$\mathbf{A}^1$-HOMOTOPY INVARIANCE IN SPECTRAL ALGEBRAIC GEOMETRY

DENIS-CHARLES CISINSKI AND ADEEL A. KHAN

Abstract. We study two different flavours of $\mathbf{A}^1$-homotopy theory in the setting of spectral algebraic geometry, and compare them to classical $\mathbf{A}^1$-homotopy theory. As an application we show that the spectral analogue of Weibel's homotopy invariant K-theory collapses to the classical theory. Along the way we give a new construction of nonconnective algebraic K-theory of stable ∞-categories via a generalization of the Bass–Thomason–Trobaugh construction.

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1. Introduction

This paper establishes in a systematic way why fundamental invariants from derived geometry, such as Serre's Tor formula, or virtual fundamental classes, have a natural interpretation in homotopy invariant (co)homology theories. In fact, we provide an explicit way to interpret Lurie's spectral geometry into Voevodsky's motivic homotopy theory as follows.

Let R be a commutative ring or connective $\mathcal{E}_\infty$-ring spectrum, and let $R[T]$ denote the polynomial algebra over R in one variable T. A peculiarity of the world of $\mathcal{E}_\infty$-ring spectra is that the polynomial algebra $R[T]$ is *free* as an $\mathcal{E}_\infty$-R-algebra only when R is of characteristic zero (an $\mathcal{E}_\infty$-$\mathbf{Q}$-algebra). If we write $R\{T\}$ for the free $\mathcal{E}_\infty$-R-algebra on one generator T (in degree zero), then in general there is only a comparison homomorphism $R\{T\} \to R[T]$. The $\mathcal{E}_\infty$-ring $R\{T\}$ is *smooth* over R (its cotangent complex $L_{R\{T\}/R}$ is free) but not flat, while the $\mathcal{E}_\infty$-ring $R[T]$ is usually not smooth but instead *fibre-smooth*: that is, it is flat over R, and $\pi_0(R[T]) \simeq \pi_0(R)[T]$ is smooth over $\pi_0(R)$ in the sense of ordinary commutative algebra. Let $\mathrm{CAlg}_R^{\mathrm{sm}}$ denote the ∞-category of smooth $\mathcal{E}_\infty$-algebras over R, and $\mathrm{CAlg}_R^{\mathrm{fibsm}}$ denote the ∞-category of fibre-smooth $\mathcal{E}_\infty$-algebras over R. Our first main result reads as follows:

Theorem A. *Let R be a connective $\mathcal{E}_\infty$-ring. Consider the following ∞-categories:*

(i) *The ∞-category $\mathbf{H}(R)$ of Nisnevich sheaves of spaces $\mathcal{F} : \mathrm{CAlg}_R^{\mathrm{sm}} \to \mathrm{Spc}$ for which the canonical map $\mathcal{F}(A) \to \mathcal{F}(A\{T\})$ is invertible for every $A \in \mathrm{CAlg}_R^{\mathrm{sm}}$.*

(ii) *The ∞-category $\mathbf{H}^\flat(R)$ of Nisnevich sheaves of spaces $\mathcal{F} : \mathrm{CAlg}_R^{\mathrm{fibsm}} \to \mathrm{Spc}$ for which the canonical map $\mathcal{F}(A) \to \mathcal{F}(A[T])$ is invertible for every $A \in \mathrm{CAlg}_R^{\mathrm{fibsm}}$.*

(iii) *The ∞-category $\mathbf{H}^{\mathrm{cl}}(\pi_0(R))$ of Nisnevich sheaves of spaces $\mathcal{F} : \mathrm{CAlg}_{\pi_0(R)}^{\mathrm{fibsm}} \to \mathrm{Spc}$ for which the canonical map $\mathcal{F}(A) \to \mathcal{F}(A[T])$ is invertible for every $A \in \mathrm{CAlg}_{\pi_0(R)}^{\mathrm{fibsm}}$.*

Then (i) and (iii) are equivalent. If R is an $\mathcal{E}_\infty$-$\mathbf{Z}$-algebra, then all three are equivalent.

See Theorems 2.7.2 and 3.4.1. The result is nontrivial even for ordinary commutative rings R (viewed as discrete $\mathcal{E}_\infty$-rings), in which case it asserts an equivalence $\mathbf{H}(R) \simeq \mathbf{H}^{\mathrm{cl}}(R)$. Note that $\mathrm{CAlg}_R^{\mathrm{fibsm}}$ coincides with the category of commutative rings that are smooth over R in the sense of ordinary commutative algebra, so that $\mathbf{H}^{\mathrm{cl}}(R)$ coincides with the usual $\mathbf{A}^1$-homotopy category considered by Morel and Voevodsky [MV99]. This equivalence was only known in characteristic zero (see Proposition 2.4.6 and Warning 2.4.7 in [Kha19b]). For simplicial commutative rings (regarded as $\mathcal{E}_\infty$-$\mathbf{Z}$-algebras), the equivalence (ii) $\Leftrightarrow$ (iii) is much more straightforward and was proven in the second author's thesis [Kha16].

Our second subject of discussion is a variant of Weibel's homotopy invariant K-theory for connective $\mathcal{E}_\infty$-rings. Let $\mathrm{KH}^{\mathrm{cl}}$ denote the classical variant [Wei89], defined by starting with nonconnective algebraic K-theory and forcing it to become $\mathbf{A}^1$-homotopy invariant in the sense that the canonical map

$$\mathrm{KH}^{\mathrm{cl}}(R) \to \mathrm{KH}^{\mathrm{cl}}(R[T])$$

is invertible for every commutative ring R. We define an analogous construction on the ∞-category of connective $\mathcal{E}_\infty$-rings,

$$R \mapsto \mathrm{KH}(R),$$

by forcing $\mathbf{A}^1$-homotopy invariance in the sense that $\mathrm{KH}(R) \to \mathrm{KH}(R\{T\})$ is invertible for any connective $\mathcal{E}_\infty$-ring R. We then have the following comparison, a K-theoretic incarnation of Theorem A:

Theorem B. *For every connective $\mathcal{E}_\infty$-ring R, there is a canonical isomorphism of spectra*

$$\mathrm{KH}(R) \simeq \mathrm{KH}^{\mathrm{cl}}(\pi_0(R)).$$

See Corollary 5.1.9. For connective $\mathcal{E}_\infty$-$\mathbf{Z}$-algebras one may adapt the proof, using the equivalence (i) $\Leftrightarrow$ (iii) in Theorem A, to derive the same result for the variant $\mathrm{KH}^\flat$ constructed by imposing invertibility of the maps $\mathrm{KH}^\flat(R) \to \mathrm{KH}^\flat(R[T])$. This result was communicated to us by B. Antieau and D. Gepner in 2015 in the generality of connective $\mathcal{E}_1$-rings, and has recently been recorded in the case of connective $\mathcal{E}_1$-$\mathbf{Z}$-algebras by Land–Tamme [LT19, Prop. 3.14].

An important ingredient in the proof is the observation that nonconnective K-theory of stable ∞-categories, defined as in [BGT13] following Schlichting, can also be described by a variant of the Bass–Thomason–Trobaugh construction [TT90, Sect. 6] defined over the sphere spectrum:

Theorem C. *There is an isomorphism*

$$\mathbb{K} \simeq \mathrm{K}^{\mathrm{B}}$$

of spectrum-valued functors on the ∞-category of small stable ∞-categories, where K *is algebraic K-theory,* $\mathbb{K}$ *is nonconnective algebraic K-theory, and* $(-)^{\mathrm{B}}$ *denotes a generalization of the Bass–Thomason–Trobaugh construction (Construction 4.5.5).*

See Example 4.4.4. In fact, we show in Theorem 4.4.3 that the construction $(-)^{\mathrm{B}}$ defines an equivalence between the ∞-category of connective spectrum-valued localizing invariants[1] and the ∞-category of spectrum-valued localizing invariants. This also implies for example that all operations on connective K-theory deloop to $\mathbb{K}$. The result is inspired by closely related work of Robalo in his framework of noncommutative motivic homotopy theory [Rob15] which gives similar results in the more restrictive setting of dg-categories.

[1] A connective spectrum-valued invariant E is localizing if, for any short exact sequence of stable ∞-categories $\mathbf{A}' \to \mathbf{A} \to \mathbf{A}''$, $E(\mathbf{A}')$ is identified with the connective cover of the homotopy fibre of $E(\mathbf{A}) \to E(\mathbf{A}'')$. See Definition 4.4.2.

Outline. In the body of the paper, we use the language of spectral algebraic geometry [SAG]. Given a spectral affine[2] scheme S, we may define $\mathbf{H}(S)$ as the ∞-category of $\mathbf{A}^1$-invariant Nisnevich sheaves on the site $\mathrm{Sm}_{/S}$ of smooth spectral affine schemes over S. Then $\mathbf{H}(\mathrm{Spec}(R))$ is equivalent to $\mathbf{H}(R)$ as defined above, and also to the construction given in [Kha19b], see Corollary 2.4.5 of *op. cit.* Similarly we have the variant $\mathbf{H}^\flat(S)$ defined as the ∞-category of $\mathbf{A}^{1,\flat}$-invariant Nisnevich sheaves on the site $\mathrm{Sm}^\flat_{/S}$ of fibre-smooth spectral affine schemes over S, where $\mathbf{A}^{1,\flat} = \mathrm{Spec}(\mathbf{S}[T])$.

The proof of the equivalence (i) $\Leftrightarrow$ (iii) in Theorem A is given in Sect. 2. Our starting point is the derived nil-invariance result of [Kha19b, Thm. A], which we propose to re-interpret as a sort of descent statement with respect to the "nil topology" whose coverings are morphisms of the form $X_{\mathrm{cl}} \to X$, where X_{cl} denotes the classical truncation of the spectral scheme X. Since the site $\mathrm{Sm}_{/S}$ is typically not closed under the operation $X \mapsto X_{\mathrm{cl}}$, making sense of this idea requires us to enlargen our site[3]. The first few subsections (2.1, 2.2, 2.3) develop some generalities related to $\mathbf{A}^1$-homotopy theory on various *admissible* sites (Definition 2.1.7) and how variation of site interplays with the basic operations such as inverse/direct image, product, and internal hom. For example, any *narrow* subcategory $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ as in Definition 2.1.10 gives rise to the same $\mathbf{A}^1$-homotopy theory as the smooth site $\mathrm{Sm}_{/S}$ (Example 2.2.4). The key result here is Proposition 2.2.3 which implies that any $\mathbf{A}^1$-invariant Nisnevich sheaf defined on a narrow subcategory $\mathcal{A}_{/S}$ admits a canonical extension $\mathcal{F}^+$ to an $\mathbf{A}^1$-invariant Nisnevich sheaf on any *broad* site $\mathcal{B}_{/S}$ (Definition 2.1.11) that contains $\mathcal{A}_{/S}$. Broad sites are in particular closed under classical truncation.

In Subsect. 2.4 we construct a comparison functor to the classical motivic homotopy category, which we show in Subsect. 2.5 is a left Bousfield localization (the *nil-localization*) if we work over a broad site (Theorem 2.5.3). In Subsect. 2.6 we formulate the nil-descent result alluded to above (Theorem 2.6.2). Finally, we put everything together in Subsect. 2.7 to show that nil-localization gives an equivalence

$$\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \xrightarrow{\sim} \mathbf{H}(S)$$

when we restrict to our narrow subcategory $\mathcal{A}_{/S}$. The last subsection (Subsect. 2.8) extends to the result to presheaves with values in general presentable ∞-categories $\mathbf{V}$ (e.g. presheaves of spectra or presheaves of chain complexes).

The second part of Theorem A is proven in Sect. 3. The idea of enlarging sites again plays an important role here. Working on a site large enough that it contains both the spectral affine line $\mathbf{A}^1$ and the flat affine line $\mathbf{A}^{1,\flat}$ allows us to exploit the canonical morphism $\varepsilon : \mathbf{A}^{1,\flat} \to \mathbf{A}^1$ which is a morphism of interval objects (3.3.2). The key input in the comparison is the fact that ε is a "universal" $\mathbf{A}^{1,\flat}$-equivalence, as long as we work over $\mathrm{Spec}(\mathbf{Z})$ (Lemma 3.3.4). The proof of the comparison is given in Subsect. 3.4.

Sect. 4 is independent of the first three sections and discusses localizing invariants of stable ∞-categories in the sense of [BGT13][4]. The main result, Theorem 4.4.3, asserts that every

[2] To simplify the exposition we usually only discuss the affine case. However, all our results extend to spectral schemes and algebraic spaces by descent: see Corollaries 2.7.5, 3.4.6, and 5.1.8.

[3] The same thing happens in classical algebraic geometry with the cdh topology; see [Kha19a].

[4] Note that, unlike [BGT13], we do not require localizing invariants to preserve filtered colimits.

connective spectrum-valued localizing invariant admits a unique delooping to a spectrum-valued localizing invariant. This is proven by generalizing the Bass construction to stable ∞-categories over the sphere spectrum (or any connective $\mathcal{E}_\infty$-ring).

Theorem B is proven in Sect. 5 by showing that the equivalence $\mathbf{H}(R) \simeq \mathbf{H}^{\mathrm{cl}}(\pi_0(R))$ of Theorem A or (rather its generalization to sheaves of spectra) sends KH to $\mathrm{KH}^{\mathrm{cl}}$. Unstably this boils down to representability results for the infinite loop spaces, and the stable result is deduced via Bott periodicity.

Notation and conventions. We will use the language of ∞-categories freely throughout the text. Our main references are [HTT, HA]. The ∞-category of spaces and spectra will be denoted by Spc and Spt, respectively, and a morphism in an ∞-category will be called an *isomorphism* if it is invertible (= an *equivalence* in the language of [HTT]). We also use the language of spectral algebraic geometry [SAG]. Given a spectral affine scheme $S = \mathrm{Spec}(R)$, we write $\mathrm{Aff}_{/S}$ for the ∞-category of spectral affine schemes over S, which is equivalent to the opposite of the ∞-category of connective $\mathcal{E}_\infty$-R-algebras.

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2. Comparison with classical motivic homotopy theory

2.1. **Fibred spaces.** For this subsection, we fix an affine spectral scheme S and write $\mathrm{Aff}_{/S}$ for the ∞-category of affine spectral schemes over S.

