangle$ such that s' is null-homotopic over some closed substack $Z \subseteq X$ and s'' is null-homotopic over $X \smallsetminus Z$. Then $t = s' \circ s''$ is a Σ-local equivalence which is null-homotopic over both Z and $X \smallsetminus Z$, and from the localization triangle
$$j_! j^* \mathbf{1}_X\langle * \rangle \to \mathbf{1}_X\langle * \rangle \to i_* i^* \mathbf{1}_X\langle * \rangle,$$

where $i : Z \to X$ and $j : X \smallsetminus Z \to X$ are the inclusions, we deduce that t is null-homotopic over X.

Since X has affine stabilizers, there exists a nonempty open with a global quotient presentation (see [HR, Prop. 2.6]). By (c) we may therefore assume that $X = [X_0/G]$ where $G = \mathrm{GL}_n$ for some $n \geqslant 0$ and X_0 is a quasi-affine scheme of finite type over k with G-action.

Let T (resp. B) denote the subgroup of diagonal matrices (resp. upper triangular matrices) in $G = \mathrm{GL}_n$. Since $B \subseteq G$ is a Borel subgroup, so that G/B is a complete flag, the canonical morphism $[X_0/B] \to [X_0/G] = X$ factors as a sequence of iterated projective bundles. Moreover, the canonical morphism $[X_0/T] \to [X_0/B]$ is an affine bundle (see the proof of [Tho2, Thm. 1.13]). By the above observations, it will thus suffice to produce a Σ-local equivalence $s : \mathbf{1}_X\langle * \rangle \to \mathbf{1}_X\langle * \rangle$ which is null-homotopic after $*$-inverse image to some nonempty open of $[X_0/T]$.

By [Tho1, Thm. 4.10, Rem. 4.11] (where we may assume X_0 reduced by nil-invariance), there exists a nonempty T-invariant affine open $V \subseteq X_0$ and a diagonalizable subgroup $T' \subseteq T$ such that

$$[V/T] \simeq W \times BT'$$

with W an affine scheme. Choose a geometric point v of $[V/T]$ and denote its image in X by x. The Σ-admissibility of X means that there exists an invertible sheaf $\mathscr{L} \in \Sigma$ with $\mathscr{L}|_{B\underline{\mathrm{Aut}}_X(x)}$ trivial. In particular, the further restriction $\mathscr{L}|_{BT'}$ along $BT' \to B\underline{\mathrm{Aut}}_X(x)$ is also trivial (where we identify $T' = \underline{\mathrm{Aut}}_{[V/T]}(v)$). Using [SGA3, Exp. I, 4.7.3], we can therefore write

$$\mathscr{L}|_{[V/T]} \simeq p_1^*(\mathscr{L}') \otimes p_2^*(\mathscr{L}|_{BT'}) \simeq p_1^*(\mathscr{L}'), \qquad (2.4)$$

where p_i denotes the projection of $W \times BT'$ to the ith factor, and $\mathscr{L}'$ is an invertible sheaf on W. Since W is a scheme, the endomorphism $c_1(\mathscr{L}') : \mathbf{1}_W\langle * \rangle \to \mathbf{1}_W\langle * \rangle$ is nilpotent (see e.g. [Dég, Prop. 2.1.22(1)]). In other words, there exists an integer $n \gg 0$ such that the endomorphism $s \coloneqq c_1(\mathscr{L})^n : \mathbf{1}_X\langle * \rangle \to \mathbf{1}_X\langle * \rangle$, which is a Σ-local equivalence since $\mathscr{L} \in \Sigma$, becomes null-homotopic after $*$-inverse image to $[V/T]$. $\square$

2.2. Concentration.

Theorem 2.5 (Categorical concentration)**.** *Let $X \in \mathrm{Stk}$ with affine stabilizers. Let $\Sigma \subseteq \mathrm{Pic}(X)$ be a subset of line bundles. For any closed immersion $i : Z \to X$ with quasi-compact and Σ-admissible complement $X \smallsetminus Z$, the functor*

$$i_*^\Sigma : \mathbf{D}(Z)[\Sigma^{-1}] \to \mathbf{D}(X)[\Sigma^{-1}]$$

is an equivalence.

Proof. We have $\mathbf{D}(X \smallsetminus Z)[\Sigma^{-1}] = 0$ by acyclicity (Theorem 2.1), so the claim follows from Lemma 1.12. $\square$

3. COHOMOLOGICAL CONCENTRATION

3.1. **Cohomology & Borel–Moore homology.** For any $X \in \mathbf{S}$, the *cohomology spectrum*, resp. with coefficients in $\mathscr{F} \in \mathbf{D}(X)$, is the mapping spectrum

$$C^{\bullet}(X) := C^{\bullet}(X; \mathbf{1}_X), \quad \text{resp. } C^{\bullet}(X; \mathscr{F}) := \mathrm{Maps}_{\mathbf{D}(X)}(\mathbf{1}_X, \mathscr{F}).$$

For an integer $n \in \mathbf{Z}$ we set

$$C^{\bullet}(X; \mathscr{F})\langle n \rangle := C^{\bullet}(X; \mathscr{F}\langle n \rangle) = \mathrm{Maps}_{\mathbf{D}(X)}(\mathbf{1}_X, \mathscr{F}\langle n \rangle)$$

and

$$C^{\bullet}(X; \mathscr{F})\langle * \rangle := \bigoplus_{n \in \mathbf{Z}} C^{\bullet}(X; \mathscr{F})\langle n \rangle.$$

3.2. **Cohomological concentration.** Given $X \in \mathbf{S}$ and a subset $\Sigma \subseteq \mathrm{Pic}(X)$, let

$$S_\Sigma \subseteq \pi_0 C^{\bullet}(X)\langle * \rangle$$

denote the set of first Chern classes $c_1(\mathscr{L})$ for $\mathscr{L} \in \Sigma$. For a $C^{\bullet}(X)\langle * \rangle$-module spectrum M, we denote by

$$M[\Sigma^{-1}] \simeq M \otimes_{C^{\bullet}(X)\langle * \rangle} C^{\bullet}(X)\langle * \rangle[\Sigma^{-1}]$$

its localization at S_Σ in the sense of [Lur2, Rem. 7.2.3.18].

Let $\mathrm{Mod}_{C^{\bullet}(X)\langle * \rangle[\Sigma^{-1}]}$ denote the ∞-category of $C^{\bullet}(X)\langle * \rangle[\Sigma^{-1}]$-module spectra. Given an open $U \subseteq X$, consider the exact functor $F : \mathbf{D}(U) \to \mathrm{Mod}_{C^{\bullet}(X)\langle * \rangle[\Sigma^{-1}]}$ sending

$$\mathscr{F} \mapsto C^{\bullet}(X; j_! \mathscr{F})\langle * \rangle[\Sigma^{-1}] \tag{3.1}$$

where $j : U \to X$ denotes the inclusion. For every $\mathscr{G} = g_!(\mathbf{1}_Z)$, with $g : Z \to X$ a representable finite type morphism, and every $\mathscr{L} \in \Sigma$, the morphism

$$c_1(\mathscr{L}) : C^{\bullet}(X; j_! j^* \mathscr{G})\langle * \rangle[\Sigma^{-1}] \to C^{\bullet}(X; j_! j^* \mathscr{G})\langle * \rangle[\Sigma^{-1}]$$

is invertible by $C^{\bullet}(X)\langle * \rangle[\Sigma^{-1}]$-linearity, so F inverts $\mathscr{S}_\Sigma$ and thus induces an essentially unique exact functor

$$F_\Sigma : \mathbf{D}(U)[\Sigma^{-1}] \to \mathrm{Mod}_{C^{\bullet}(X)\langle * \rangle[\Sigma^{-1}]} \tag{3.2}$$

with $F_\Sigma \circ L_\Sigma \simeq F$.

