DIMENSIONAL CLASSICALITY CRITERION FOR DERIVED STACKS

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We define (co)dimension of derived schemes and stacks on classical truncations (see [SP, Tag 04N3] or [EGA, 0_{IV}, 14.1.2, 14.2.4] for schemes, and [SP, Tags 0AFL, 0DRL] for stacks).

Proposition 1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived* 1*-Artin stacks where $\mathcal{Y}_{\mathrm{cl}}$ is Cohen–Macaulay*[1]*. If $x \in |\mathcal{X}|$ is a point at which the relative dimension of f is equal to the relative virtual dimension of f, then $\mathcal{X} \times^{\mathbf{R}}_{\mathcal{Y}} \mathcal{Y}_{\mathrm{cl}}$ is classical in a Zariski neighbourhood of x.*

We will make use of the following lemma from [KR, 2.3.12]:

Lemma 2. *Let $Z \to X$ be a quasi-smooth closed immersion of derived schemes where X_{cl} is Cohen–Macaulay. Then we have* $-\operatorname{vdim}(Z/X) \geq \operatorname{codim}(Z, X)$*, with equality if and only if $Z \times^{\mathbf{R}}_X X_{\mathrm{cl}}$ is classical in a Zariski neighbourhood of x.*

Proof of Proposition 1. The statement is invariant under replacing $\mathcal{Y}$ by $\mathcal{Y}_{\mathrm{cl}}$ and $\mathcal{X}$ by $\mathcal{X} \times^{\mathbf{R}}_{\mathcal{Y}} \mathcal{Y}_{\mathrm{cl}}$, so we may assume $\mathcal{Y}$ classical.

Suppose first that $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are schemes. Since $f : X \to Y$ is quasi-smooth, there exists for every $x \in |X|$ over y a Zariski neighbourhood $U \subseteq X$ of x, a derived scheme M which is smooth over Y, and a quasi-smooth closed immersion $U \hookrightarrow M$ over Y (see [KR, Prop. 2.3.14]). We have

$$\operatorname{vdim}_x(U_y/M_y) = \operatorname{vdim}_x(U/M),$$
$$\operatorname{codim}_x(U_y, M_y) \leq \operatorname{codim}_x(U, M).$$

Since $\operatorname{vdim}_x(U_y/\kappa(y)) = \dim_x(U_y)$ by assumption, we also have

$$-\operatorname{vdim}_x(U_y/M_y) = \dim_x(M_y) - \dim_x(U_y) = \operatorname{codim}_x(U_y, M_y)$$

where the last equality holds because M_y is catenary (see [EGA, 0_{IV}, Cor. 16.5.12; IV_2, Prop. 5.1.9]). We conclude that

$$-\operatorname{vdim}_x(U/M) \leq \operatorname{codim}_x(U, M).$$

Now Lemma 2 implies that U is classical in a Zariski neighbourhood of x.

Next suppose that $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are algebraic spaces. Choose an étale surjection $X_0 \twoheadrightarrow X$ where X_0 is a derived scheme, and let $x_0 \in |X_0|$ be a lift of the given point $x \in |X|$. Choose also an étale surjection $Y_0 \twoheadrightarrow Y$ where Y_0 is a Cohen–Macaulay scheme and a lift $y_0 \in |Y_0|$ of y. Since Y has schematic diagonal, $X_0 \times_Y Y_0$ is a derived scheme. Applying the case

[1] equivalently, $\mathcal{Y}$ admits a smooth surjection $Y \twoheadrightarrow \mathcal{Y}$ where Y_{cl} is a Cohen–Macaulay scheme

above to the morphism $X_0 \times_Y Y_0 \to Y_0$, we obtain a Zariski neighbourhood of $(x_0, y_0) \in X_0 \times_Y Y_0$ which is classical. Its image along the étale morphism $X_0 \times_Y Y_0 \twoheadrightarrow X_0 \twoheadrightarrow X$ is then a Zariski neighbourhood of $x \in X$ which is classical.

Finally we consider the general case. Choose a smooth surjection $X \twoheadrightarrow \mathcal{X}$ where X is a derived scheme, a lift $x_0 \in |X_0|$ of the given point $x \in |X|$, a smooth surjection $Y_0 \twoheadrightarrow Y$ where Y_0 is a Cohen–Macaulay scheme, and a lift $y_0 \in |Y_0|$ of y. Since Y has representable diagonal, $X \times_{\mathcal{Y}} Y$ is a derived algebraic space. Hence the previous case applied to the morphism $X \times_{\mathcal{Y}} Y \to Y$ yields a Zariski neighbourhood of $(x_0, y_0) \in X \times_{\mathcal{Y}} Y$ which is classical. Its image along the smooth morphism $X \times_{\mathcal{Y}} Y \twoheadrightarrow X \twoheadrightarrow \mathcal{X}$ is then a Zariski neighbourhood of $x \in \mathcal{X}$ which is classical. □

References

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, Publ. Math. IHÉS **4** (Chapter 0, 1–7, and I, 1–10), **8** (II, 1–8), **11** (Chapter 0, 8–13, and III, 1–5), **17** (III, 6–7), **20** (Chapter 0, 14–23, and IV, 1), **24** (IV, 2–7), **28** (IV, 8–15), and **32** (IV, 16–21), 1960–1967.

[KR] A. A. Khan, D. Rydh, *Virtual Cartier divisors and blow-ups.* arXiv:1802.05702 (2018).

[SP] The Stacks Project. `https://stacks.math.columbia.edu`.