Definition 2.1.1. A morphism of affine spectral schemes $X \to S$ is called *smooth* (resp. *étale*) if it is of finite presentation and the relative cotangent complex $\mathcal{L}_{X/S}$ is a locally free $\mathcal{O}_X$-module of finite rank (resp. is zero).

Remark 2.1.2. From [SAG, Prop. 11.2.2.1] it follows that a morphism of affine spectral schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth if and only if $A \to B$ is differentially smooth in the sense of [SAG, Def. 11.2.2.2].

Example 2.1.3. Let $\mathbf{S}$ denote the sphere spectrum. For every integer $n \geqslant 0$, we write $\mathbf{S}\{T_1, \dots, T_n\}$ for the free $\mathcal{E}_\infty$-algebra on n generators T_i (in degree zero). We let $\mathbf{A}^n$ denote the affine spectral scheme $\mathrm{Spec}(\mathbf{S}\{T_1, \dots, T_n\})$ and refer to it as n-dimensional *spectral affine space* (over the sphere spectrum). The morphism $\mathbf{A}^n \to \mathrm{Spec}(\mathbf{S})$ is smooth, and we have a canonical isomorphism $(\mathbf{A}^n)_{\mathrm{cl}} \simeq \mathbf{A}^n_{\mathrm{cl}}$, where $\mathbf{A}^n_{\mathrm{cl}}$ denotes the classical affine space over $\mathrm{Spec}(\mathbf{Z})$.

Remark 2.1.4. If $X \in \mathrm{Aff}_{/S}$ is smooth over S, then Zariski-locally on X there exists an étale S-morphism $X \to S \times \mathbf{A}^n$ for some $n \geqslant 0$. This follows from [SAG, Prop. 11.2.2.1].

Definition 2.1.5 (Nisnevich excision)**.**

(i) A *Nisnevich square* over $X \in \mathrm{Aff}_{/S}$ is a cartesian square of affine spectral schemes

$$\begin{array}{ccc} W & \longrightarrow & V \\ \big\downarrow & & \big\downarrow p \\ U & \xrightarrow{\ j\ } & X \end{array} \tag{2.1.a}$$

where j is an open immersion, p is étale, and there exists a closed immersion $Z \hookrightarrow X$ complementary to j such that the induced morphism $p^{-1}(Z) \to Z$ is invertible.

(ii) We say that a presheaf of spaces $\mathcal{F}$ on $\mathrm{Aff}_{/S}$ satisfies *Nisnevich excision* if it is reduced, i.e. the space $\Gamma(\varnothing, \mathcal{F})$ is contractible, and for any $X \in \mathrm{Aff}_{/S}$ and any Nisnevich square over X of the form (2.1.a), the induced square of spaces

$$\begin{array}{ccc} \Gamma(X,\mathcal{F}) & \xrightarrow{\ j^*\ } & \Gamma(U,\mathcal{F}) \\ \big\downarrow p^* & & \big\downarrow \\ \Gamma(V,\mathcal{F}) & \longrightarrow & \Gamma(W,\mathcal{F}) \end{array}$$

is cartesian.

Definition 2.1.6 ($\mathbf{A}^1$-invariance)**.** Let $\mathcal{F}$ be a presheaf of spaces on $\mathrm{Aff}_{/S}$. We say that $\mathcal{F}$ satisfies $\mathbf{A}^1$-*homotopy invariance* if for every $X \in \mathrm{Aff}_{/S}$, the canonical map of spaces

$$p^* : \Gamma(X,\mathcal{F}) \to \Gamma(X \times \mathbf{A}^1, \mathcal{F})$$

is invertible, where $p : X \times \mathbf{A}^1 \to X$ is the projection of the spectral affine line over X.

We will need to consider presheaves defined on smaller subcategories of $\mathrm{Aff}_{/S}$. The following definition identifies the minimal conditions under which Definitions 2.1.5 and 2.1.6 make sense.

Definition 2.1.7. We say that a full subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ is *admissible* if it is essentially small and satisfies the following conditions:

(i) The affine spectral scheme S (viewed over S via the identity) belongs to $\mathcal{C}_{/S}$.

(ii) If X belongs to $\mathcal{C}_{/S}$ and Y is étale over X, then Y belongs to $\mathcal{C}_{/S}$.

(iii) If X belongs to $\mathcal{C}_{/S}$, then $X \times \mathbf{A}^n$ belongs to $\mathcal{C}_{/S}$ for every $n \geqslant 0$.

Example 2.1.8. The full subcategory $\mathrm{Sm}_{/S} \subseteq \mathrm{Aff}_{/S}$ of *smooth* affine spectral S-schemes is admissible. This follows from the fact that étale morphisms are smooth, the morphism $\mathbf{A}^n \to \mathrm{Spec}(\mathbf{S})$ is smooth for every $n \geqslant 0$, and the class of smooth morphisms is stable under composition and base change.

Example 2.1.9. Let $\mathcal{A}^0_{/S} \subseteq \mathrm{Aff}_{/S}$ denote the full subcategory spanned by $X \in \mathrm{Aff}_{/S}$ which admit an étale morphism

$$X \to S \times \mathbf{A}^n$$

over S. Then $\mathcal{A}^0_{/S}$ is admissible. For the third condition, note that if X admits an étale S-morphism to $S \times \mathbf{A}^n$, then $X \times \mathbf{A}^m$ admits (for every m) an étale S-morphism

$$X \times \mathbf{A}^m \to S \times \mathbf{A}^n \times \mathbf{A}^m \xrightarrow{\mathrm{pr}} S \times \mathbf{A}^n,$$

and hence also belongs to $\mathcal{A}^0_{/S}$. Note that $\mathcal{A}^0_{/S}$ is in fact the minimal admissible subcategory of $\mathrm{Aff}_{/S}$.

Definition 2.1.10. Let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. We say that $\mathcal{A}_{/S}$ is *narrow* if it is contained in the full subcategory $\mathrm{Sm}_{/S}$.

Definition 2.1.11. Let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. We say that $\mathcal{A}_{/S}$ is *broad* if it satisfies the following further condition:

(iv) For every $X \in \mathcal{A}_{/S}$, the classical truncation X_{cl} also belongs to $\mathcal{A}_{/S}$.

Note that there is a minimal broad subcategory of $\mathrm{Aff}_{/S}$, which is the closure of Example 2.1.9 under the operations (ii), (iii), and (iv) (constructed by transfinite iteration).

Remark 2.1.12. Note that any broad subcategory contains the minimal narrow subcategory $\mathcal{A}^0_{/S}$ (Example 2.1.9). Note also that, as long as S is not discrete, no admissible subcategory is both narrow and broad, since the morphism $S_{\mathrm{cl}} \to S$ is smooth if and only if it is an isomorphism.

Definition 2.1.13. Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a full subcategory. A $\mathcal{C}$-*fibred space* over S is a presheaf of spaces on $\mathcal{C}_{/S}$.

Definition 2.1.14. Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. We say that a $\mathcal{C}$-fibred space $\mathcal{F}$ satisfies *Nisnevich excision* and $\mathbf{A}^1$*-homotopy invariance* if it satisfies the conditions of Definitions 2.1.5 and 2.1.6, respectively (imposed only on objects $X \in \mathcal{C}_{/S}$). We say that $\mathcal{F}$ is a $\mathcal{C}$-*fibred motivic space* over S if it is both Nisnevich excisive and $\mathbf{A}^1$-homotopy invariant.

We denote by

$$\mathrm{Spc}(\mathcal{C}_{/S}) \quad \text{and } \mathbf{H}(\mathcal{C}_{/S})$$

the ∞-category of $\mathcal{C}$-fibred spaces over S and its full subcategory of motivic objects.

Example 2.1.15. In case of the admissible subcategory $\mathrm{Sm}_{/S} \subseteq \mathrm{Aff}_{/S}$ (Example 2.1.8), we write

$$\mathrm{Spc}(S) \coloneqq \mathrm{Spc}(\mathrm{Sm}_{/S}), \quad \mathbf{H}(S) \coloneqq \mathbf{H}(\mathrm{Sm}_{/S}).$$

With this definition we have $\mathbf{H}(\mathrm{Spec}(R)) \simeq \mathbf{H}(R)$ for any connective $\mathcal{E}_\infty$-ring R, where the right-hand side is as defined in Theorem A.

Remark 2.1.16. For any admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$, the full subcategories of Nisnevich-excisive, $\mathbf{A}^1$-invariant, and motivic $\mathcal{C}$-fibred spaces are each left Bousfield localizations of the ∞-category of $\mathcal{C}$-fibred spaces. The following assertions are proven in the same way as their analogues for Sm-fibred spaces (cf. [Kha19b, Sect. 2]):

(i) The Nisnevich localization functor $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{Nis}}(\mathcal{F})$ is exact (follows from [Kha19b, Thm. 2.2.7]).

(ii) The $\mathbf{A}^1$-localization functor $\mathcal{F} \mapsto \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})$ admits the following description (see [Kha19b, Rem. 2.3.5], [Hoy17, Prop. 3.4]): for every $\mathcal{C}$-fibred space $\mathcal{F}$, the space of sections over any $X \in \mathcal{C}_{/S}$ is computed by a sifted colimit:
$$\Gamma\big(X, \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})\big) \simeq \varinjlim \Gamma(\mathbf{A}^n \times X, \mathcal{F}), \tag{2.1.b}$$
indexed by the opposite of the full subcategory $\mathbf{A}_X \subseteq \mathrm{Aff}_{/X}$ whose objects are spectral affine spaces $X \times \mathbf{A}^n$ $(n \geqslant 0)$.

(iii) The motivic localization functor $\mathcal{F} \mapsto \mathbf{L}(\mathcal{F})$ can be computed as the transfinite composite
$$\mathbf{L}(\mathcal{F}) \simeq \varinjlim_{n \geqslant 0} (\mathrm{L}_{\mathbf{A}^1} \circ \mathrm{L}_{\mathrm{Nis}})^{\circ n}(\mathcal{F}), \tag{2.1.c}$$
for any $\mathcal{F} \in \mathrm{Spc}(\mathcal{C}_{/S})$ (cf. [Kha19b, Rem. 2.4.3]). Moreover, $\mathbf{H}(\mathcal{C}_{/S})$ has universality of colimits.

(iv) The ∞-category $\mathbf{H}(\mathcal{C}_{/S})$ of $\mathcal{C}$-fibred motivic spaces is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_S(X)$, where $\mathrm{h}_S(X)$ is the presheaf on $\mathcal{C}_{/S}$ represented by $X \in \mathcal{C}_{/S}$ (cf. [Kha19b, Prop. 2.4.4]).

2.2. **Extension of fibred spaces.** As in Subsect. 2.1, we fix an affine spectral scheme S. We also fix the following data:

Notation 2.2.1. Fix an inclusion $\mathcal{C}_{/S} \subseteq \mathcal{D}_{/S}$ of admissible subcategories of $\mathrm{Aff}_{/S}$. We consider the ∞-categories
$$\mathrm{Spc}(\mathcal{C}_{/S}) \quad \text{and } \mathbf{H}(\mathcal{C}_{/S}),$$
$$\mathrm{Spc}(\mathcal{D}_{/S}) \quad \text{and } \mathbf{H}(\mathcal{D}_{/S}),$$
as in Definition 2.1.14.

In this subsection we show that there are fully faithful embeddings
$$\mathrm{Spc}(\mathcal{C}_{/S}) \hookrightarrow \mathrm{Spc}(\mathcal{D}_{/S}), \quad \mathbf{H}(\mathcal{C}_{/S}) \hookrightarrow \mathbf{H}(\mathcal{D}_{/S}).$$

Notation 2.2.2. Let $\iota : \mathcal{C}_{/S} \hookrightarrow \mathcal{D}_{/S}$ denote the inclusion functor. Restriction along ι defines a functor $\iota^* : \mathrm{Spc}(\mathcal{D}_{/S}) \to \mathrm{Spc}(\mathcal{C}_{/S})$, whose left adjoint $\iota_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{D}_{/S})$ is given by left Kan extension of ι. The latter is uniquely characterized by the property of commutativity with colimits, and the identity $\iota_!\mathrm{h}_S(X) \simeq \mathrm{h}_S(X)$ for $X \in \mathcal{C}_{/S}$. In particular, $\iota_!$ is fully faithful with essential image generated under colimits by objects of the form $\mathrm{h}_S(X)$, with $X \in \mathcal{C}_{/S}$. Similarly, ι^* also admits a fully faithful right adjoint ι_* given by right Kan extending ι.

Proposition 2.2.3. *In the notation of 2.2.2, the assignment $\mathcal{F} \mapsto \mathbf{L}\iota_!(\mathcal{F})$ induces a fully faithful functor of ∞-categories*

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{D}_{/S}),$$

whose essential image is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_S(X)$, where X belongs to $\mathcal{C}_{/S}$.

Example 2.2.4. Suppose $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ is a *narrow* subcategory, and consider the inclusion $\iota : \mathcal{A}_{/S} \hookrightarrow \mathrm{Sm}_{/S}$. Then the fully faithful embedding

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}_{/S}) \hookrightarrow \mathbf{H}(\mathrm{Sm}_{/S}) = \mathbf{H}(S)$$

is an *equivalence*. Indeed, by [Kha19b, Prop. 2.4.4], $\mathbf{H}(S)$ is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_S(X)$, where X belongs to the minimal admissible subcategory $\mathcal{A}^0_{/S}$ (Example 2.1.9), and hence also to $\mathcal{A}_{/S}$.

We will deduce Proposition 2.2.3 from the following lemma:

Lemma 2.2.5. *The functors $\iota_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{D}_{/S})$ and $\iota^* : \mathrm{Spc}(\mathcal{D}_{/S}) \to \mathrm{Spc}(\mathcal{C}_{/S})$ preserve Nisnevich-local and $\mathbf{A}^1$-local equivalences.*

Proof. Since ι preserves Nisnevich squares and $\mathbf{A}^1$-projections, it follows that $\iota_!$ preserves Nisnevich-local and $\mathbf{A}^1$-local equivalences. The definition of admissibility implies that ι is also cocontinuous with respect to the Nisnevich topology, so it follows that ι^* preserves Nisnevich-local equivalences (see [SGA 4, Exp. III, Prop. 2.2] or [Kha19b, Def. 3.1.5]).