Now if $\mathscr{F} \in \mathbf{D}(U)$ such that $L_\Sigma(\mathscr{F}) \simeq 0$, then also $F(\mathscr{F}) \simeq F_\Sigma(L_\Sigma(\mathscr{F})) \simeq 0$. In particular, categorical acyclicity (Theorem 2.1) yields:

Corollary 3.3 (Cohomological acyclicity). *Let $X \in \mathrm{Stk}$ with affine stabilizers and $\Sigma \subseteq \mathrm{Pic}(X)$ a subset of line bundles. If $U \subseteq X$ is a Σ-admissible quasi-compact open, then we have*

$$C^{\bullet}(X; j_! \mathscr{F})\langle * \rangle[\Sigma^{-1}] \simeq 0$$

for all $\mathscr{F} \in \mathbf{D}(U)$, where $j : U \to X$ denotes the inclusion. In particular, we have $C^{\bullet}(U; \mathscr{F})\langle * \rangle[\Sigma^{-1}] \simeq 0$.

For the second statement, apply the first with X replaced by U (and Σ by $j^* \Sigma$, j by the identity).

Corollary 3.4 (Cohomological concentration)**.** *Let $X \in$ Stk with affine stabilizers and $\Sigma \subseteq \mathrm{Pic}(X)$ a subset of line bundles. Let $i : Z \to X$ be a closed immersion such that $X \smallsetminus Z$ is Σ-admissible and quasi-compact. Then for every $\mathscr{F} \in \mathbf{D}(X)$, the unit $\mathscr{F} \to i_* i^* \mathscr{F}$ and counit $i_* i^! \mathscr{F} \to \mathscr{F}$ induce isomorphisms*

$$\mathrm{C}^\bullet(X; \mathscr{F})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(Z; i^* \mathscr{F})\langle * \rangle[\Sigma^{-1}] \tag{3.5}$$

$$\mathrm{C}^\bullet(Z; i^! \mathscr{F})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(X; \mathscr{F})\langle * \rangle[\Sigma^{-1}]. \tag{3.6}$$

Proof. Consider the localization triangle

$$\mathrm{C}^\bullet(X; j_! j^* \mathscr{F})\langle * \rangle \to \mathrm{C}^\bullet(X; \mathscr{F})\langle * \rangle \to \mathrm{C}^\bullet(X; i_* i^* \mathscr{F})\langle * \rangle$$

where j is the inclusion of $U = X \smallsetminus Z$. It remains exact after applying the exact functor $(-)[\Sigma^{-1}]$. Since U is Σ-admissible and quasi-compact, the left-most term vanishes after localization by Corollary 3.3. This shows the first isomorphism (3.5).

Similarly, we have the localization triangle

$$\mathrm{C}^\bullet(X; i_* i^! \mathscr{F})\langle * \rangle \to \mathrm{C}^\bullet(X; \mathscr{F})\langle * \rangle \to \mathrm{C}^\bullet(X; j_* j^* \mathscr{F})\langle * \rangle$$

where the right-most term $\mathrm{C}^\bullet(X; j_* j^* \mathscr{F})\langle * \rangle \simeq \mathrm{C}^\bullet(U; j^* \mathscr{F})\langle * \rangle$ vanishes after localization by Corollary 3.3. □

Remark 3.7. The second isomorphism (3.6) recovers concentration in Borel–Moore homology (as in [AKLPR]): taking $\mathscr{F} = a_X^!(\Lambda)$ where $a_X : X \to \mathrm{Spec}(k)$ is the projection and $\Lambda \in \mathbf{D}(\mathrm{Spec}(k))$, we get the isomorphism

$$i_* : \mathrm{C}_\bullet^{\mathrm{BM}}(Z; \Lambda)\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}_\bullet^{\mathrm{BM}}(X; \Lambda)\langle * \rangle[\Sigma^{-1}]$$

where $\mathrm{C}_\bullet^{\mathrm{BM}}(X; \Lambda) \coloneqq \mathrm{C}^\bullet(X; a_X^!(\Lambda))$ and similarly for Z.

Variant 3.8. We can replace all occurrences of $\mathrm{C}^\bullet(X; -) = \mathrm{Maps}(\mathbf{1}_X, -)$ above by $a_{X,*}(-)$, where $a_X : X \to \mathrm{Spec}(k)$ is the projection. In other words, we can stop short of taking derived global sections $R\Gamma(\mathrm{Spec}(k), -) : \mathbf{D}(\mathrm{Spec}(k)) \to \mathrm{Spt}$. Thus for example when $\mathbf{D}$ is the weave of motives (or cobordism motives), we get isomorphisms

$$i^* : \mathrm{M}(X)[\Sigma^{-1}] \to \mathrm{M}(Z)[\Sigma^{-1}],$$

$$i_* : \mathrm{M}^{\mathrm{BM}}(Z)[\Sigma^{-1}] \to \mathrm{M}^{\mathrm{BM}}(X)[\Sigma^{-1}]$$

under the assumptions of Corollary 3.4, where the notation is as in (0.8).

4. Categorical localization formula

In this section we prove a categorification of the Atiyah–Bott–Berline–Vergne localization formula (see Theorem 4.5). This is a warm-up for the virtual version, to be proven in Sect. 6.

We fix a diagonalizable group scheme T over k and assume k has no nontrivial idempotents. We denote by $\Sigma \subseteq \mathrm{Pic}(BT)$ the set of all nontrivial line bundles; by abuse of notation, we also write Σ for the set $f^* \Sigma \subseteq \mathrm{Pic}(\mathcal{X})$ for any $f : \mathcal{X} \to BT$.

4.1. **Invertibility of Euler classes.** We will use the following corollary of categorical concentration (Theorem 2.5), which follows just as in [AKLPR, Thm. 3.1]:

Theorem 4.1 (T-Equivariant categorical concentration). *Let $X \in \mathrm{Stk}$ be quasi-compact and $Z \to X$ a T-equivariant closed immersion such that for every geometric point $x \in X \smallsetminus Z$ we have[12] $\mathrm{St}_X^T(x) \subsetneq T_x$. Denote by $i : \mathcal{Z} \to \mathcal{X}$ the induced morphism between $\mathcal{Z} \coloneqq [Z/T]$ and $\mathcal{X} \coloneqq [X/T]$. Then the functor*

$$i_* : \mathbf{D}(\mathcal{Z})[\Sigma^{-1}] \to \mathbf{D}(\mathcal{X})[\Sigma^{-1}]$$

is an equivalence.