For $\mathbf{A}^1$-local equivalences it will suffice to show that, for any $X \in \mathcal{D}_{/S}$, the canonical morphism

$$\iota^*\mathrm{h}_S(X \times \mathbf{A}^1) \to \iota^*\mathrm{h}_S(X)$$

is an $\mathbf{A}^1$-local equivalence of $\mathcal{C}$-fibred spaces. By universality of colimits it suffices to show that, for any $Y \in \mathcal{C}_{/S}$ and any morphism $\varphi : \mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(X)$ (corresponding to a morphism $Y \to X$ in $\mathcal{D}_{/S}$), the base change

$$\iota^*\mathrm{h}_S(X \times \mathbf{A}^1) \underset{\iota^*\mathrm{h}_S(X)}{\times} \mathrm{h}_S(Y) \to \mathrm{h}_S(Y)$$

is an $\mathbf{A}^1$-local equivalence. Since the morphism φ factors as $\mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(X)$, the morphism in question is a base change of the morphism

$$\iota^*\mathrm{h}_S(X \times \mathbf{A}^1) \underset{\iota^*\mathrm{h}_S(X)}{\times} \iota^*\mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(Y),$$

which itself is identified with the canonical morphism

$$\mathrm{h}_S(Y \times \mathbf{A}^1) \to \mathrm{h}_S(Y),$$

since ι^* and h_S commute with limits and $\iota^*\iota_! = \mathrm{id}$. This is an $\mathbf{A}^1$-local equivalence, so the claim follows. □

Proof of Proposition 2.2.3. Since $\iota_!$ preserves motivic equivalences (Lemma 2.2.5), its right adjoint ι^* preserves motivic spaces and induces a functor $\iota^* : \mathbf{H}(\mathcal{D}_{/S}) \to \mathbf{H}(\mathcal{C}_{/S})$, right adjoint to $\mathbf{L}\iota_!$. Similarly, Lemma 2.2.5 also implies that the right Kan extension functor ι_* preserves motivic spaces and defines a right adjoint to $\iota^* : \mathbf{H}(\mathcal{D}_{/S}) \to \mathbf{H}(\mathcal{C}_{/S})$. Now the fully faithfulness of $\mathbf{L}\iota_!$, which is equivalent to invertibility of the unit map $\iota^*\mathbf{L}\iota_! \to \mathrm{id}$, follows by passage to left adjoints from the fully faithfulness of ι_* (which is equivalent to invertibility of the counit map $\iota^*\iota_* \to \mathrm{id}$). The description of the essential image follows from [HTT, Lem. 5.5.8.14]. □

Corollary 2.2.6. *There is a canonical invertible natural transformation*

$$\mathrm{L}_{\mathbf{A}^1}\,\iota^* \to \iota^*\,\mathrm{L}_{\mathbf{A}^1}\,.$$

Proof. It follows from Lemma 2.2.5 that $\iota^*\,\mathrm{L}_{\mathbf{A}^1}$ takes $\mathbf{A}^1$-invariant values, so the natural transformation in question is induced by the canonical map $\mathrm{id} \to \mathrm{L}_{\mathbf{A}^1}$. The fact that it is invertible follows from the formula (2.1.b), which is valid for both $\mathcal{C}$- and $\mathcal{D}$-fibred spaces. □

2.3. **Functoriality.** We now record the various functorialities of $\mathcal{C}$-fibred motivic spaces as the base varies; this works exactly as in the case $\mathcal{C} = \mathrm{Sm}$ treated in [Kha19b, Subsect. 2.5]. We then discuss the compatibility of these operations, as well as products and internal homs (Remark 2.3.7), under the operation of extension along an inclusion of admissible subcategories (Proposition 2.2.3).

Notation 2.3.1. Let $f : T \to S$ be a morphism of affine spectral schemes. Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory, and choose also an admissible subcategory $\mathcal{C}_{/T} \subseteq \mathrm{Aff}_{/T}$ which contains the base changes $X \times_S T$ of every $X \in \mathcal{C}_{/S}$. A minimal such can be constructed as in Definition 2.1.11.

Construction 2.3.2. Under the notation of 2.3.1, the base change functor $\mathrm{Aff}_{/S} \to \mathrm{Aff}_{/T}$ restricts to $\mathcal{C}_{/S} \to \mathcal{C}_{/T}$.

(i) The direct image functor f_* on $\mathcal{C}$-fibred spaces is given by restriction along the base change functor $\mathcal{C}_{/S} \to \mathcal{C}_{/T}$. The latter preserves Nisnevich covering families and $\mathbf{A}^1$-projections, so f_* preserves motivic spaces. Its left adjoint f^* on motivic spaces is characterized uniquely by commutativity with colimits and the formula $f^*_{\mathcal{D}}(\mathbf{L}\mathrm{h}_S(X)) \simeq \mathbf{L}\mathrm{h}_T(X \times_S T)$ for $X \in \mathcal{C}_{/S}$.

(ii) Suppose that the morphism $f : T \to S$ exhibits T as an object of the full subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$. Then the base change functor $\mathcal{C}_{/S} \to \mathcal{C}_{/T}$ admits a left adjoint, the forgetful functor

$$(X \to T) \mapsto (X \to T \xrightarrow{f} S)$$

which preserves Nisnevich covering families and $\mathbf{A}^1$-projections. In this case f^* is given by restriction along this forgetful functor, hence preserves motivic spaces and admits a left adjoint $f_\sharp$ characterized uniquely by commutativity with colimits and the formula $f^{\mathcal{D}}_\sharp(\mathbf{L}\mathrm{h}_T(X)) \simeq \mathbf{L}\mathrm{h}_S(X)$ for $X \in \mathcal{C}_{/T}$.

We now discuss the compatibility of these operations under the embedding of Proposition 2.2.3. For this we fix the following notation.

Notation 2.3.3. Let $f : T \to S$ be a morphism of affine spectral schemes. Fix an inclusion of admissible subcategories $\mathcal{C}_{/S} \subseteq \mathcal{D}_{/S}$ as in Notation 2.2.1. Fix similarly an inclusion $\mathcal{C}_{/T} \subseteq \mathcal{D}_{/T}$ of admissible subcategories both satisfying the condition of Notation 2.3.1. Write $\mathbf{H}(\mathcal{C}_{/S})$, $\mathbf{H}(\mathcal{D}_{/S})$, $\mathbf{H}(\mathcal{C}_{/T})$, and $\mathbf{H}(\mathcal{D}_{/T})$ for the ∞-categories of motivic fibred spaces formed with respect to these choices.

Remark 2.3.4. The condition of Notation 2.3.1 guarantees that the base change functor $X \mapsto X \times_S T$ commutes with the inclusions $\iota : \mathcal{C}_{/S} \to \mathcal{D}_{/S}$ and $\iota : \mathcal{C}_{/T} \to \mathcal{D}_{/T}$. From this it follows that the functor f_* commutes with ι^* and that f^* commutes with $\mathbf{L}\iota_!$. That is, we have commutative squares

$$\begin{array}{ccc} \mathbf{H}(\mathcal{C}_{/S}) & \xrightarrow{\mathbf{L}\iota_!} & \mathbf{H}(\mathcal{D}_{/S}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathbf{H}(\mathcal{C}_{/T}) & \xrightarrow{\mathbf{L}\iota_!} & \mathbf{H}(\mathcal{D}_{/T}), \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathcal{D}_{/T}) & \xrightarrow{\iota^*} & \mathbf{H}(\mathcal{C}_{/T}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathbf{H}(\mathcal{D}_{/S}) & \xrightarrow{\iota^*} & \mathbf{H}(\mathcal{C}_{/S}). \end{array}$$

Similarly, if f exhibits T as an object of $\mathcal{C}_{/S}$, then $f_\sharp$ commutes with $\mathbf{L}\iota_!$ and f^* commutes with ι^*.

The following compatibility is less obvious:

Proposition 2.3.5. *With notation as in 2.3.3, assume that $\mathcal{C}_{/S}$ is* narrow. *Let $i : Z \hookrightarrow S$ be a closed immersion of affine spectral schemes with affine open complement. Then there is a canonical invertible natural transformation*

$$\mathbf{L}\iota_! \circ i_*^{\mathcal{C}} \to i_*^{\mathcal{D}} \circ \mathbf{L}\iota_! \tag{2.3.a}$$

of functors $\mathbf{H}(\mathcal{C}_{/Z}) \to \mathbf{H}(\mathcal{D}_{/S})$, where the decorations indicate whether the functor is defined on $\mathcal{C}$-fibred or $\mathcal{D}$-fibred motivic spaces.

The Sm-fibred localization theorem [Kha19b, Thm. 3.2.2] implies the same for $\mathcal{C}$-fibred motivic spaces (for any narrow $\mathcal{C}$). We will deduce Proposition 2.3.5 by combining this with the following $\mathcal{D}$-fibred variant:

Theorem 2.3.6 (Localization). *Let the notation be as in Proposition 2.3.5, and let $j : U \to S$ be the open immersion complementary to i. Let $\mathcal{F} \in \mathbf{H}(\mathcal{D}_{/S})$ be a $\mathcal{D}$-fibred motivic space over S. If $\mathcal{F}$ belongs to the essential image of the functor $\mathbf{L}\iota_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{D}_{/S})$ (Proposition 2.2.3), then there is a cocartesian square*

$$\begin{array}{ccc} j_\sharp j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_\sharp(\mathrm{pt}_U) & \longrightarrow & i_* i^*(\mathcal{F}) \end{array}$$

of $\mathcal{D}$-fibred motivic spaces over S.

Proof. From [Kha19b, Prop. 3.1.4] it follows that the functor $i_* : \mathbf{H}(\mathcal{D}_{/Z}) \to \mathbf{H}(\mathcal{D}_{/S})$ commutes with contractible[5] colimits (exactly as in the proof of Thm. 3.1.1 in *loc. cit*). By Proposition 2.2.3 we may assume that $\mathcal{F}$ is the motivic localization of some $X \in \mathcal{C}_{/S}$. Since $\mathcal{C}_{/S}$ is narrow, we can moreover assume by Example 2.2.4 that it belongs to $\mathcal{A}^0_{/S}$ (Example 2.1.9), i.e., that it admits an étale morphism to some $\mathbf{A}^n_S$, $n \geqslant 0$. Then we can proceed exactly as in [Kha19b, Subsect. 4.3]. □

Proof of Proposition 2.3.5. The natural transformation (2.3.a) is the composite

$$\mathbf{L}\iota_! \circ i^{\mathcal{C}}_* \xrightarrow{\text{unit}} i^{\mathcal{D}}_* i^*_{\mathcal{D}} \mathbf{L}\iota_! \circ i^{\mathcal{C}}_* \simeq i^{\mathcal{D}}_* \mathbf{L}\iota_! \circ i^*_{\mathcal{C}} i^{\mathcal{C}}_* \xrightarrow{\text{counit}} i^{\mathcal{D}}_* \circ \mathbf{L}\iota_!,$$

where the identification $\mathbf{L}\iota_! \circ i^*_{\mathcal{C}} \simeq i^*_{\mathcal{D}} \circ \mathbf{L}\iota_!$ comes from Remark 2.3.4. By [Kha19b, Prop. 2.4.4] it will suffice to show that the canonical morphism

$$\mathbf{L}\iota_! \circ i^{\mathcal{C}}_*(\mathbf{L}\mathrm{h}_Z(X)) \to i^{\mathcal{D}}_*(\mathbf{L}\mathrm{h}_Z(X))$$

is invertible for all $X \in \mathcal{C}_{/Z}$. Using [Kha19b, Prop. 3.1.4], we may assume that X is of the form $Y \times_S Z$ for some $Y \in \mathcal{C}_{/S}$. Then we conclude by comparing the description of $i^{\mathcal{C}}_* i^*_{\mathcal{C}}(\mathbf{L}\mathrm{h}_S(Y))$ given by [Kha19b, Thm. 3.2.2], and the description of $i^{\mathcal{D}}_* i^*_{\mathcal{D}}(\mathbf{L}\mathrm{h}_S(Y))$ provided by Theorem 2.3.6. □

Finally, we discuss the compatibility of the functors $\mathbf{L}\iota_!$ and ι^* with products and internal homs. Note that, just as in the $\mathcal{C}$-fibred case [Kha19b, Rem. 2.4.2], the full subcategory $\mathbf{H}(\mathcal{D}_{/S}) \subseteq \mathrm{Spc}(\mathcal{D}_{/S})$ is closed under formation of internal homs.

Remark 2.3.7.

(i) Since the functor $\iota^* : \mathbf{H}(\mathcal{D}_{/S}) \to \mathbf{H}(\mathcal{C}_{/S})$ preserves limits (see proof of Proposition 2.2.3), it is symmetric monoidal with respect to the cartesian product.

(ii) The functor $\mathbf{L}\iota_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{D}_{/S})$ is also symmetric monoidal. Indeed, since $\mathbf{L}$ preserves finite products by Remark 2.1.16(iii), it suffices to show the claim for $\iota_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{D}_{/S})$. For this we may reduce to representables which is obvious.

(iii) For any $\mathcal{C}$-fibred motivic space $\mathcal{F} \in \mathbf{H}(\mathcal{D}_{/S})$ and $\mathcal{D}$-fibred motivic space $\mathcal{G} \in \mathbf{H}(\mathcal{D}_{/S})$, there is a canonical isomorphism

$$\iota^* \underline{\mathrm{Hom}}(\mathbf{L}\iota_!(\mathcal{F}), \mathcal{G}) \to \underline{\mathrm{Hom}}(\mathcal{F}, \iota^*(\mathcal{G}))$$

of $\mathcal{C}$-fibred motivic spaces, where $\underline{\mathrm{Hom}}$ is taken in $\mathbf{H}(\mathcal{D}_{/S})$ on the left and in $\mathbf{H}(\mathcal{C}_{/S})$ on the right. This follows by adjunction from (ii).

2.4. **Classical fibred spaces.** In this subsection we set up, for a classical affine scheme S, a classical variant of the ∞-category of $\mathcal{C}$-fibred motivic spaces. For S a spectral affine scheme we then define a pair of adjoint functors

$$\mathbf{L}v_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{C}_{/S}).$$

Later we will focus on understanding these adjunctions when $\mathcal{C}_{/S}$ is broad (Subsect. 2.5) and narrow (Subsect. 2.7).