Suppose now that X and Z are *smooth*, or more generally that the inclusion $Z \to X$ is quasi-smooth (i.e., the relative cotangent complex $\mathscr{L}_{Z/X}$ lies in $\mathbf{D}_{\mathrm{perf}}^{[-1,-1]}(Z)$). In this case there is a canonical natural transformation

$$\mathrm{tr}_i : i_* i^* \langle -c \rangle \to \mathrm{id} \tag{4.2}$$

called the *trace* (see [DJK, 4.3.1] and [Kha1, Rem. 3.8]), where c is the rank of $\mathscr{L}_{Z/X}[-1]$. Recall also the Euler transformation

$$\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}} : \mathrm{id} \to \langle c \rangle \tag{4.3}$$

associated with the locally free sheaf $\mathscr{N}_{\mathcal{Z}/\mathcal{X}} = \mathscr{L}_{\mathcal{Z}/\mathcal{X}}[-1]$, which can be described as the composite

$$\mathrm{id} \simeq \pi_! \pi^* \langle c \rangle \xrightarrow{\mathrm{unit}} \pi_! 0_! 0^* \pi^* \langle c \rangle \simeq \langle c \rangle$$

where $\pi : \mathbf{V}_{\mathcal{Z}}(\mathscr{N}_{\mathcal{Z}/\mathcal{X}}) \to \mathcal{Z}$ is the projection of the normal bundle and the isomorphism comes from homotopy invariance and purity.

By (1.11), tr_i and $\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}}$ descend to natural transformations

$$\mathrm{tr}_i^\Sigma : i_*^\Sigma i_\Sigma^* \langle -c \rangle \to \mathrm{id}_{\mathbf{D}(\mathcal{X})[\Sigma^{-1}]}, \qquad \mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}}^\Sigma : \mathrm{id}_{\mathbf{D}(\mathcal{Z})[\Sigma^{-1}]} \to \langle c \rangle.$$

When Z has finite stabilizers and trivial T-action, the following implies that $\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}}^\Sigma$ is invertible as long as $\mathscr{N}_{\mathcal{Z}/\mathcal{X}}$ has no fixed part (see [AKLPR, Def. 5.2] for the definitions of the *fixed* and *moving* parts of a complex on $\mathcal{Z} \simeq Z \times BT$).

Corollary 4.4. *Let $X \in \mathrm{Stk}$ be quasi-compact and let $\mathscr{E}$ be a connective coherent complex on $\mathcal{X} \coloneqq X \times BT$, i.e., a coherent complex on X that is equivariant with respect to the trivial T-action. If X has finite stabilizers and $\mathscr{E}$ has no fixed part, then the zero section of the derived cone $\mathcal{E} \coloneqq \mathbf{V}_{\mathcal{X}}(\mathscr{E}) \to \mathcal{X}$ induces a canonical equivalence*

$$0_* : \mathbf{D}(\mathcal{X})[\Sigma^{-1}] \to \mathbf{D}(\mathcal{E})[\Sigma^{-1}].$$

Proof. By derived invariance, we may assume that $\mathscr{E}$ is 0-truncated (i.e., a coherent sheaf). Then [AKLPR, Prop. 3.8] shows that the conditions of Theorem 4.1 are satisfied under our assumptions. □

---

[12]Here T_x is the fibre $T \otimes_k k(x)$ and $\mathrm{St}_X^T(x)$ is the T-*stabilizer* at a geometric point $x \in X$, defined as the image of the map $\alpha : \underline{\mathrm{Aut}}_{\mathcal{X}}(x) \to T_x$ induced by the T-action. See [AKLPR, Def. A.4].

4.2. **Categorical localization formula.** When i is quasi-smooth, the "categorical localization formula" computes the inverses of the isomorphism $\mathrm{id} \to i_*^\Sigma i_\Sigma^*$ in terms of the trace:

Theorem 4.5 (Categorical localization formula)**.** *Let $X \in \mathrm{Stk}$ be and $Z \to X$ a T-equivariant closed immersion such that $\mathrm{St}_X^T(x) \subsetneq T_x$ for every geometric point $x \in X \smallsetminus Z$, and such that the T-action on Z is trivial. Denote by $i : \mathcal{Z} \to \mathcal{X}$ the induced morphism between $\mathcal{Z} := [Z/T] \simeq Z \times BT$ and $\mathcal{X} := [X/T]$. If $Z \to X$ is quasi-smooth, Z has finite stabilizers, and $\mathscr{N}_{\mathcal{Z}/\mathcal{X}}$ has no fixed part, then the natural transformation of endofunctors of $\mathbf{D}(\mathcal{X})[\Sigma^{-1}]$*

$$i_*^\Sigma i_\Sigma^* \xrightarrow{\mathrm{eul}^{-1}} i_*^\Sigma i_\Sigma^* \langle -c \rangle \xrightarrow{\mathrm{tr}_i^\Sigma} \mathrm{id},$$

where eul^{-1} is an inverse of $\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}}^\Sigma$, is an inverse to the unit $\mathrm{id} \to i_*^\Sigma i_\Sigma^*$.

In view of the invertibility of $\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}}$ on $\mathbf{D}(\mathcal{Z})[\Sigma^{-1}]$, Theorem 4.5 is a consequence of the following categorification of the self-intersection formula:

Proposition 4.6 (Categorical self-intersection formula)**.** *Let $i : \mathcal{Z} \to \mathcal{X}$ be a quasi-smooth closed immersion in* Stk*. Then the composite natural transformation*

$$i_* i^* \langle -c \rangle \xrightarrow{\mathrm{tr}_i} \mathrm{id} \xrightarrow{\mathrm{unit}} i_* i^*$$

is canonically identified with $\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}} : i_* i^* \langle -c \rangle \to i_* i^*$.

This statement is easily derived from [Kha1, Cor. 3.17]; see also Corollary 6.4 below.

Corollary 4.7. *For i as in Proposition 4.6, the composite*

$$i^* \langle -c \rangle \xrightarrow{i^* \mathrm{unit}} i^* i_* i^* \langle -c \rangle \xrightarrow{i^* \mathrm{tr}_i} i^*$$

is canonically identified with $\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}} : i^* \langle -c \rangle \to i^*$.

Proof. By the triangle identities, it will suffice to identify the composite

$$i^* \langle -c \rangle \xrightarrow{i^* \mathrm{unit}} i^* i_* i^* \langle -c \rangle \xrightarrow{i^* \mathrm{tr}_i} i^* \xrightarrow{i^* \mathrm{unit}} i^* i_* i^* \xrightarrow{\mathrm{counit}(i^*)} i^*$$

since the last two arrows collapse to the identity. By Proposition 4.6 this is identified with the total composite in the commutative diagram below:

$$
\begin{array}{ccccc}
i^* \langle -c \rangle & \xrightarrow{i^* \mathrm{unit}} & i^* i_* i^* \langle -c \rangle & \xrightarrow{\mathrm{eul}} & i^* i_* i^* \\
& \diagdown\diagdown & \downarrow {\scriptstyle \mathrm{counit}(i^*)} & & \downarrow {\scriptstyle \mathrm{counit}(i^*)} \\
& & i^* \langle -c \rangle & \xrightarrow{\mathrm{eul}} & i^*,
\end{array}
$$

whence the claim. $\qquad\square$

Proof of Theorem 4.5. All operations are considered on $\mathbf{D}(-)[\Sigma^{-1}]$ and we omit the decorations "Σ" from the notation. Proposition 4.6 yields an identification between the composite

$$i_* i^* \xrightarrow{\mathrm{eul}^{-1}} i_* i^* \langle -c \rangle \xrightarrow{\mathrm{tr}_i} \mathrm{id} \xrightarrow{\mathrm{unit}} i_* i^*$$

and the identity. The other composite

$$\mathrm{id} \xrightarrow{\mathrm{unit}} i_* i^* \xrightarrow{\mathrm{eul}^{-1}} i_* i^* \langle -c \rangle \xrightarrow{\mathrm{tr}_i} \mathrm{id},$$

which is equivalent to

$$\mathrm{id} \xrightarrow{\mathrm{eul}^{-1}} \langle -c \rangle \xrightarrow{\mathrm{unit}} i_* i^* \langle -c \rangle \xrightarrow{\mathrm{tr}_i} \mathrm{id},$$

is identity after applying i^* on the left by Corollary 4.7. Since i^* is an equivalence (Theorem 4.1), the claim follows. $\square$