The following is a classical analogue of Definitions 2.1.7 and 2.1.10.

Definition 2.4.1. Let S be an affine scheme. Denote by $\mathrm{AffCl}_{/S}$ the category of classical affine schemes over S. We say that a full subcategory $\mathcal{C}_{/S} \subseteq \mathrm{AffCl}_{/S}$ is *admissible* if it is essentially small and satisfies the following conditions:

(i) The affine scheme S (viewed over S via the identity) belongs to $\mathcal{C}_{/S}$.

(ii) If X belongs to $\mathcal{C}_{/S}$ and Y is étale over X, then Y belongs to $\mathcal{C}_{/S}$.

(iii) If X belongs to $\mathcal{C}_{/S}$, then $X \times \mathbf{A}^n_{\mathrm{cl}}$ belongs to $\mathcal{C}_{/S}$ for every $n \geqslant 0$.

For example, the full subcategory $\mathrm{SmCl}_{/S} \subseteq \mathrm{AffCl}_{/S}$ of *smooth* affine schemes over S (where smoothness is understood in the sense of classical algebraic geometry) is admissible. We say that an admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{AffCl}_{/S}$ is *narrow* if it is contained in $\mathrm{SmCl}_{/S}$.

Example 2.4.2. Let S be a *spectral* affine scheme and $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ an admissible subcategory. Then the full subcategory $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ spanned by the classical truncations X_{cl} of all objects $X \in \mathcal{C}_{/S}$ is admissible. The first condition follows from Definition 2.1.7(i), the second follows from Definition 2.1.7(ii) and [HA, Thm. 7.5.0.6], and the third follows from Definition 2.1.7(iii) and the fact that $\mathbf{A}^n_{\mathrm{cl}} \simeq (\mathbf{A}^n)_{\mathrm{cl}}$.

Definition 2.4.3. Let S be an affine scheme and let $\mathcal{C}^{\mathrm{cl}}_{/S} \subseteq \mathrm{AffCl}_{/S}$ be an admissible subcategory. A $\mathcal{C}^{\mathrm{cl}}$-*fibred space* over S is a presheaf of spaces on $\mathcal{C}^{\mathrm{cl}}_{/S}$. We say that a $\mathcal{C}^{\mathrm{cl}}$-fibred space $\mathcal{F}$ over S satisfies *Nisnevich excision* if it is reduced, and for any $X \in \mathcal{C}^{\mathrm{cl}}_{/S}$ and any Nisnevich square Q over X, the induced square of spaces $\Gamma(Q, \mathcal{F})$ is cartesian. We say that $\mathcal{F}$ satisfies $\mathbf{A}^1_{\mathrm{cl}}$-*homotopy invariance* if for any $X \in \mathcal{C}^{\mathrm{cl}}_{/S}$, the canonical map $\Gamma(X, \mathcal{F}) \to \Gamma(X \times \mathbf{A}^1_{\mathrm{cl}}, \mathcal{F})$ is invertible, where $\mathbf{A}^1_{\mathrm{cl}} = \mathrm{Spec}(\mathbf{Z}[T]) \simeq (\mathbf{A}^1)_{\mathrm{cl}}$ denotes the classical affine line. A *motivic* $\mathcal{C}^{\mathrm{cl}}$-*fibred space* is a $\mathcal{C}^{\mathrm{cl}}$-fibred space that satisfies Nisnevich excision and $\mathbf{A}^1_{\mathrm{cl}}$-homotopy invariance.

Example 2.4.4. Let $S = \mathrm{Spec}(R)$ be a spectral affine scheme. Then the ∞-category of motivic SmCl-fibred spaces $\mathbf{H}(\mathrm{SmCl}_{/S_{\mathrm{cl}}})$ is equivalent to $\mathbf{H}^{\mathrm{cl}}(\pi_0(R))$ as defined in Theorem A.

For the remainder of this subsection, we fix the following notation:

Notation 2.4.5. Let S be a spectral affine scheme. Fix an admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ (Definition 2.1.7). Let $\mathcal{C}^{\mathrm{cl}}_{/S}$ be the induced admissible subcategory of $\mathrm{AffCl}_{/S}$ as in Example 2.4.2. We denote by

$$\mathrm{Spc}(\mathcal{C}_{/S}), \quad \text{resp. } \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}),$$

the ∞-category of $\mathcal{C}$-fibred spaces over S, resp. of $\mathcal{C}^{\mathrm{cl}}$-fibred spaces over S_{cl}, and by

$$\mathbf{H}(\mathcal{C}_{/S}), \quad \text{resp. } \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}),$$

the full subcategory of motivic objects.

Construction 2.4.6. The operation of passing to classical truncations,

$$(X \to S) \mapsto (X_{\mathrm{cl}} \to S_{\mathrm{cl}}),$$

defines a canonical functor $v : \mathrm{Aff}_{/S} \to \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ which restricts to $v : \mathcal{C}_{/S} \to \mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. Denote by $v^* : \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathrm{Spc}(\mathcal{C}_{/S})$ the functor of restriction along v, and by $v_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ its left adjoint given by left Kan extension of v. Recall that $v_!$ is uniquely characterized by commutativity with colimits and the formula $v_!(\mathrm{h}_S(X)) \simeq \mathrm{h}_{S_{\mathrm{cl}}}(X_{\mathrm{cl}})$ for all $X \in \mathcal{C}_{/S}$.

Lemma 2.4.7.

(i) *The functor $v_!$ preserves Nisnevich-local equivalences, $\mathbf{A}^1$-local equivalences, and motivic equivalences.*

(ii) *The functor v^* preserves Nisnevich excisive spaces and Nisnevich-local equivalences. In particular, it commutes with $\mathrm{L}_{\mathrm{Nis}}$; that is, there is a canonical invertible natural transformation $\mathrm{L}_{\mathrm{Nis}}\, v^* \to v^*\, \mathrm{L}_{\mathrm{Nis}}$.*

(iii) *The functor v^* sends $\mathbf{A}^1_{\mathrm{cl}}$-invariant $\mathcal{C}^{\mathrm{cl}}$-fibred spaces to $\mathbf{A}^1$-invariant $\mathcal{C}$-fibred spaces. In particular, it sends motivic $\mathcal{C}^{\mathrm{cl}}$-fibred spaces to motivic $\mathcal{C}$-fibred spaces.*

In particular, we find that the functors $v_!$ and v^* descend to a pair of adjoint functors

$$\mathbf{L}v_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{C}_{/S}). \tag{2.4.a}$$

Proof. The first claim follows from the fact that v preserves Nisnevich squares and sends $\mathbf{A}^1$ to $\mathbf{A}^1_{\mathrm{cl}}$. By adjunction it follows that v^* preserves Nisnevich-excisive spaces and sends $\mathbf{A}^1_{\mathrm{cl}}$-invariant spaces to $\mathbf{A}^1$-invariant spaces. It remains to show that v^* preserves Nisnevich-local equivalences. For this it is sufficient to check that the functor v is cocontinuous for the Nisnevich topology, i.e., that for all $X \in \mathcal{C}_{/S}$, any Nisnevich covering of X_{cl} lifts to a Nisnevich covering of X (cf. [Kha19b, Def. 3.1.5]). This follows from [HA, Thm. 7.5.0.6]. □

2.5. **Nil-localization.** In this subsection we study the adjunction (2.4.a) when the admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ is *broad*. In this case, we find that the classical construction $\mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ is a left Bousfield localization of the spectral variant $\mathbf{H}(\mathcal{C}_{/S})$ (Theorem 2.5.3).

Notation 2.5.1. Let S be a spectral affine scheme. Fix a broad subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ (Definition 2.1.11), and let $\mathcal{B}^{\mathrm{cl}}_{/S}$ be the induced admissible subcategory of $\mathrm{AffCl}_{/S}$ as in Example 2.4.2. Consider the ∞-categories

$$\mathbf{H}(\mathcal{B}_{/S}) \subseteq \mathrm{Spc}(\mathcal{B}_{/S}), \quad \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \subseteq \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}).$$

Definition 2.5.2. A $\mathcal{B}$-fibred space $\mathcal{F} \in \mathrm{Spc}(\mathcal{B}_{/S})$ is called *nil-local* if for any $X \in \mathcal{B}_{/S}$, the canonical map of spaces

$$\Gamma(X, \mathcal{F}) \to \Gamma(X_{\mathrm{cl}}, \mathcal{F})$$

is invertible.

Theorem 2.5.3. *The functor* $v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}_{/S})$ *is fully faithful, and induces an equivalence*

$$v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}_{\mathrm{nil}}(\mathcal{B}_{/S})$$

from the ∞*-category of* $\mathcal{B}^{\mathrm{cl}}$*-fibred motivic spaces over* S_{cl} *to the* ∞*-category* $\mathbf{H}_{\mathrm{nil}}(\mathcal{B}_{/S}) \subseteq \mathbf{H}(\mathcal{B}_{/S})$ *of nil-local* $\mathcal{B}$*-fibred motivic spaces over* S*. In particular, the functor* $\mathbf{L}v_! : \mathbf{H}(\mathcal{B}_{/S}) \to \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ *is a left Bousfield localization.*

The proof of Theorem 2.5.3 relies on an analysis of the behaviour of the functors

$$v_! : \mathrm{Spc}(\mathcal{B}_{/S}) \to \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathrm{Spc}(\mathcal{B}_{/S})$$

with respect to $\mathbf{A}^1$-local and Nisnevich-local equivalences, specializing Lemma 2.4.7 to the broad case:

Proposition 2.5.4.

(i) *The functor* $v_!$ *preserves Nisnevich excisive spaces and Nisnevich-local equivalences. In particular, it commutes with* $\mathrm{L}_{\mathrm{Nis}}$*; that is, there is a canonical invertible natural transformation* $\mathrm{L}_{\mathrm{Nis}}\, v_! \to v_!\, \mathrm{L}_{\mathrm{Nis}}$.

(ii) *The functor* v^* *preserves Nisnevich excisive spaces and Nisnevich-local equivalences. In particular, it commutes with* $\mathrm{L}_{\mathrm{Nis}}$*; that is, there is a canonical invertible natural transformation* $\mathrm{L}_{\mathrm{Nis}}\, v^* \to v^*\, \mathrm{L}_{\mathrm{Nis}}$.

(iii) *The functor* $v_!$ *sends* $\mathbf{A}^1$*-local equivalences to* $\mathbf{A}^1_{\mathrm{cl}}$*-local equivalences.*

(iv) *The canonical natural transformations*

$$\mathrm{L}_{\mathbf{A}^1}\, v^* \to v^* \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} \quad \textit{and} \quad v_!\, \mathrm{L}_{\mathbf{A}^1}\, v^* \to \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$$

are invertible.

(v) *The functor* $v_!$ *preserves motivic equivalences.*

(vi) *The canonical natural transformations*

$$\mathbf{L}v^* \to v^*\mathbf{L} \quad \textit{and} \quad v_!\mathbf{L}v^* \to \mathbf{L}$$

are invertible.

The key feature of the broad case is the existence of a left adjoint u to the functor $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$:

Remark 2.5.5. Any affine scheme X_0 over S_{cl} can be viewed as a discrete affine spectral scheme over S (by composition with the canonical morphism $S_{\mathrm{cl}} \to S$). This defines a functor $u : \mathrm{AffCl}_{/S_{\mathrm{cl}}} \to \mathrm{Aff}_{/S}$, left adjoint to v. Note that u is fully faithful (as the unit map $X_0 \to vu(X_0)$ is always invertible). Since $\mathcal{B}$ is *broad*, our choice of $\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ (Example 2.4.2) guarantees that u restricts to a functor $u : \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \to \mathcal{B}_{/S}$. Restriction along u defines a functor $u^* : \mathrm{Spc}(\mathcal{B}_{/S}) \to \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$, which admits fully faithful left and right adjoints $u_!$ and u_*, respectively. By adjunction, we have identifications $v_! \simeq u^*$ and $v^* \simeq u_*$. In particular, it follows that the co-unit

$$v_! v^* \to \mathrm{id}$$

is invertible.

Remark 2.5.6. Note that a $\mathcal{B}$-fibred space $\mathcal{F} \in \mathrm{Spc}(\mathcal{B}_{/S})$ is nil-local if and only if it belongs to the essential image of v^*, or equivalently if and only if the unit map $\mathcal{F} \to v^* v_!(\mathcal{F}) \simeq v^* u^*(\mathcal{F})$ is invertible. Indeed the counit map $uv(X) \to X$ is the inclusion of the classical truncation, so the map $\Gamma(X, \mathcal{F}) \to \Gamma(X_{\mathrm{cl}}, \mathcal{F})$ is canonically identified with the map

$$\Gamma(X, \mathcal{F}) \to \Gamma(X, v^* u^*(\mathcal{F}))$$

for every $X \in \mathcal{B}_{/S}$.

Proof of Proposition 2.5.4. We already know from Lemma 2.4.7 that $v_!$ sends Nisnevich-local equivalences to Nisnevich-local equivalences, $\mathbf{A}^1$-local equivalences to $\mathbf{A}^1_{\mathrm{cl}}$-local equivalences, and motivic equivalences to motivic equivalences. We also know that its right adjoint v^* sends Nisnevich excisive spaces to Nisnevich excisive spaces, $\mathbf{A}^1_{\mathrm{cl}}$-invariant spaces to $\mathbf{A}^1$-invariant spaces, and motivic spaces to motivic spaces.

Let $u : \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \to \mathcal{B}_{/S}$ be as in Remark 2.5.5, so that $v_! \simeq u^*$. Since u preserves Nisnevich squares, the functor $u_!$ preserves Nisnevich-local equivalences. Hence its right adjoint $u^* \simeq v_!$ preserves Nisnevich excisive spaces. This proves claim (i).