5. VIRTUAL EULER AND TRACE

Our next goal is to extend the categorical localization formula (Theorem 4.5) to the case of *quasi*-smooth stacks. Since the closed immersion $i : \mathcal{Z} \to \mathcal{X}$ need no longer be quasi-smooth, we begin in this section by making sense of the trace tr_i as well as the Euler transformation $\mathrm{eul}_{\mathcal{N}_{\mathcal{Z}/\mathcal{X}}}$ associated with the perfect complex

$$\mathcal{N}_{\mathcal{Z}/\mathcal{X}} = \mathcal{L}_{\mathcal{Z}/\mathcal{X}}[-1] \in \mathbf{D}_{\mathrm{perf}}^{[-1,0]}(\mathcal{X}).$$

We maintain the notation T and $\Sigma \subseteq \mathrm{Pic}(BT)$ of the previous section. Throught this section and in the rest of this paper we work under the assumption that all our stacks have affine stabilizers.

5.1. **Quasi-smooth bundles.** Given a perfect complex $\mathscr{E} \in \mathbf{D}_{\mathrm{perf}}(X)$, the associated "generalized vector bundle" $\pi : E = \mathbf{V}_X(\mathscr{E}) \to X$ is the derived stack with functor of points

$$(T \xrightarrow{t} X) \mapsto \mathrm{Maps}_{\mathbf{D}_{\mathrm{perf}}(X)}(t^* \mathscr{E}, \mathscr{O}_T).$$

In Sect. 4 we made use of the fact that when $\mathscr{E}$ is of Tor-amplitude $\leqslant 0$ and virtual rank r, the projection π is smooth so that we have an invertible natural transformation

$$\mathrm{tr}_\pi : \pi_! \pi^* \langle r \rangle \to \mathrm{id},$$

see [Kha1, Prop. A.10, Thm. A.13, Constr. A.16].

Now suppose $\mathscr{E}$ is a perfect complex on $\mathcal{X} \coloneqq X \times BT$, i.e., a perfect complex on X which is equivariant with respect to the trivial T-action. Suppose that $\mathscr{E}$ is of Tor-amplitude $\leqslant 1$, so that $\pi : \mathbf{V}_{\mathcal{X}}(\mathscr{E}) \to \mathcal{X}$ is only *quasi*-smooth. We still have the trace (see [Kha1, §3.1, Rem. 3.8])

$$\mathrm{tr}_\pi : \pi_! \pi^* \langle r \rangle \to \mathrm{id}$$

but it is typically not invertible. However, we will show that this becomes true Σ-locally, as long as the fixed part $\mathscr{E}^{\mathrm{fix}}$ is of Tor-amplitude $\leqslant 0$.

Proposition 5.1. *Let $X \in \mathrm{Stk}$ be quasi-compact and $\mathcal{X} = X \times BT$. Let $\mathscr{E} \in \mathbf{D}_{\mathrm{perf}}^{\geqslant -1}(\mathcal{X})$ of virtual rank r with $\mathscr{E}^{\mathrm{fix}} \in \mathbf{D}_{\mathrm{perf}}^{\geqslant 0}(\mathcal{X})$. If X has finite stabilizers, then the trace of $\pi : \mathbf{V}_{\mathcal{X}}(\mathscr{E}) \to \mathcal{X}$ induces an isomorphism*

$$\mathrm{tr}_\pi : \pi_!^\Sigma \pi_\Sigma^* \langle r \rangle \to \mathrm{id}$$

of endofunctors of $\mathbf{D}(\mathcal{X})[\Sigma^{-1}]$.

Proof. Note that $\pi : \mathcal{E} := \mathbf{V}_{\mathcal{X}}(\mathscr{E}) \to \mathcal{X}$ factors through $\pi^{\mathrm{mov}} : \mathcal{E}^{\mathrm{mov}} \to \mathcal{X}$ and $\mathcal{E} \to \mathcal{E}^{\mathrm{mov}}$, which is a torsor under the vector bundle stack $\pi^{\mathrm{fix}} : \mathcal{E}^{\mathrm{fix}} \to \mathcal{X}$. By homotopy invariance for vector bundle stacks [Kha1, Prop. A.10] we may replace $\mathscr{E}$ by $\mathscr{E}^{\mathrm{mov}}$ and assume that $\mathscr{E}$ has no fixed part.

Using the localization triangle and stratifying X by global quotient stacks, we may assume that X has the resolution property. Arguing as in the proof of [Kha1, Prop. A.10] by induction on the Tor-amplitude of $\mathscr{E}$, we reduce to the case where

$$\mathscr{E} = \mathrm{Cofib}(\mathscr{E}^{-1} \to \mathscr{E}^0) \in \mathbf{D}_{\mathrm{perf}}^{[-1,0]}(\mathcal{X})$$

with $\mathscr{E}^{-1}, \mathscr{E}^0 \in \mathbf{D}_{\mathrm{perf}}^{[0,0]}(\mathcal{X})$. We have a commutative diagram

$$(5.2)$$

where $\mathcal{E}_0 = \mathbf{V}_{\mathcal{X}}(\mathscr{E}^0)$ and $\mathcal{E}_1 = \mathbf{V}_{\mathcal{X}}(\mathscr{E}^{-1})$ and the square is homotopy cartesian.

All operations below are considered on $\mathbf{D}(-)[\Sigma^{-1}]$ and we omit the decorations "Σ". Since $\mathscr{E}_1$ has no fixed part and X has finite stabilizers, $0_{\mathcal{E}_1}^*$ and $0_{\mathcal{E}_1,!}$ are equivalences (Corollary 4.4), so it will suffice to show that

$$\mathrm{tr}_\pi : 0_{\mathcal{E}_1,!}\pi_!\pi^*\langle r \rangle \to 0_{\mathcal{E}_1,!}$$

is invertible. Under the base change isomorphism, this is identified with (see [DJK, Cor. 2.5.6])

$$\mathrm{tr}_p : p_!p^!0_{\mathcal{E}_1,!}\langle r \rangle \to 0_{\mathcal{E}_1,!}.$$

Thus it is sufficient to show that $\mathrm{tr}_p : p_!p^*\langle r \rangle \to \mathrm{id}$ is invertible.

Since $0_{\mathcal{E}_1}^*$ and $0_{\mathcal{E}_1,!}$ are equivalences, the same holds for $\pi_{\mathcal{E}_1}^*$ and $\pi_{\mathcal{E}_1,!}$ by functoriality. Thus it will suffice to show that

$$\pi_{\mathcal{E}_0,!}\pi_{\mathcal{E}_0}^*\langle r \rangle \simeq \pi_{\mathcal{E}_1,!}p_!p^*\pi_{\mathcal{E}_1}^*\langle r \rangle \xrightarrow{\mathrm{tr}_p} \pi_{\mathcal{E}_1,!}\pi_{\mathcal{E}_1}^*$$

is invertible. By functoriality of the trace, this follows from the invertibility of $\mathrm{tr}_{\mathcal{E}_0}$ and $\mathrm{tr}_{\mathcal{E}_1}$ (which holds since $\mathscr{E}_0$ and $\mathscr{E}_1$ are of Tor-amplitude $[0,0]$, see the discussion above). $\square$

5.2. **Virtual Euler and trace.**

We use Proposition 5.1 to define generalizations of the Euler and trace transformations at the level of localized equivariant derived categories.

Construction 5.3 (Euler transformation). Let $X \in \mathrm{Stk}$ be quasi-compact and write $\mathcal{X} = X \times BT$. Let $\mathscr{E} \in \mathbf{D}_{\mathrm{perf}}^{[-1,0]}(\mathcal{X})$ of virtual rank r, with $\mathscr{E}^{\mathrm{fix}} \in \mathbf{D}_{\mathrm{perf}}^{[0,0]}(\mathcal{X})$. The *Euler transformation* associated with $\mathscr{E}$ is the natural transformation

$$\mathrm{eul}_{\mathscr{E}}^{\Sigma} : \mathrm{id}_{\mathbf{D}(\mathcal{X})[\Sigma^{-1}]} \to \langle r \rangle$$

of endofunctors of $\mathbf{D}(\mathfrak{X})[\Sigma^{-1}]$, defined as follows. Denote by $\pi : \mathbf{V}_{\mathfrak{X}}(\mathscr{E}) \to \mathfrak{X}$ the projection and by $0 : \mathfrak{X} \to \mathbf{V}_{\mathfrak{X}}(\mathscr{E})$ the zero section. Note that the latter is a closed immersion since $\mathscr{E} \in \mathbf{D}_{\mathrm{perf}}^{\leq 0}(\mathfrak{X})$, so that the unit of the adjunction $(0_!^{\Sigma}, 0_{\Sigma}^*)$ gives rise to a canonical natural transformation

$$\mathrm{id} \simeq \pi_!^{\Sigma} \pi_{\Sigma}^* \langle r \rangle \xrightarrow{\mathrm{unit}} \pi_!^{\Sigma} 0_!^{\Sigma} 0_{\Sigma}^* \pi_{\Sigma}^* \langle r \rangle \simeq \langle r \rangle$$

where the first isomorphism is the inverse of the trace (Proposition 5.1).

Proposition 5.4. *In the situation of Construction 5.3, suppose moreover that $\mathscr{E}^{\mathrm{fix}} \simeq 0$. Then the unit $\mathrm{id} \to 0_!^{\Sigma} 0_{\Sigma}^*$ is invertible. In particular, the Euler transformation $\mathrm{eul}_{\mathscr{E}}^{\Sigma} : \mathrm{id}_{\mathbf{D}(\mathfrak{X})[\Sigma^{-1}]} \to \langle r \rangle$ is invertible.*

Proof. When $\mathscr{E}$ is of Tor-amplitude $[0,0]$, i.e., $\pi : E := \mathbf{V}_{\mathfrak{X}}(\mathscr{E}) \to \mathfrak{X}$ is a vector bundle, the zero section $0 : \mathfrak{X} \to E$ satisfies the assumptions of Theorem 4.1 by [AKLPR, Prop. 3.8] since $\mathscr{E}^{\mathrm{fix}} \simeq 0$ and X has finite stabilizers.

In general, we stratify X by global quotient stacks and use the localization triangle to reduce to the case where X has the resolution property. We can then write

$$\mathscr{E} \simeq \mathrm{Cofib}(\mathscr{E}^{-1} \to \mathscr{E}^0) \in \mathbf{D}_{\mathrm{perf}}^{[-1,0]}(\mathfrak{X})$$

with $\mathscr{E}^{-1}, \mathscr{E}^0 \in \mathbf{D}_{\mathrm{perf}}^{[0,0]}(\mathfrak{X})$. We adopt the notation of (5.2). Since s^* and $s_!$ are equivalences by Theorem 4.1 (we omit the decorations "Σ" throughout the proof), it will suffice to show that the lower horizontal arrow below is invertible:

$$
\begin{array}{ccc}
\mathrm{id} & \xrightarrow{\mathrm{unit}} & 0_{E_0,!} 0_{E_0}^* \\
{\scriptstyle\mathrm{unit}}\downarrow & & \| \\
s_! s^* & \xrightarrow{\mathrm{unit}} & s_! 0_! 0^* s^*
\end{array}
$$

But the lower horizontal and right-hand vertical arrows are invertible by Theorem 4.1, so the claim follows. $\qquad\square$

Construction 5.5 (Trace transformation)**.** Let $X \in \mathrm{Stk}$ be quasi-compact with T-action, and let $Z \to X$ be a T-equivariant closed immersion where the T-action on Z is trivial. Write $\mathfrak{Z} := [Z/T] \simeq Z \times BT$ and $\mathfrak{X} := [X/T]$. Assume that $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}} \in \mathbf{D}_{\mathrm{perf}}^{[-2,-1]}(\mathfrak{Z})$, $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}}^{\mathrm{fix}} \in \mathbf{D}_{\mathrm{perf}}^{[-1,-1]}(\mathfrak{Z})$, and Z has finite stabilizers. Let $i : \mathfrak{Z} \to \mathfrak{X}$ denote the inclusion, $\pi : N_{\mathfrak{Z}/\mathfrak{X}} := \mathbf{V}_{\mathfrak{Z}}(\mathscr{N}_{\mathfrak{Z}/\mathfrak{X}}) \to \mathfrak{Z}$ the derived normal bundle, and c the virtual rank of $\mathscr{N}_{\mathfrak{Z}/\mathfrak{X}} = \mathscr{L}_{\mathfrak{Z}/\mathfrak{X}}[-1]$.

Recall that there is a specialization transformation (see [Kha1, Constr. 3.1], [KR1, Constr. 8.7])

$$\mathrm{sp}_i : i_! \pi_! \pi^* i^* \to \mathrm{id}$$

defined by deformation to the derived normal bundle. It gives rise to a natural transformation sp_i^{Σ} on localized functors by (1.11). The *trace transformation*

$$\mathrm{tr}_i^{\Sigma} : i_*^{\Sigma} i_{\Sigma}^* \langle -c \rangle \to \mathrm{id} \tag{5.6}$$

of functors $\mathbf{D}(\mathfrak{X})[\Sigma^{-1}] \to \mathbf{D}(\mathfrak{X})[\Sigma^{-1}]$ is defined as the composite

$$i_*^{\Sigma} i_{\Sigma}^* \langle -c \rangle \simeq i_*^{\Sigma} \pi_!^{\Sigma} \pi_{\Sigma}^* i_{\Sigma}^* \xrightarrow{\mathrm{sp}_i^{\Sigma}} \mathrm{id}$$

where the isomorphism $\pi_!^{\Sigma}\pi_{\Sigma}^* \simeq \langle -c \rangle$ is the inverse of tr_π (Proposition 5.1).