Consider claim (iv). Since v^* preserves $\mathbf{A}^1$-invariant spaces, the natural transformation $\mathrm{id} \to \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$ induces a transformation

$$\mathrm{L}_{\mathbf{A}^1} v^* v_! \to \mathrm{L}_{\mathbf{A}^1} v^* \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} v_! \simeq v^* \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} v_! \tag{2.5.a}$$

which we claim is invertible. For every $X \in \mathcal{B}_{/S}$, let $\mathcal{I}_X \subseteq \mathrm{Aff}_{/X}$ denote the non-full subcategory whose objects are spectral affine spaces $\mathbf{A}^n \times X$ over X, and whose morphisms are projections. A variant of the formula (2.1.b) (see [Hoy17, Prop. 3.4]) then yields the functorial isomorphisms

$$\begin{aligned}\Gamma\big(X, \mathrm{L}_{\mathbf{A}^1} v^* u^*(\mathcal{F})\big) &\simeq \varinjlim_{\mathcal{I}_X^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n \times X, v^* u^*(\mathcal{F})\big) \\ &\simeq \varinjlim_{\mathcal{I}_X^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}, \mathcal{F}\big),\end{aligned}$$

by Remark 2.5.6. Similarly, if we write $\mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/X_{\mathrm{cl}}}$ for the subcategory of classical affine spaces $\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}$ and projections between them, then [Hoy17, Prop. 3.4] again yields

$$\begin{aligned}\Gamma\big(X, v^*\mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} u^*(\mathcal{F})\big) &\simeq \Gamma\big(X_{\mathrm{cl}}, \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} u^*(\mathcal{F})\big)\\ &\simeq \varinjlim_{(\mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}})^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}, u^*(\mathcal{F})\big)\\ &\simeq \varinjlim_{(\mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}})^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}, \mathcal{F}\big).\end{aligned}$$

Since $v : \mathcal{B}_{/X} \to \mathcal{B}^{\mathrm{cl}}_{/X_{\mathrm{cl}}}$ induces an equivalence $\mathcal{I}_X \simeq \mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}}$, it follows that (2.5.a) is invertible. Applying v^* on the right, we deduce that the canonical transformation $\mathrm{L}_{\mathbf{A}^1} v^* \to v^*\mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$ is also invertible (since $v_! v^* \simeq \mathrm{id}$ by Remark 2.5.5). Applying $v_!$ on the left, we also obtain the invertible transformation $v_!\, \mathrm{L}_{\mathbf{A}^1}\, v^* \to \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$.

Claim (vi) follows from claims (ii) and (iv) in view of the formula (2.1.c) (and the analogous formula for $\mathcal{B}^{\mathrm{cl}}$-fibred spaces). □

Proof of Theorem 2.5.3. By Remark 2.5.5 we know that the functor

$$v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}_{/S})$$

is fully faithful. Its essential image is spanned by objects $\mathcal{F} \in \mathbf{H}(\mathcal{B}_{/S})$ for which the unit map $\mathcal{F} \to v^*\mathbf{L}v_!(\mathcal{F})$ is invertible. By Remark 2.5.6, this condition implies that $\mathcal{F}$ is nil-local. Conversely if $\mathcal{F}$ is nil-local, so that the unit map $\mathcal{F} \to v^*v_!(\mathcal{F})$ is invertible (again by Remark 2.5.6), then using Proposition 2.5.4(vi) we see that the induced map $\mathcal{F} \simeq \mathbf{L}(\mathcal{F}) \to \mathbf{L}v^*v_!(\mathcal{F}) \simeq v^*\mathbf{L}v_!(\mathcal{F})$ is also invertible. □

2.6. Nil descent.

Notation 2.6.1. Let S be a spectral affine scheme. Fix a narrow subcategory $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ and a broad subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ containing $\mathcal{A}_{/S}$. We consider the ∞-categories

$$\mathbf{H}(\mathcal{A}_{/S}) \subseteq \mathrm{Spc}(\mathcal{A}_{/S}), \quad \mathbf{H}(\mathcal{B}_{/S}) \subseteq \mathrm{Spc}(\mathcal{B}_{/S}).$$

Let $\iota : \mathcal{A}_{/S} \to \mathcal{B}_{/S}$ denote the inclusion, so that we have the fully faithful functor (see Proposition 2.2.3)

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{B}_{/S}).$$

We are now in a position to state and prove the following result:

Theorem 2.6.2. *Let $\mathcal{F} \in \mathbf{H}(\mathcal{A}_{/S})$ be an $\mathcal{A}$-fibred motivic space. Then the $\mathcal{B}$-fibred motivic space $\mathbf{L}\iota_!(\mathcal{F}) \in \mathbf{H}(\mathcal{B}_{/S})$ is nil-local (Definition 2.5.2).*

Proof. Set $\mathcal{F}^+ = \mathbf{L}\iota_!(\mathcal{F})$. Let $X \in \mathcal{B}_{/S}$ with structural morphism $f : X \to S$ and choose subcategories $\mathcal{A}_{/X} \subseteq \mathcal{B}_{/X}$ of $\mathrm{Aff}_{/X}$, narrow and broad, respectively, and both satisfying the condition of Notation 2.3.1. By adjunction, there are canonical isomorphisms

$$\begin{aligned}\Gamma(X,\mathcal{F}^+) &\simeq \mathrm{Maps}(\mathrm{pt}_X, f_{\mathcal{B}}^*(\mathcal{F}^+)),\\ \Gamma(X_{\mathrm{cl}},\mathcal{F}^+) &\simeq \mathrm{Maps}(\mathrm{h}_X(X_{\mathrm{cl}}), f_{\mathcal{B}}^*(\mathcal{F}^+))\\ &\simeq \mathrm{Maps}(i_\sharp^{\mathcal{B}} i_{\mathcal{B}}^*(\mathrm{pt}_X), f_{\mathcal{B}}^*(\mathcal{F}^+))\\ &\simeq \mathrm{Maps}(\mathrm{pt}_X, i_*^{\mathcal{B}} i_{\mathcal{B}}^* f_{\mathcal{B}}^*(\mathcal{F}^+)),\end{aligned}$$

where all the mapping spaces are formed in $\mathbf{H}(\mathcal{B}_{/X})$. Under these identifications the map $\Gamma(X,\mathcal{F}^+)\to\Gamma(X_{\mathrm{cl}},\mathcal{F}^+)$ is induced by the unit morphism

$$f_{\mathcal{B}}^*(\mathcal{F}^+)\to i_*^{\mathcal{B}} i_{\mathcal{B}}^* f_{\mathcal{B}}^*(\mathcal{F}^+)$$

in $\mathbf{H}(\mathcal{B}_{/X})$. By Remark 2.3.4 and Proposition 2.3.5 this morphism is the image by $\mathbf{L}\iota_!$ of the unit morphism

$$f_{\mathcal{A}}^*(\mathcal{F})\to i_*^{\mathcal{A}} i_{\mathcal{A}}^* f_{\mathcal{A}}^*(\mathcal{F})$$

in $\mathbf{H}(\mathcal{A}_{/X})$. Since i is a closed immersion with empty complement, this morphism is invertible by the nilpotent invariance property of $\mathbf{H}(\mathcal{A}_{/X})$, see [Kha19b, Cor. 3.2.7] (which applies to any narrow subcategory and not just $\mathrm{Sm}_{/X}$). □

2.7. The comparison.

Notation 2.7.1. Let S be a spectral affine scheme. We again fix narrow and broad subcategories $\mathcal{A}_{/S}\subseteq\mathrm{Aff}_{/S}$ and $\mathcal{B}_{/S}\subseteq\mathrm{Aff}_{/S_{\mathrm{cl}}}$ as in Notation 2.6.1. We let $\mathcal{A}_{/S}^{\mathrm{cl}}$ and $\mathcal{B}_{/S}^{\mathrm{cl}}$ be the induced admissible subcategories of $\mathrm{AffCl}_{/S}$ as in Example 2.4.2. To simplify notation set

$$\underline{\mathrm{Spc}}(S) := \mathrm{Spc}(\mathcal{B}_{/S}),\quad \underline{\mathrm{Spc}}^{\mathrm{cl}}(S_{\mathrm{cl}}) := \mathrm{Spc}(\mathcal{B}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}),\quad \underline{\mathbf{H}}(S) := \mathbf{H}(\mathcal{B}_{/S}),\quad \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) := \mathbf{H}(\mathcal{B}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}),$$

and similarly

$$\mathrm{Spc}(S) := \mathrm{Spc}(\mathcal{A}_{/S}),\quad \mathrm{Spc}^{\mathrm{cl}}(S_{\mathrm{cl}}) := \mathrm{Spc}(\mathcal{A}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}),\quad \mathbf{H}(S) := \mathbf{H}(\mathcal{A}_{/S}),\quad \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) := \mathbf{H}(\mathcal{A}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}).$$

Recall from Example 2.2.4 that this notation agrees with that of Example 2.1.15, even though $\mathcal{A}_{/S}$ is allowed to be any narrow subcategory.

Let $v:\mathcal{B}_{/S}\to\mathcal{B}_{/S_{\mathrm{cl}}}$ and $w:\mathcal{A}_{/S}\to\mathcal{A}_{/S_{\mathrm{cl}}}$ be the classical truncation functors as in Construction 2.4.6. In this subsection we will prove the following result, which gives the equivalence between (i) and (iii) in Theorem A.

Theorem 2.7.2. *The adjunction of (2.4.a),*

$$\mathbf{L}w_!:\mathbf{H}(S)\to\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}),\quad w^*:\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})\to\mathbf{H}(S),$$

is an equivalence of ∞-categories.

The proof will combine the $\mathcal{B}$-fibred nil-localization statement (Theorem 2.5.3) and nil descent for $\mathcal{A}$-fibred spaces (Theorem 2.6.2), as well as Proposition 2.2.3 and the following classical analogue of the latter:

Proposition 2.7.3. *Consider the inclusion functor* $\iota : \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \hookrightarrow \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. *Let* $\iota_! : \mathrm{Spc}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \underline{\mathrm{Spc}}^{\mathrm{cl}}(S_{\mathrm{cl}})$ *denote the left Kan extension of* ι, *left adjoint to the restriction functor* $\iota^* : \underline{\mathrm{Spc}}^{\mathrm{cl}}(S) \to \mathrm{Spc}^{\mathrm{cl}}(S)$. *Then the assignment* $\mathcal{F} \mapsto \mathbf{L}\iota_!(\mathcal{F})$ *induces a fully faithful functor of* ∞*-categories*

$$\mathbf{L}\iota_! : \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}),$$

whose essential image is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_{S_{\mathrm{cl}}}(X)$, *where* $X \in \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ *admits an étale* S_{cl}*-morphism to a classical affine space* $S_{\mathrm{cl}} \times \mathbf{A}^n_{\mathrm{cl}}$, *for some* $n \geqslant 0$.

Proof. Same proof as Proposition 2.2.3. □

Remark 2.7.4. Since $w : \mathcal{A}_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ is the restriction of $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$, we have commutative squares

$$\begin{array}{ccccccc} \mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \quad & \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{v^*} & \underline{\mathbf{H}}(S) \\ \downarrow{\scriptstyle \mathbf{L}\iota_!} & & \downarrow{\scriptstyle \mathbf{L}\iota_!} & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \underline{\mathbf{H}}(S) & \xrightarrow{\mathbf{L}v_!} & \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}), & & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S). \end{array} \tag{2.7.a}$$

Proof of Theorem 2.7.2. We show that the adjunction $(\mathbf{L}w_!, w^*)$ is an equivalence. For any $\mathcal{A}$-fibred motivic space $\mathcal{F} \in \mathbf{H}(S)$, the $\mathcal{B}$-fibred motivic space $\mathbf{L}\iota_!(\mathcal{F})$ is nil-local by Theorem 2.6.2. Therefore by Theorem 2.5.3 the canonical map

$$\mathbf{L}\iota_!(\mathcal{F}) \to v^*\mathbf{L}v_!\mathbf{L}\iota_!(\mathcal{F})$$

is invertible. Applying ι^* and using Proposition 2.2.3, we deduce that the canonical map

$$\mathcal{F} \to \iota^*v^*\mathbf{L}v_!\mathbf{L}\iota_!(\mathcal{F})$$

is invertible. This map is identified with the unit $\mathcal{F} \to w^*\mathbf{L}w_!(\mathcal{F})$ under the identifications (Remark 2.7.4 and Proposition 2.7.3)

$$\iota^*v^*\mathbf{L}v_!\mathbf{L}\iota_! \simeq w^*\iota^*\mathbf{L}\iota_!\mathbf{L}w_! \simeq w^*\mathbf{L}w_!.$$

Since $\mathcal{F} \in \mathbf{H}(S)$ was arbitrary, this shows that the unit

$$\mathrm{id} \to w^*\mathbf{L}w_!$$

is invertible, hence $\mathbf{L}w_!$ is fully faithful. It remains to show that $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$ is generated under colimits by objects of the form $\mathbf{L}\mathrm{h}_{S_{\mathrm{cl}}}(X_{\mathrm{cl}})$, where $X \in \mathcal{A}_{/S}$. But this follows from the definition of $\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ (Example 2.4.2). □

The next few results are corollaries of Theorems 2.5.3 and 2.7.2.

Corollary 2.7.5. *For any quasi-compact quasi-separated spectral algebraic space* S, *there are canonical equivalences of* ∞*-categories*

$$\mathbf{H}(S) \simeq \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}),$$

where $\mathbf{H}(S)$ *is as in* [Kha19b, Def. 2.4.1] *and* $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$ *its classical variant.*

Proof. Classical truncation defines a functor w from the ∞-category of smooth spectral algebraic spaces over S to the category of smooth classical algebraic spaces over S_{cl}. This induces an adjunction

$$\mathbf{L}w_! : \mathbf{H}(S) \to \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}), \quad w^* : \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \mathbf{H}(S)$$

which globalizes that of Theorem 2.7.2. To show that the unit and counit maps are invertible, we may use Nisnevich descent [Kha19b, Prop. 2.5.7] to reduce to the affine case proven in Theorem 2.7.2. □

Corollary 2.7.6. *Let S be an affine spectral scheme. Then for any $\mathcal{A}$-fibred space $\mathcal{F} \in \mathrm{Spc}(S)$, the canonical map in $\mathbf{H}(S)$*

$$\mathbf{L}(\mathcal{F}) \to w^*\mathbf{L}w_!(\mathcal{F})$$

is invertible.

Proof. By Theorem A the canonical map $\mathbf{L}(\mathcal{F}) \xrightarrow{\sim} w^*\mathbf{L}w_!(\mathbf{L}(\mathcal{F}))$ is invertible. By Proposition 2.5.4 (v) the canonical map $w_!(\mathcal{F}) \to w_!(\mathbf{L}(\mathcal{F}))$ is a motivic equivalence, whence the claim. □

Corollary 2.7.7. *Let S be an affine spectral scheme. Then there is a commutative square*

$$\begin{CD}
\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) @>{w^*}>> \mathbf{H}(S) \\
@V{\mathbf{L}\iota_!}VV @VV{\mathbf{L}\iota_!}V \\
\underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) @>{v^*}>> \underline{\mathbf{H}}(S).
\end{CD}$$

Proof. The square is obtained by horizontally passing to right adjoints in the left-hand square in (2.7.a), and thus commutes up to a natural transformation which we claim is invertible. Note that both clockwise and counterclockwise composites factor through the full subcategory $\underline{\mathbf{H}}_{\mathrm{nil}}(S)$ of nil-local objects by Theorem 2.6.2 and Remark 2.5.6. By Theorem 2.5.3 it will therefore suffice to show that the natural transformation becomes invertible after post-composition with $\mathbf{L}v_! : \underline{\mathbf{H}}(S) \to \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}})$. This is immediate from the fact that v^* is fully faithful (Theorem 2.5.3) and the commutativity of the right-hand square in (2.7.a). □

Corollary 2.7.8. *Let S be an affine spectral scheme. Then there is a commutative square*

$$\begin{CD}
\underline{\mathbf{H}}_{\mathrm{nil}}(S) @>{\mathbf{L}v_!}>> \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) \\
@V{\iota^*}VV @VV{\iota^*}V \\
\mathbf{H}(S) @>{\mathbf{L}w_!}>> \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}).
\end{CD}$$

Proof. The square is obtained by horizontally passing to left adjoints in the right-hand square in (2.7.a) (and restricting to the full subcategory $\underline{\mathbf{H}}_{\mathrm{nil}}(S) \subseteq \underline{\mathbf{H}}(S)$), and thus commutes up to a natural transformation which we claim is invertible. By Theorem 2.5.3 it will suffice to show this after pre-composition with $v^* : \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \underline{\mathbf{H}}(S)$. This is immediate from the fact that v^* is fully faithful (Theorem 2.5.3) and the commutativity of the right-hand square in (2.7.a). □

Remark 2.7.9. The discussion of Subsect. 2.3 makes sense in the classical setting and provides $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$ with the same functorialities as S varies. The equivalence of Theorem A is compatible with all the operations $f_\sharp$, f^*, f_*, as well as $\times$, and $\underline{\mathrm{Hom}}$:

(i) Let $f : T \to S$ be a morphism of affine spectral schemes. Then we have commutative squares

$$
\begin{array}{ccc}
\mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*_{\mathrm{cl}}} \\
\mathbf{H}(T) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}),
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(T) \\
\downarrow{\scriptstyle f_{\mathrm{cl},*}} & & \downarrow{\scriptstyle f_*} \\
\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S).
\end{array}
$$

Indeed, the left-hand square is induced by the commutative square

$$
\begin{array}{ccc}
\mathcal{A}_{/S} & \xrightarrow{w} & \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \\
\downarrow & & \downarrow \\
\mathcal{A}_{/T} & \xrightarrow{w} & \mathcal{A}^{\mathrm{cl}}_{/T_{\mathrm{cl}}},
\end{array}
$$

where the upper horizontal arrow is (derived) base change along f, and the lower horizontal arrow is classical base change along f_{cl}. The right-hand square is obtained by passage to right adjoints.

(ii) Since the horizontal arrows in the squares above are equivalences (Theorem A), the squares are horizontally right- and left-adjointable, respectively. In other words, they give rise to further commutative squares

$$
\begin{array}{ccc}
\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S) \\
\downarrow{\scriptstyle f^*_{\mathrm{cl}}} & & \downarrow{\scriptstyle f^*} \\
\mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(T),
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{H}(T) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{\mathrm{cl},*}} \\
\mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}).
\end{array}
$$

(iii) Similarly f is smooth, then we have commutative squares

$$
\begin{array}{ccc}
\mathbf{H}(T) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) \\
\downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_{\mathrm{cl},\sharp}} \\
\mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}),
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(T) \\
\downarrow{\scriptstyle f_{\mathrm{cl},\sharp}} & & \downarrow{\scriptstyle f_\sharp} \\
\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S).
\end{array}
$$

Here the left-hand square is induced by the commutative square

$$
\begin{array}{ccc}
\mathcal{A}_{/T} & \xrightarrow{v} & \mathcal{A}^{\mathrm{cl}}_{/T_{\mathrm{cl}}} \\
\downarrow & & \downarrow \\
\mathcal{A}_{/S} & \xrightarrow{v} & \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}.
\end{array}
$$

The right-hand square comes from the horizontal right-adjointability of the left-hand one.

(iv) Finally, consider the operations $\times$ and $\underline{\mathrm{Hom}}$. Note that w^* is cartesian monoidal since it is a right adjoint. Its left adjoint $\mathbf{L}w_!$ is also monoidal, since $w_!$ is clearly monoidal (as can be checked on representables) and $\mathbf{L}$ preserves finite products by Remark 2.1.16(iii). Then by adjunction we have a canonical isomorphism

$$w^*\big(\underline{\mathrm{Hom}}(\mathbf{L}w_!(\mathcal{F}),\mathcal{G})\big) \to \underline{\mathrm{Hom}}(\mathcal{F}, v^*(\mathcal{G}))$$

in $\mathbf{H}(S)$, for any $\mathcal{F} \in \mathbf{H}(S)$ and $\mathcal{G} \in \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$. This induces in turn for every $\mathcal{F},\mathcal{G} \in \mathbf{H}(S)$ isomorphisms

$$\begin{aligned}\underline{\mathrm{Hom}}(\mathbf{L}w_!(\mathcal{F}),\mathbf{L}w_!(\mathcal{G})) &\simeq \mathbf{L}w_!w^*\big(\underline{\mathrm{Hom}}(\mathbf{L}w_!(\mathcal{F}),\mathbf{L}w_!(\mathcal{G}))\big)\\ &\simeq \mathbf{L}w_!\big(\underline{\mathrm{Hom}}(\mathcal{F}, v^*\mathbf{L}w_!(\mathcal{G}))\big)\\ &\simeq \mathbf{L}w_!\big(\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})\big)\end{aligned}$$

where the first and third isomorphisms come from Theorem A.

2.8. **V-linear motivic objects.** Let S be an affine spectral scheme and let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. By replacing the ∞-category of spaces in Definition 2.1.14 with any given presentable ∞-category $\mathbf{V}$, we can define a $\mathbf{V}$-linear variant of the construction $\mathbf{H}(\mathcal{C}_{/S})$:

Definition 2.8.1. A $\mathcal{C}$-*fibred motivic* $\mathbf{V}$-*object* is a $\mathbf{V}$-valued presheaf $(\mathcal{C}_{/S})^{\mathrm{op}} \to \mathbf{V}$ satisfying $\mathbf{A}^1$-homotopy invariance and Nisnevich excision. We write $\mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}$ for the ∞-category of motivic $\mathcal{C}$-fibred $\mathbf{V}$-objects.

Similarly, given an admissible subcategory $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$, we may consider the ∞-category $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})_{\mathbf{V}}$ of $\mathcal{C}^{\mathrm{cl}}$-fibred motivic $\mathbf{V}$-objects. We have $\mathbf{V}$-linear analogues of each of the categories defined in *loc. cit.*:

$$\mathrm{Spc}(\mathcal{C}_{/S})_{\mathbf{V}}, \quad \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathbf{V}}, \quad \mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}, \quad \mathbf{H}^{\mathrm{cl}}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathbf{V}}.$$

Example 2.8.2. Taking $\mathbf{V}$ to be the stable presentable ∞-category Spt of spectra, we obtain ∞-categories of fibred motivic spectra. We will refer to these as fibred *motivic* S^1-*spectra*, to distinguish them from the notion of motivic spectra with respect to the Thom space of the trivial line bundle.

Remark 2.8.3. If $\mathbf{V}$ is *stable*, then so is $\mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}$ for any admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$.

Remark 2.8.4. The ∞-category $\mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}$ can also be described as the tensor product of presentable ∞-categories $\mathbf{H}(\mathcal{C}_{/S})\otimes\mathbf{V}$ in the sense of [HA, Sect. 4.8]. An analogous description holds for the classical variant $\mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$, for $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ admissible. It follows that when $\mathbf{V}$ is presentably symmetric monoidal, these categories also inherit presentably symmetric monoidal structures. Remark 2.3.7 then carries over to the $\mathbf{V}$-linear setting.

The description of Remark 2.8.4 immediately gives the following generalization of the comparison result of Theorem A (or rather the more precise statement proven in Subsect. 2.7):

Theorem 2.8.5. *Let S be a spectral affine scheme and let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a* narrow *subcategory. Let $\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S}$ be as in Example 2.4.2 and let $w : \mathcal{A}_{/S} \to \mathcal{A}_{/S_{\mathrm{cl}}}$ denote the classical truncation functor (Construction 2.4.6). Then for any presentable ∞-category $\mathbf{V}$, the adjunction*

$$\mathbf{L}w_! : \mathbf{H}(S)_{\mathbf{V}} \to \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})_{\mathbf{V}}, \quad w^* : \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})_{\mathbf{V}} \to \mathbf{H}(S)_{\mathbf{V}}$$

is an equivalence of ∞-categories. Moreover, this equivalence is compatible with the operations f^*, f_* for any morphism $f : T \to S$, with $f_\sharp$ when f exhibits T as an object of $\mathcal{A}_{/S}$, with products and with internal homs.

Finally, let us note the following two properties which are specific to the stable case.

Proposition 2.8.6. *Let $\mathbf{V}$ be a* stable *presentable ∞-category. Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. Then on the ∞-category of $\mathcal{C}$-fibred $\mathbf{V}$-objects, the $\mathbf{A}^1$-localization functor $\mathrm{L}_{\mathbf{A}^1}$ preserves the property of Nisnevich excision. In particular, the motivic localization functor $\mathbf{L}$ can be computed by the formula*

$$\mathbf{L} \simeq \mathrm{L}_{\mathbf{A}^1}\,\mathrm{L}_{\mathrm{Nis}}\,.$$

Proof. Let $\mathcal{F}$ be a $\mathcal{C}$-fibred $\mathbf{V}$-object. If $\mathcal{F}$ is Nisnevich-excisive, then its $\mathbf{A}^1$-localization $\mathrm{L}_{\mathbf{A}^1}(\mathcal{F})$ is still Nisnevich-excisive, in view of the formula (2.1.b) and the fact that colimits commute with finite limits in stable ∞-categories. Therefore the claim follows from Remark 2.1.16 (iii). □

Corollary 2.8.7. *Let $\mathbf{V}$ be a* stable *presentable ∞-category. Let $\mathcal{F} \in \mathbf{H}(S)_{\mathbf{V}}$ be an $\mathcal{A}$-fibred $\mathbf{V}$-object over S (where $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ is narrow). If $\mathcal{F}$ is Nisnevich-excisive, then the canonical map*

$$\mathrm{L}_{\mathbf{A}^1}(\mathcal{F}) \to w^*\,\mathrm{L}_{\mathbf{A}^1}\,w_!(\mathcal{F})$$

is invertible.

Proof. It follows from Theorem 2.8.5 (cf. Corollary 2.7.6) that the canonical map $\mathbf{L}(\mathcal{F}) \to w^*\mathbf{L}w_!(\mathcal{F})$ is invertible. Since $\mathcal{F}$ and hence $w_!(\mathcal{F})$ are Nisnevich-excisive, we conclude by Proposition 2.8.6. □

3. Comparison with $\mathbf{A}^{1,\flat}$-motivic homotopy theory

3.1. Flat affine spaces.

Notation 3.1.1. Let R be an $\mathcal{E}_\infty$-ring. Denote by $R[T_1, \ldots, T_n]$ denote the polynomial R-algebra in n variables $T_1, \ldots, T_n$ (in degree zero). This is by definition the monoid algebra $R[\mathbf{N}^n] = R \otimes \Sigma^\infty_+(\mathbf{N}^n)$, where $\mathbf{N}$ is the set of natural numbers, viewed as a discrete (additive) $\mathcal{E}_\infty$-monoid space. Note that we have canonical isomorphisms $\pi_*(R[T_1, \ldots, T_n]) \simeq \pi_*(R) \otimes_{\pi_0(R)} \pi_0(R)[T_1, \ldots, T_n]$, so that $R[T_1, \ldots, T_n]$ is flat over R.

Definition 3.1.2. For every $n \geqslant 0$, let $\mathbf{A}^{n,\flat}$ denote the affine spectral scheme $\mathrm{Spec}(\mathbf{S}[T_1,\ldots,T_n])$, where $\mathbf{S}$ is the sphere spectrum. Note that we have a canonical isomorphism $(\mathbf{A}^{n,\flat})_{\mathrm{cl}} \simeq \mathbf{A}^n_{\mathrm{cl}}$. We refer to $\mathbf{A}^{n,\flat}$ as the *flat affine space* (over the sphere spectrum). If S is classical, then $S \times \mathbf{A}^{n,\flat} = S \times \mathbf{A}^n_{\mathrm{cl}}$.

Remark 3.1.3. The affine spectral schemes $\mathbf{A}^{n,\flat}$ are equipped with the following additional structure:

(i) The flat affine line $\mathbf{A}^{1,\flat}$ has the structure of a commutative monoid under the operation of multiplication. This is induced by the cocommutative comonoid structure on the commutative monoid $\mathbf{N}$. For example, the multiplication morphism $\mathbf{A}^{1,\flat} \times \mathbf{A}^{1,\flat} \to \mathbf{A}^{1,\flat}$ corresponds to the diagonal $\Sigma^\infty_+(\mathbf{N}) \to \Sigma^\infty_+(\mathbf{N}\times\mathbf{N}) \simeq \Sigma^\infty_+(\mathbf{N}) \otimes \Sigma^\infty_+(\mathbf{N})$. Similarly the counit of $\mathbf{N}$ induces an $\mathcal{E}_\infty$-ring homomorphism $\mathbf{S}[T] \simeq \Sigma^\infty_+(\mathbf{N}) \to \Sigma^\infty_+(\mathrm{pt}) \simeq \mathbf{S}$ which corresponds to the unit section $s_1 : \mathrm{Spec}(\mathbf{S}) \to \mathbf{A}^{1,\flat}$.