Remark 5.7. It is immediate from the constructions that if $\mathscr{E} \in \mathbf{D}_{\mathrm{perf}}^{[-1,0]}(\mathfrak{X})$ in fact belongs to $\mathbf{D}_{\mathrm{perf}}^{[0,0]}$, then the Euler transformation $\mathrm{eul}_{\mathscr{E}} : \mathrm{id} \to \langle r \rangle$ on $\mathbf{D}(\mathfrak{X})[\Sigma^{-1}]$ (Construction 5.3) is the restriction of the usual one on $\mathbf{D}(\mathfrak{X})$. Similarly, if $i : \mathfrak{Z} \to \mathfrak{X}$ is in fact quasi-smooth (i.e., $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}} \in \mathbf{D}_{\mathrm{perf}}^{[-2,-1]}(\mathfrak{Z})$ belongs to $\mathbf{D}_{\mathrm{perf}}^{[-1,-1]}(\mathfrak{Z})$), then the trace transformations on Σ-localized categories are the restrictions of the usual ones on unlocalized categories.

6. Categorical virtual localization formula

We now prove the categorical virtual localization formula, extending Theorem 4.5 to the case of quasi-smooth stacks.

Let $X \in \mathrm{Stk}$ be quasi-compact with T-action, and let $Z \to X$ be a T-equivariant closed immersion such that Z contains all T-fixed points (that is, for every geometric point $x \in X \setminus Z$ we have $\mathrm{St}_X^T(x) \neq T_x$). Denote by $i : \mathfrak{Z} \to \mathfrak{X}$ the induced morphism between $\mathfrak{Z} := [Z/T]$ and $\mathfrak{X} := [X/T]$. The categorical concentration theorem (Theorem 4.1) implies that the unit $\mathrm{id} \to i_* L_\Sigma i^*$ and counit $i_* i^! \to \mathrm{id}$ are invertible on $\mathbf{D}(\mathfrak{X})[\Sigma^{-1}]$. The "categorical localization formula" computes their inverses in terms of the trace (Construction 5.5), in the situation where the latter is defined:

Theorem 6.1 (Categorical virtual localization formula). *Let $X \in \mathrm{Stk}$ be quasi-compact with T-action, and let $Z \to X$ be a T-equivariant closed immersion such that $\mathrm{St}_X^T(x) \subsetneq T_x$ for every geometric point $x \in X \setminus Z$, and such that the T-action on Z is trivial. Denote by $i : \mathfrak{Z} \to \mathfrak{X}$ the induced morphism between $\mathfrak{Z} := [Z/T] \simeq Z \times BT$ and $\mathfrak{X} := [X/T]$. Assume that $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}} \in \mathbf{D}_{\mathrm{perf}}^{[-2,-1]}(\mathfrak{Z})$, $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}}^{\mathrm{fix}} \simeq 0$, and Z has finite stabilizers. Then the natural transformation of endofunctors of $\mathbf{D}(\mathfrak{X})[\Sigma^{-1}]$*

$$i_*^{\Sigma} i_{\Sigma}^* \xrightarrow{\ \mathrm{eul}^{-1}\ } i_*^{\Sigma} i_{\Sigma}^* \langle -c \rangle \xrightarrow{\ \mathrm{tr}_i^{\Sigma}\ } \mathrm{id},$$

where eul^{-1} is an inverse of $\mathrm{eul}_{\mathscr{N}_{\mathfrak{Z}/\mathfrak{X}}}^{\Sigma}$, is an inverse to the unit $\mathrm{id} \to i_*^{\Sigma} i_{\Sigma}^*$.

The proof of Theorem 6.1 will require the following categorifications of Proposition 1.15 and Corollary 1.16 in [AKLPR]. A closed immersion $i : Z \to X$ is *homotopically smooth* if its conormal complex $\mathscr{N}_{Z/X} = \mathscr{L}_{Z/X}[-1]$ is perfect; its derived normal bundle is $N_{Z/X} := \mathbf{V}_Z(\mathscr{N}_{Z/X}) \to X$.

Proposition 6.2. *Suppose given a commutative square*

$$\begin{array}{ccc} Z' & \xrightarrow{\ i'\ } & X' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow{\ i\ } & X \end{array}$$

in Stk *where i and i' are homotopically smooth closed immersions, q is proper, and the square is cartesian on classical truncations. Denote by*

$\pi' : N_{Z'/X'} \to Z'$, $\pi : N_{Z/X} \to Z$ the derived normal bundles to i' and i, respectively, and by $N_\Delta : N_{Z'/X'} \to N_{Z/X}$ the induced (proper) morphism. Then the diagram

$$
\begin{array}{ccc}
i_! \pi_! \pi^* i^* & \xrightarrow{\quad \mathrm{sp}_i \quad} & \mathrm{id} \\
\scriptstyle{\mathrm{unit}} \downarrow & & \downarrow \scriptstyle{\mathrm{unit}} \\
i_! \pi_! N_{\Delta,!} N_\Delta^* \pi^* i^* \;=\!=\; q_! i'_! \pi'_! \pi'^* i'^* q^* & \xrightarrow{\quad \mathrm{sp}_{i'} \quad} & q_! q^*,
\end{array}
$$

where the lower isomorphism is by functoriality, commutes up to canonical homotopy.

Proof. The assumptions imply that the induced morphism $D_\Delta : D_{Z'/X'} \to D_{Z/X}$ on the derived deformations to the normal bundle is proper. The commutative diagram of complementary closed/open immersions

$$
\begin{array}{ccccc}
N_{Z'/X'} & \xrightarrow{\;u'\;} & D_{Z'/X'} & \xleftarrow{\;v'\;} & X' \times \mathbf{G}_m \\
\downarrow \scriptstyle{N_\Delta} & & \downarrow \scriptstyle{D_\Delta} & & \downarrow \scriptstyle{q \times \mathrm{id}} \\
N_{Z/X} & \xrightarrow{\;u\;} & D_{Z/X} & \xleftarrow{\;v\;} & X \times \mathbf{G}_m
\end{array}
$$

gives rise to the commutative diagram

$$
\begin{array}{ccccc}
v_! v^* & \longrightarrow & \mathrm{id} & \longrightarrow & u_! u^* \\
\downarrow & & \downarrow & & \downarrow \\
v_! (q \times \mathrm{id})_! (q \times \mathrm{id})^* v^* & \longrightarrow & D_{\Delta,!} D_\Delta^* & \longrightarrow & u_! N_{\Delta,!} N_\Delta^* u^* \\
\| & & \| & & \| \\
D_{\Delta,!} v'_! v'^* D_\Delta^* & \longrightarrow & D_{\Delta,!} D_\Delta^* & \longrightarrow & D_{\Delta,!} u'_! u'^* D_\Delta^*
\end{array}
$$

where the rows are localization triangles and the upper vertical arrows are unit transformations. Now the claim follows by unravelling the definition of the specialization transformation. $\square$

Corollary 6.3. *Let $i : Z \to X$ be a homotopically smooth closed immersion in* Stk. *Then the diagram*

$$
\begin{array}{ccc}
i_! \pi_! \pi^* i^* & \xrightarrow{\quad \mathrm{unit} \quad} & i_! \pi_! 0_! 0^* \pi^* i^* \\
\| & & \| \\
i_! \pi_! \pi^* i^* & \xrightarrow{\;\mathrm{sp}_i\;} \mathrm{id} \xrightarrow{\;\mathrm{unit}\;} & i_! i^*
\end{array}
$$

commutes up to canonical homotopy, where $0 : X \to N_{Z/X}$ denotes the zero section to the derived normal bundle $\pi : N_{Z/X} \to X$.