(ii) The flat affine line $\mathbf{A}^{1,\flat}$ also admits a zero section $s_0 : \mathrm{Spec}(\mathbf{S}) \to \mathbf{A}^{1,\flat}$, which can be constructed as follows. Identify the discrete pointed $\mathcal{E}_\infty$-monoid space pt_+ with the set $\{0,1\}$, viewed as a multiplicative monoid with base point 0 and identity element 1. Since $\mathbf{N}$ is freely generated as a (discrete) commutative monoid by the element $1 \in \mathbf{N}$, either choice of element $i \in \{0,1\}$ gives rise to a unique homomorphism $\sigma'_i : \mathbf{N}_+ \to \mathrm{pt}_+$ of pointed commutative monoids such that $\sigma'_i(1) = i$. Regarding σ'_i as a homomorphism of discrete pointed $\mathcal{E}_\infty$-monoid spaces, application of the symmetric monoidal functor Σ^∞ produces $\mathcal{E}_\infty$-ring homomorphisms

$$\sigma_i : \mathbf{S}[T] \simeq \Sigma^\infty(\mathbf{N}_+) \to \Sigma^\infty(\mathrm{pt}_+) \simeq \mathbf{S}$$

for each $i \in \{0,1\}$. For $i = 1$ this is the same homomorphism defining the unit section s_1, and we let s_0 denote the section $\mathrm{Spec}(\mathbf{S}) \to \mathrm{Spec}(\mathbf{S}[T]) = \mathbf{A}^{1,\flat}$ corresponding to σ_0.

(iii) For every $n \geqslant 0$, the flat affine space $\mathbf{A}^{n,\flat}$ admits the structure of a module over $\mathbf{A}^{1,\flat}$. This is induced by the canonical comodule structure on the commutative monoid $\mathbf{N}^n$ over the comonoid $\mathbf{N}$, where the coaction homomorphism sends $\mathbf{N}^n \to \mathbf{N} \times \mathbf{N}^n$

$$(k_1, k_2, \ldots, k_n) \mapsto (k_1 + \cdots + k_n, k_1, k_2, \ldots, k_n).$$

In particular, there is a "scalar multiplication" morphism

$$\mathbf{A}^{1,\flat} \times \mathbf{A}^{n,\flat} \to \mathbf{A}^{n,\flat}.$$

(iv) After base change along $\mathrm{Spec}(\mathbf{Z}) \to \mathrm{Spec}(\mathbf{S})$, the flat affine spaces $\mathrm{Spec}(\mathbf{Z}) \times \mathbf{A}^{n,\flat} \simeq \mathbf{A}^n_{\mathrm{cl}}$ become abelian groups under addition.

Remark 3.1.4. The zero section of $\mathbf{A}^{1,\flat}$ is compatible with the multiplicative structure, in the sense that $\mathbf{A}^{1,\flat}$ defines an *interval object* in the sense of Morel–Voevodsky [MV99, Sect. 2.3].

3.2. $\mathrm{Sm}^\flat$**-fibred motivic spaces.** Unlike $S \times \mathbf{A}^n$, the flat affine spaces $S \times \mathbf{A}^{n,\flat}$ are not smooth over S in the sense of Definition 2.1.1 (except under the conditions of Remark 3.3.1). However, they are smooth in the following sense:

Definition 3.2.1. A morphism of affine spectral schemes $X \to S$ is called *fibre-smooth* if it is almost of finite presentation, flat, and on classical truncations induces a morphism $X_{\mathrm{cl}} \to S_{\mathrm{cl}}$ that is smooth in the sense of classical algebraic geometry.

Remark 3.2.2. From [SAG, Cor. 11.2.4.2] and [EGA IV$_4$, § 17.3] it follows that a morphism of affine spectral schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is fibre-smooth if and only if $A \to B$ is fibre-smooth in the sense of [SAG, Def. 11.2.3.1].

Remark 3.2.3. Let $f : X \to S$ be a fibre-smooth morphism. Then Zariski-locally on X, there exists an étale S-morphism $X \to S \times \mathbf{A}^{n,\flat}$ for some $n \geqslant 0$. This follows from Remark 11.2.3.5 and Proposition 11.2.4.1 of [SAG], combined with [HA, Thm. 7.5.0.6]. Contrast with Remark 2.1.4.

The following is the same as Definition 2.1.7 except for the last condition.

Definition 3.2.4. We say that a full subcategory $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ is $\flat$*-admissible* if it is essentially small and satisfies the following conditions:

(i) The affine spectral scheme S (viewed over S via the identity) belongs to $\mathcal{C}^\flat_{/S}$.

(ii) If X belongs to $\mathcal{C}^\flat_{/S}$ and Y is étale over X, then Y belongs to $\mathcal{C}^\flat_{/S}$.

(iii) If X belongs to $\mathcal{C}^\flat_{/S}$, then $X \times \mathbf{A}^{n,\flat}$ belongs to $\mathcal{C}^\flat_{/S}$ for every $n \geqslant 0$.

A $\flat$*-narrow* subcategory $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ is a $\flat$-admissible subcategory which is contained in the full subcategory $\mathrm{Sm}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ of fibre-smooth spectral affine schemes over S.

Example 3.2.5. Let $\mathcal{A}^{\flat,0}_{/S} \subseteq \mathrm{Aff}_{/S}$ denote the full subcategory spanned by $X \in \mathrm{Aff}_{/S}$ for which the structural morphism $X \to S$ factors through an étale morphism

$$X \to S \times \mathbf{A}^{n,\flat}$$

over S. Then $\mathcal{A}^{\flat,0}_{/S}$ is the minimal $\flat$-admissible subcategory of $\mathrm{Aff}_{/S}$ (same proof as Example 2.1.9).

Remark 3.2.6. Let S be a spectral affine scheme. Let $\mathcal{C}^\flat \subseteq \mathrm{Aff}_{/S}$ be a $\flat$-admissible subcategory. As in Example 2.4.2, the full subcategory $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ spanned by classical truncations of objects in $\mathcal{C}^\flat$ is admissible.

Definition 3.2.7. Let S be a spectral affine scheme and $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ a $\flat$-admissible subcategory. Let $\mathcal{F}$ be a $\mathcal{C}^\flat$-fibred space (Definition 2.1.13), i.e., a presheaf of spaces on $\mathcal{C}^\flat_{/S}$. We say that $\mathcal{F}$ is Nisnevich excisive if it is reduced and sends Nisnevich squares Q to cartesian squares $\Gamma(Q, \mathcal{F})$ (Definition 2.1.5). We say that $\mathcal{F}$ satisfies $\mathbf{A}^{1,\flat}$*-homotopy invariance* if for any $X \in \mathcal{C}^\flat_{/S}$, the canonical map $\Gamma(X, \mathcal{F}) \to \Gamma(X \times \mathbf{A}^{1,\flat}, \mathcal{F})$ is invertible, where $\mathbf{A}^{1,\flat}$ denotes the flat affine line (Definition 3.1.2). We say that $\mathcal{F}$ is *motivic* if it is Nisnevich excisive and $\mathbf{A}^{1,\flat}$-homotopy invariant. We denote by $\mathbf{H}(\mathcal{C}^\flat_{/S}) \subseteq \mathrm{Spc}(\mathcal{C}^\flat_{/S})$ the full subcategory of motivic $\mathcal{C}^\flat$-fibred spaces.

Notation 3.2.8. If $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ is both admissible and $\flat$-admissible, then there is possible ambiguity in the terminology "$\mathcal{C}$-fibred motivic space" and in the notation $\mathbf{H}(\mathcal{C}_{/S})$. To maintain the distinction we introduce the following convention: we write $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ for the same full subcategory when it is to be regarded as a $\flat$-admissible subcategory. Thus a *motivic* $\mathcal{C}$*-fibred space* is an $\mathbf{A}^1$-invariant Nisnevich-excisive $\mathcal{C}$-fibred space, while a *motivic* $\mathcal{C}^\flat$*-fibred space* is an $\mathbf{A}^{1,\flat}$-invariant Nisnevich-excisive $\mathcal{C}$-fibred space. In particular, $\mathbf{H}(\mathcal{C}_{/S})$ and $\mathbf{H}(\mathcal{C}^\flat_{/S})$ are two distinct full subcategories of $\mathrm{Spc}(\mathcal{C}_{/S}) = \mathrm{Spc}(\mathcal{C}^\flat_{/S})$.

Remark 3.2.9. For any $\flat$-admissible subcategory $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$, the full subcategories of Nisnevich-excisive, $\mathbf{A}^{1,\flat}$-homotopy invariant, and motivic $\mathcal{C}^\flat$-fibred spaces are each left Bousfield localizations of the ∞-category of $\mathcal{C}^\flat$-fibred spaces. We write $\mathrm{L}_{\mathrm{Nis}}$, $\mathrm{L}_{\mathbf{A}^{1,\flat}}$, and $\mathbf{L}^\flat$ for the respective localization functors. The remarks in 2.1.16 remain valid *mutatis mutandis* up to replacing the spectral affine spaces $\mathbf{A}^n$ by the flat affine spaces $\mathbf{A}^{n,\flat}$ in item (ii).

Proposition 3.2.10. *Let S be a spectral affine scheme. Fix an inclusion $\mathcal{C}^\flat_{/S} \subseteq \mathcal{D}^\flat_{/S}$ of $\flat$-admissible subcategories of $\mathrm{Aff}_{/S}$. Denote by $\iota : \mathcal{C}^\flat_{/S} \hookrightarrow \mathcal{D}^\flat_{/S}$ the inclusion functor and by $\iota_! : \mathrm{Spc}(\mathcal{C}^\flat_{/S}) \to \mathrm{Spc}(\mathcal{D}^\flat_{/S})$ its left Kan extension. Then the assignment $\mathcal{F} \mapsto \mathbf{L}^\flat \iota_!(\mathcal{F})$ induces a fully faithful functor of ∞-categories*

$$\mathbf{L}^\flat \iota_! : \mathbf{H}(\mathcal{C}^\flat_{/S}) \to \mathbf{H}(\mathcal{D}^\flat_{/S}),$$

whose essential image is generated under sifted colimits by objects of the form $\mathbf{L}^\flat(\mathrm{h}_S(X))$, where X belongs to $\mathcal{C}_{/S}$.

Proof. Same proof as Proposition 2.2.3. □

Remark 3.2.11. Let $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ be $\flat$-admissible. Using the interval structure on $\mathbf{A}^{1,\flat}$ (Remarks 3.1.3 and 3.1.4), we can make sense of $\mathbf{A}^{1,\flat}$*-homotopies* between morphisms of $\mathcal{C}^\flat$-fibred spaces, and therefore of *strict* $\mathbf{A}^{1,\flat}$*-homotopy equivalences* (see [Kha19b, Def. 2.3.6]).

3.3. $\mathbf{A}^1$-homotopies vs. $\mathbf{A}^{1,\flat}$-homotopies.

Remark 3.3.1. For any connective $\mathcal{E}_\infty$-ring R and integer $n \geqslant 0$, there is a canonical homomorphism of $\mathcal{E}_\infty$-R-algebras $R\{T_1, \ldots, T_n\} \to R[T_1, \ldots, T_n]$ determined by the assignment $T_i \mapsto T_i$. This gives rise to canonical morphisms

$$\varepsilon_S : S \times \mathbf{A}^{n,\flat} \to S \times \mathbf{A}^n$$

for every affine spectral scheme S, which are invertible if and only if either $n = 0$ or S is of characteristic zero (i.e., $S = \mathrm{Spec}(R)$ with R an $\mathcal{E}_\infty$-$\mathbf{Q}$-algebra).

Remark 3.3.2. The map $\varepsilon : \mathbf{A}^{1,\flat} \to \mathbf{A}^1$ is compatible with interval structures. That is, it preserves the zero and unit sections, and is compatible with the multiplicative structures in the sense that the diagram

$$\begin{array}{ccc} \mathbf{A}^{1,\flat} \times \mathbf{A}^{1,\flat} & \longrightarrow & \mathbf{A}^{1,\flat} \\ \downarrow & & \downarrow \\ \mathbf{A}^1 \times \mathbf{A}^1 & \longrightarrow & \mathbf{A}^1 \end{array}$$

commutes. By restriction of scalars, we may therefore regard $\mathbf{A}^n$ as a module over $\mathbf{A}^{1,\flat}$. Note that for every $n \geq 0$, the morphism $\varepsilon : \mathbf{A}^{n,\flat} \to \mathbf{A}^n$ is then $\mathbf{A}^{1,\flat}$-linear.

Note also that over $\mathrm{Spec}(\mathbf{Z})$, ε defines a group homomorphism

$$\mathbf{A}^n_{\mathrm{cl}} \simeq \mathrm{Spec}(\mathbf{Z}) \times \mathbf{A}^{n,\flat} \to \mathrm{Spec}(\mathbf{Z}) \times \mathbf{A}^n$$

with respect to the additive structures (Remark 3.1.3(iv)).

Lemma 3.3.3. *Let S be an affine spectral scheme. Let $\mathcal{C}_{/S}$ be an admissible and $\flat$-admissible subcategory of* $\mathrm{Aff}_{/S}$. *Then we have:*

(i) *Every $\mathbf{A}^1$-local equivalence between $\mathcal{C}$-fibred spaces over S is an $\mathbf{A}^{1,\flat}$-local equivalence.*

(ii) *Every $\mathbf{A}^{1,\flat}$-homotopy invariant $\mathcal{C}$-fibred space over S is $\mathbf{A}^1$-homotopy invariant.*

In particular, there is an inclusion

$$\mathbf{H}(\mathcal{C}^\flat_{/S}) \subseteq \mathbf{H}(\mathcal{C}_{/S})$$

of subcategories of $\mathrm{Spc}(\mathcal{C}_{/S})$.