Proof. The self-intersection square

$$
\begin{array}{ccc}
Z & =\!\!=\!\!= & Z \\
\| & & \downarrow \scriptstyle{i} \\
Z & \xrightarrow{\;i\;} & X
\end{array}
$$

satisfies the assumptions of Proposition 6.2. $\square$

Corollary 6.4 (Categorical virtual self-intersection formula)**.** *Let $X \in \mathrm{Stk}$ be quasi-compact with T-action, and let $Z \to X$ be a T-equivariant closed immersion such that the T-action on Z is trivial. Denote by $i : \mathcal{Z} \to \mathcal{X}$ the induced morphism between $\mathcal{Z} := [Z/T] \simeq Z \times BT$ and $\mathcal{X} := [X/T]$. Assume that $\mathscr{L}_{\mathcal{Z}/\mathcal{X}} \in \mathbf{D}_{\mathrm{perf}}^{[-2,-1]}(\mathcal{Z})$, $\mathscr{L}_{\mathcal{Z}/\mathcal{X}}^{\mathrm{fix}} \in \mathbf{D}_{\mathrm{perf}}^{[-1,-1]}(\mathcal{Z})$, and Z has finite stabilizers. Then the composite natural transformation*

$$i_*^{\Sigma} i_{\Sigma}^* \langle -c \rangle \xrightarrow{\mathrm{tr}_i^{\Sigma}} \mathrm{id} \xrightarrow{\mathrm{unit}} i_*^{\Sigma} i_{\Sigma}^*$$

of endofunctors of $\mathbf{D}(\mathcal{X})[\Sigma^{-1}]$, is canonically identified with

$$\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}}^{\Sigma} : i_*^{\Sigma} i_{\Sigma}^* \langle -c \rangle \to i_*^{\Sigma} i_{\Sigma}^*.$$

Proof. Consider the following diagram of endofunctors of $\mathbf{D}(\mathcal{X})[\Sigma^{-1}]$

$$
\begin{array}{ccccc}
i_* i^! \langle -c \rangle & \xleftarrow{\mathrm{tr}_\pi} & i_* \pi_* \pi^! i^! & \xrightarrow{\mathrm{unit}} & i_* \pi_* 0_* 0^! \pi^! i^! \\
\| & & \| & & \| \\
i_* i^! \langle -c \rangle & \xleftarrow{\mathrm{tr}_\pi} & i_* \pi_* \pi^! i^! & \xrightarrow{\mathrm{sp}_{\mathcal{Z}/\mathcal{X}}} \mathrm{id} \xrightarrow{\mathrm{unit}} & i_* i^!,
\end{array}
$$

where we omit the decorations "Σ". The right-hand square commutes by Corollary 6.3. By construction of tr_i and $\mathrm{eul}_{\mathscr{N}_{\mathcal{Z}/\mathcal{X}}}$, the commutativity of the outer composite rectangle yields the identification desired. $\qquad\square$

Proof of Theorem 6.1. Repeat the proof of Theorem 4.5, substituting in Corollary 6.4 for Proposition 4.6. $\qquad\square$

7. Cohomological localization and integration formulas

We now derive from Theorem 6.1 the localization formula in cohomology and Borel–Moore homology. The latter recovers [AKLPR, Cor. 5.27].

We begin by defining Euler classes and Gysin maps. Recall that for every $\mathcal{X}$ over BT we have a canonical functor (3.2)

$$F_\Sigma : \mathbf{D}(\mathcal{X})[\Sigma^{-1}] \to \mathrm{Mod}_{\mathrm{C}^\bullet(\mathcal{X})\langle *\rangle[\Sigma^{-1}]} \tag{7.1}$$

with the property that $F_\Sigma(L_\Sigma \mathscr{F}) \simeq \mathrm{C}^\bullet(\mathcal{X}; \mathscr{F})\langle *\rangle[\Sigma^{-1}]$ for any $\mathscr{F} \in \mathbf{D}(\mathcal{X})$. Dually, for any fixed $\mathscr{G} \in \mathbf{D}(\mathcal{X})$ there is a canonical functor

$$G_\Sigma^{\mathscr{G}} : \mathbf{D}(\mathcal{X})[\Sigma^{-1}] \to (\mathrm{Mod}_{\mathrm{C}^\bullet(\mathcal{X})\langle *\rangle[\Sigma^{-1}]})^{\mathrm{op}} \tag{7.2}$$

with $G_\Sigma^{\mathscr{G}}(L_\Sigma \mathscr{F}) \simeq \mathrm{Maps}_{\mathbf{D}(\mathcal{X})}(\mathscr{F}, \mathscr{G}\langle *\rangle)[\Sigma^{-1}]$.

Construction 7.3 (Euler class)**.** Let $X \in \mathrm{Stk}$ be quasi-compact with finite stabilizers. Suppose given a perfect complex

$$\mathscr{E} \in \mathbf{D}_{\mathrm{perf}}^{[-1,0]}(\mathcal{X}) \quad \text{with } \mathscr{E}^{\mathrm{fix}} \in \mathbf{D}_{\mathrm{perf}}^{[0,0]}(\mathcal{X})$$

on $\mathcal{X} := X \times BT$. The Euler transformation $\mathrm{eul}_{\mathscr{E}}^{\Sigma}$ (Construction 5.3) gives rise to a morphism

$$\mathrm{eul}_{\mathscr{E}} : L_\Sigma \mathbf{1}_{\mathcal{X}} \to L_\Sigma \mathbf{1}_{\mathcal{X}} \langle r \rangle,$$

where r is the virtual rank of $\mathscr{E}$. Applying F_Σ yields a canonical morphism of the form

$$\mathrm{C}^\bullet(\mathfrak{X}; \mathbf{1}_\mathfrak{X})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(\mathfrak{X}; \mathbf{1}_\mathfrak{X})\langle * \rangle[\Sigma^{-1}]$$

which by $\mathrm{C}^\bullet(\mathfrak{X}; \mathbf{1}_\mathfrak{X})\langle * \rangle[\Sigma^{-1}]$-linearity amounts to an element

$$e(\mathscr{E}) \in \mathrm{C}^\bullet_T(X)\langle r \rangle[\Sigma^{-1}] \simeq \mathrm{C}^\bullet(\mathfrak{X})\langle r \rangle[\Sigma^{-1}], \tag{7.4}$$

we call the *Euler class* (or *top Chern class*). If $\mathscr{E}^{\mathrm{fix}} \simeq 0$, then $\mathrm{eul}_{\mathscr{E}}^\Sigma$ is invertible (Proposition 5.4), hence $e(\mathscr{E}) \in \mathrm{C}^\bullet(\mathfrak{X})\langle * \rangle[\Sigma^{-1}]$ is invertible. By cap product, $e(\mathscr{E})$ gives rise for every $\mathrm{C}^\bullet(\mathfrak{X})\langle * \rangle$-module spectrum M to a map

$$e(\mathscr{E}) : M[\Sigma^{-1}] \to M[\Sigma^{-1}],$$

invertible when $\mathscr{E}^{\mathrm{fix}} \simeq 0$.

Construction 7.5 (Gysin maps)**.** Let $X \in \mathrm{Stk}$ be quasi-compact with T-action, and let $Z \to X$ be a T-equivariant closed immersion where the T-action on Z is trivial. Write $\mathfrak{Z} := [Z/T] \simeq Z \times BT$ and $\mathfrak{X} := [X/T]$. Assume that $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}} \in \mathbf{D}_{\mathrm{perf}}(\mathfrak{Z})^{[-2,-1]}$, $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}}^{\mathrm{fix}} \in \mathbf{D}_{\mathrm{perf}}(\mathfrak{Z})^{[-1,-1]}$, and Z has finite stabilizers. Let $i : \mathfrak{Z} \to \mathfrak{X}$ denote the inclusion and let c be the virtual rank of $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}}[-1]$.

(i) For any object $\mathscr{F} \in \mathbf{D}(\mathfrak{X})$, consider the morphism

$$L_\Sigma i_! i^* \mathscr{F} \langle -c \rangle \simeq i_!^\Sigma i_\Sigma^* L_\Sigma \mathscr{F} \langle -c \rangle \xrightarrow{\mathrm{tr}_i^\Sigma} L_\Sigma \mathscr{F}$$

given by the trace transformation (5.6). Applying F_Σ (7.1) yields a canonical map of $\mathrm{C}^\bullet(\mathfrak{X})\langle * \rangle[\Sigma^{-1}]$-module spectra

$$i_! : \mathrm{C}^\bullet(\mathfrak{Z}; i^* \mathscr{F})\langle * \rangle[\Sigma^{-1}] \simeq \mathrm{C}^\bullet(\mathfrak{X}; i_! i^* \mathscr{F})\langle * \rangle[\Sigma^{-1}]$$
$$\to \mathrm{C}^\bullet(\mathfrak{X}; \mathscr{F})\langle * \rangle[\Sigma^{-1}]. \tag{7.6}$$

We call (7.6) the *Gysin push-forward* along i.

(ii) The trace (5.6) gives rise to a morphism

$$L_\Sigma i_! \mathbf{1}_\mathfrak{Z} \langle -c \rangle \to L_\Sigma \mathbf{1}_\mathfrak{X}.$$

For any $\mathscr{F} \in \mathbf{D}(\mathfrak{X})$ we may apply the functor $G_\Sigma^{\mathscr{F}}$ (7.2) to get a canonical map of $\mathrm{C}^\bullet(\mathfrak{X})\langle * \rangle[\Sigma^{-1}]$-modules

$$\mathrm{Maps}_{\mathbf{D}(\mathfrak{X})}(i_! \mathbf{1}_\mathfrak{Z} \langle -c \rangle, \mathscr{F} \langle * \rangle)[\Sigma^{-1}] \to \mathrm{Maps}_{\mathbf{D}(\mathfrak{X})}(\mathbf{1}_\mathfrak{X}, \mathscr{F} \langle * \rangle)[\Sigma^{-1}].$$

By adjunction, this amounts to a canonical $\mathrm{C}^\bullet(\mathfrak{X})\langle * \rangle[\Sigma^{-1}]$-module map

$$i^! : \mathrm{C}^\bullet(\mathfrak{X}; \mathscr{F})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(\mathfrak{Z}; i^! \mathscr{F})[\Sigma^{-1}]. \tag{7.7}$$

We think of this as a Gysin pull-back in Borel–Moore homology (compare Remark 3.7).

Corollary 7.8 (Co/homological localization formula)**.** *Let $X \in \mathrm{Stk}$ be quasi-compact with T-action, and let $Z \to X$ be a T-equivariant closed immersion such that $\mathrm{St}_X^T(x) \not\subseteq T_x$ for every geometric point $x \in X \smallsetminus Z$, and such that the T-action on Z is trivial. Denote by $i : \mathfrak{Z} \to \mathfrak{X}$ the induced morphism between $\mathfrak{Z} := [Z/T] \simeq Z \times BT$ and $\mathfrak{X} := [X/T]$. Assume that $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}} \in \mathbf{D}_{\mathrm{perf}}^{[-2,-1]}(\mathfrak{Z})$, $\mathscr{L}_{\mathfrak{Z}/\mathfrak{X}}^{\mathrm{fix}} \simeq 0$, and Z has finite stabilizers. Then for every $\mathscr{F} \in \mathbf{D}(\mathfrak{X})$ we have:*

(i) *There is a canonical identification*

$$i_! \left(- \cap e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})^{-1} \right) \simeq (i^*)^{-1}$$

of maps $\mathrm{C}^\bullet(\mathcal{Z}; \mathscr{F})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(\mathcal{X}; \mathscr{F})\langle * \rangle[\Sigma^{-1}]$.

(ii) *There is a canonical identification*

$$i^!(-) \cap e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})^{-1} \simeq (i_*)^{-1}$$

of maps $\mathrm{C}^\bullet(\mathcal{X}; \mathscr{F})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(\mathcal{Z}; i^!\mathscr{F})\langle * \rangle[\Sigma^{-1}]$.

Proof. Follows immediately from Theorem 6.1 by unravelling definitions. $\square$

Corollary 7.9 (Integration formula)**.** *Let the notation be as in Corollary 7.8. Suppose that $\mathcal{X}$ is proper representable over some $S \in \mathrm{Stk}$. Denote by $f : \mathcal{X} \to S$ and $g : \mathcal{Z} \to S$ the structural morphisms. Then for every $\mathscr{F} \in \mathbf{D}(S)$ we have:*

(i) *If $f : \mathcal{X} \to S$ is quasi-smooth, then there is a canonical identification*[13]

$$f_!(-) \simeq f_! i_! (i^*(-) \cap e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})^{-1})$$

of maps $\mathrm{C}^\bullet(\mathcal{X}; f^*\mathscr{F})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(S; \mathscr{F})\langle * \rangle[\Sigma^{-1}]$.

(ii) *There is a canonical identification*

$$f_*(-) \simeq g_* (i^!(-) \cap e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})^{-1})$$

of maps

$$\mathrm{C}^\bullet(\mathcal{X}; f^!\mathscr{F})\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^\bullet(S; \mathscr{F})\langle * \rangle[\Sigma^{-1}].$$

Proof. Since f is proper representable, there is a canonical isomorphism $f_! \simeq f_*$. In particular, one has a cohomological Gysin push-forward $f_!$ (where f is quasi-smooth) and Borel–Moore homological push-forwards f_* and g_*, respectively. $\square$

Remark 7.10. For lisse-extended weaves satisfying proper descent (out of the examples listed in (1.2.1), this includes the Betti and étale weaves as well as motives with rational coefficients), Corollary 7.9 holds more generally without assuming that $\mathcal{X} \to S$ is representable. This is because in that case we have the canonical isomorphism of functors $f_! \simeq f_*$ for any proper morphism f. See [Kha1, Thm. A.7], [Kha5].

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[13]Here $i_!$ is the Gysin map of (7.6). When g is quasi-smooth, we can simplify $f_! i_! \simeq g_!$.

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