Proof. By adjunction, the two statements are equivalent. To prove the first it will suffice to show that for every $X \in \mathcal{C}_{/S}$, the morphism $\mathrm{h}_S(X \times \mathbf{A}^1) \to \mathrm{h}_S(X)$ is an $\mathbf{A}^{1,\flat}$-local equivalence. In fact, we claim that every strict $\mathbf{A}^1$-homotopy equivalence is a strict $\mathbf{A}^{1,\flat}$-homotopy equivalence (hence *a fortiori* an $\mathbf{A}^{1,\flat}$-local equivalence). Indeed, the canonical map $\varepsilon_S : S \times \mathbf{A}^{1,\flat} \to S \times \mathbf{A}^1$ (Remark 3.3.1) is a morphism of interval objects (Remark 3.3.2), so composition with ε_S sends elementary $\mathbf{A}^1$-homotopies to elementary $\mathbf{A}^{1,\flat}$-homotopies. □

The following key lemma shows that the comparison morphism $\varepsilon : \mathbf{A}^{n,\flat} \to \mathbf{A}^n$ is a "universal" $\mathbf{A}^{1,\flat}$-local equivalence, at least over $\mathrm{Spec}(\mathbf{Z})$. The reason for this restriction is that the proof uses the additive structure on the flat affine spaces (Remark 3.1.3(iv)).

Lemma 3.3.4. *Let S be a spectral affine scheme defined over* $\mathrm{Spec}(\mathbf{Z})$. *Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible and $\flat$-admissible subcategory. Given $X \in \mathcal{C}_{/S}$ and an S-morphism $f : X \to \mathbf{A}^n_S$, consider the cartesian square*

$$\begin{array}{ccc} X^\flat & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ S \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_S} & S \times \mathbf{A}^n. \end{array}$$

Then the morphism $\mathrm{h}_S(X^\flat) \to \mathrm{h}_S(X)$ *is an* $\mathbf{A}^{1,\flat}$*-local equivalence of* $\mathcal{C}$*-fibred spaces.*

Proof. Note that f induces a section $s : X \to (S \times \mathbf{A}^n) \times_S X = X \times \mathbf{A}^n$ of the projection $X \times \mathbf{A}^n \to X$, and similarly g induces a section $t : X \to X \times \mathbf{A}^{n,\flat}$. These fit into a factorization of the given square:

$$\begin{array}{ccc} X^\flat & \longrightarrow & X \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ X \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_X} & X \times \mathbf{A}^n \\ \downarrow & & \downarrow \\ S \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_S} & S \times \mathbf{A}^n \end{array}$$

Since the lower square is cartesian, so is the upper square. Therefore we may as well assume that $X = S$, in which case the claim becomes that $\mathrm{h}_X(X^\flat)$ is $\mathbf{A}^{1,\flat}$-contractible. Since S is defined over $\mathrm{Spec}(\mathbf{Z})$, ε is a group homomorphism with respect to the additive structures (Remark 3.3.2). Therefore it gives rise to an isomorphism between $X^\flat$ and the fibre of ε over the zero section. Thus we may also assume that s is the zero section.

Since $s : X \to \mathbf{A}^n_X$ lifts to the zero section $s : X \to \mathbf{A}^n_{\flat,X}$, it induces an X-morphism $X \to X^\flat$ and hence a base point of the $\mathcal{C}$-fibred space $\mathrm{h}_X(X^\flat)$. It will suffice to exhibit an $\mathbf{A}^{1,\flat}$-homotopy contracting $\mathrm{h}_X(X^\flat)$ to this base point. Recall that $\mathbf{A}^{1,\flat}$ acts compatibly on $\mathbf{A}^n_{\flat,X}$ and $\mathbf{A}^n_X$ (Remarks 3.1.3 and 3.3.2). The action on the latter restricts along the zero section $s : X \to \mathbf{A}^n_X$ to the trivial action on X. The induced $\mathbf{A}^{1,\flat}$-action on $X^\flat$ is a morphism

$$\mathbf{A}^{1,\flat} \times X^\flat \to X^\flat,$$

which induces the $\mathbf{A}^{1,\flat}$-homotopy desired. ◻

Corollary 3.3.5. *Let S be a spectral affine scheme over* $\mathrm{Spec}(\mathbf{Z})$*. Let* $\mathcal{A}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ *be a* $\flat$*-narrow subcategory and let* $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ *be an admissible and* $\flat$*-admissible subcategory containing* $\mathcal{A}^\flat_{/S}$*. Let* $\iota : \mathcal{A}^\flat_{/S} \hookrightarrow \mathcal{C}_{/S}$ *denote the inclusion. Then the essential image of the fully faithful functor (Proposition 3.2.10)*

$$\mathbf{L}^\flat \iota_! : \mathbf{H}(\mathcal{A}^\flat_{/S}) \to \mathbf{H}(\mathcal{C}^\flat_{/S})$$

is generated under sifted colimits by objects of the form $\mathbf{L}^\flat \mathrm{h}_S(X)$*, where* $X \in \mathrm{Sm}_{/S}$ *is affine and admits an étale S-morphism to the* spectral *affine space* $\mathbf{A}^n_S$*, for some* $n \geqslant 0$*.*

Proof. Let $\mathcal{E}$ and $\mathcal{E}_\flat$ denote the full subcategories of $\mathbf{H}(\mathcal{C}^\flat_{/S})$ generated under sifted colimits by objects of the form $\mathbf{L}^\flat \mathrm{h}_S(X)$, where X admits an étale S-morphism to $S \times \mathbf{A}^n$, respectively to $S \times \mathbf{A}^{n,\flat}$, for some $n \geqslant 0$. We may as well assume $\mathcal{A}^\flat_{/S}$ is the minimal $\flat$-admissible subcategory (Example 3.2.5), spanned by $X \in \mathrm{Aff}_{/S}$ which admit an étale morphism $X \to S \times \mathbf{A}^{n,\flat}$ for some $n \geqslant 0$. Then by Proposition 3.2.10, $\mathcal{E}_\flat$ is identified with the essential image of the functor in question, so it will suffice to show that $\mathcal{E} = \mathcal{E}_\flat$.

Let $X \in \mathrm{Aff}_{/S}$. Suppose X admits an étale S-morphism $f : X \to S \times \mathbf{A}^n$ for some $n \geqslant 0$. The base change of f along $\varepsilon_S : S \times \mathbf{A}^{n,\flat} \to S \times \mathbf{A}^n$ is an étale morphism $X^\flat \to S \times \mathbf{A}^{n,\flat}$. By

Lemma 3.3.4, the canonical morphism $X^\flat \to X$ induces an isomorphism $\mathbf{L}^\flat \mathrm{h}_S(X^\flat) \simeq \mathbf{L}^\flat \mathrm{h}_S(X)$, hence in particular $\mathbf{L}^\flat \mathrm{h}_S(X) \in \mathcal{E}_\flat$. This shows $\mathcal{E} \subseteq \mathcal{E}_\flat$.

For the other direction, suppose $X \in \mathrm{Aff}_{/S}$ admits an étale S-morphism $g : X \to S \times \mathbf{A}^{n,\flat}$ for some $n \geqslant 0$. Since ε_S is an isomorphism on classical truncations, it follows from [HA, Thm. 7.5.0.6] that there exists $Y \in \mathrm{Aff}_{/S}$, an étale S-morphism $f : Y \to S \times \mathbf{A}^n$, and a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ S \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_S} & S \times \mathbf{A}^n. \end{array}$$

By Lemma 3.3.4, there is an isomorphism $\mathbf{L}^\flat \mathrm{h}_S(X) \simeq \mathbf{L}^\flat \mathrm{h}_S(Y)$, hence in particular $\mathbf{L}^\flat \mathrm{h}_S(X) \in \mathcal{E}$. This shows $\mathcal{E}_\flat \subseteq \mathcal{E}$. □

3.4. **The comparison.** In this subsection we prove the following statement, which in particular yields the equivalence between (ii) and (iii) in Theorem A.

Theorem 3.4.1. *Let S be a spectral affine scheme defined over* $\mathrm{Spec}(\mathbf{Z})$*. Let $\mathcal{A}^\flat_{/S}$ be any $\flat$-narrow subcategory of* $\mathrm{Aff}_{/S}$ *and let $w^\flat : \mathcal{A}^\flat_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S}$ be the restriction of the classical truncation functor $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. Then the adjunction*

$$\mathbf{L}w^\flat_! : \mathbf{H}(\mathcal{A}^\flat_{/S}) \to \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S}) \quad w^* : \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S}) \to \mathbf{H}(\mathcal{A}^\flat_{/S})$$

is an equivalence.

We fix the following notation for this subsection.

Notation 3.4.2. Let S be a spectral affine scheme. Let $\mathcal{A}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ be a $\flat$-narrow subcategory and $\mathcal{B} \subseteq \mathrm{Aff}_{/S}$ be a $\flat$-admissible and broad subcategory containing $\mathcal{A}^\flat_{/S}$. Let $\iota : \mathcal{A}^\flat_{/S} \hookrightarrow \mathcal{B}_{/S}$ denote the inclusion. Let $\mathcal{B}^\flat_{/S}$ be as in Remark 3.2.8 and let $\mathcal{B}^{\mathrm{cl}}_{/S} \subseteq \mathrm{AffCl}_{/S}$ be as in Example 2.4.2. We write

$$\mathbf{H}(\mathcal{B}_{/S}), \quad \mathbf{H}(\mathcal{B}^\flat_{/S}), \quad \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$$

for the ∞-categories of motivic spaces formed respectively out of the admissible subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$, the $\flat$-admissible subcategory $\mathcal{B}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$, and the admissible subcategory $\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S}$.

Remark 3.4.3. By construction the classical truncation functor $v : \mathrm{Aff}_{/S} \to \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ (Construction 2.4.6) restricts to a functor $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. Since the latter preserves Nisnevich squares and sends $\mathbf{A}^{1,\flat}$ to $\mathbf{A}^1_{\mathrm{cl}}$, one sees as in Lemma 2.4.7 that v induces an adjunction

$$\mathbf{L}v_! : \mathbf{H}(\mathcal{B}^\flat_{/S}) \to \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}^\flat_{/S}).$$

Proposition 2.5.4 also holds *mutatis mutandis* for the functors

$$v_! : \mathrm{Spc}(\mathcal{B}^\flat_{/S}) \to \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathrm{Spc}(\mathcal{B}^\flat_{/S}).$$

In particular, v^* commutes with $\mathbf{L}^\flat$. By Remark 2.5.6 this implies that $\mathbf{L}^\flat$ preserves nil-local objects.

Corollary 3.4.4. *The functor*

$$v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}^{\flat}_{/S})$$

is fully faithful, with essential image spanned by the nil-local objects of $\mathbf{H}(\mathcal{B}^{\flat}_{/S})$.

Proof. Follows immediately by combining Theorem 2.5.3 with Lemma 3.3.3. □

Corollary 3.4.5. *If S is defined over* $\mathrm{Spec}(\mathbf{Z})$, *then for every $\mathcal{F} \in \mathbf{H}(\mathcal{A}^{\flat}_{/S})$, the $\mathcal{B}^{\flat}$-fibred motivic space $\mathbf{L}^{\flat}\iota_!(\mathcal{F}) \in \mathbf{H}(\mathcal{B}^{\flat}_{/S})$ is nil-local.*

Proof. By Theorem 2.6.2, every object in the essential image of

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}^{0}_{/S}) \to \mathbf{H}(\mathcal{B}_{/S})$$

is nil-local, where $\mathcal{A}^0_{/S}$ is the minimal admissible subcategory of $\mathrm{Aff}_{/S}$ (Example 2.1.9). It follows from Remark 3.4.3 that the same holds for objects in the essential image of

$$\mathbf{L}^{\flat}\mathbf{L}\iota_! \simeq \mathbf{L}^{\flat}\iota_! : \mathbf{H}(\mathcal{A}^{0}_{/S}) \to \mathbf{H}(\mathcal{B}^{\flat}_{/S}),$$

where the isomorphism is due to Lemma 3.3.3. By Corollary 3.3.5 this coincides with the essential image of the functor in question. □

Proof of Theorem 3.4.1. We are free to choose any $\flat$-admissible and broad subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ containing $\mathcal{A}^{\flat}_{/S}$ as in Notation 3.4.2. By Proposition 3.2.10 we have the commutative diagram

$$\begin{array}{ccc} \mathbf{H}(\mathcal{A}^{\flat}_{/S}) & \xrightarrow{\mathbf{L}w^{\flat}_!} & \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \\ \downarrow{\scriptstyle \mathbf{L}^{\flat}\iota_!} & & \downarrow{\scriptstyle \mathbf{L}\iota_!} \\ \mathbf{H}(\mathcal{B}^{\flat}_{/S}) & \xrightarrow{\mathbf{L}v_!} & \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \end{array}$$

where the vertical arrows are fully faithful. Write $\langle \mathcal{A}^{\flat}_{/S} \rangle$ for the essential image of the left-hand vertical arrow, and $\langle \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \rangle$ for that of the right-hand vertical arrow. It will suffice to show that $\mathbf{L}v_!$ restricts to an equivalence $\langle \mathcal{A}^{\flat}_{/S} \rangle \simeq \langle \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \rangle$. By Corollary 3.4.5, every object in the source is nil-local. Thus the claim follows from Corollary 3.4.4. □

Corollary 3.4.6. *For any quasi-compact quasi-separated spectral algebraic space S over* $\mathrm{Spec}(\mathbf{Z})$, *there are canonical equivalences of ∞-categories*

$$\mathbf{H}^{\flat}(S) \simeq \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}).$$

Here $\mathbf{H}^{\flat}(S)$ is the ∞-category of $\mathbf{A}^{1,\flat}$-invariant Nisnevich sheaves on the site of (quasi-compact quasi-separated) fibre-smooth spectral algebraic spaces over S (where fibre-smoothness is defined as in [SAG, Def. 11.2.5.5]*).*

Proof. Follows from Theorem 3.4.1 by Nisnevich descent as in Corollary 2.7.5. □

4. The Bass construction

4.1. **Localizing invariants.** We briefly establish our notations and conventions for localizing invariants. We work with R-linear stable ∞-categories, over a fixed connective $\mathcal{E}_\infty$-ring R, although the discussion makes sense in the greater generality of $\mathrm{Perf}(S)$-linear ∞-categories, for any spectral algebraic space S.

Notation 4.1.1. Let Stab denote the ∞-category of small stable ∞-categories. For any connective $\mathcal{E}_\infty$-ring R, let Stab_R denote the ∞-category of small stable R-linear ∞-categories.

Notation 4.1.2. Let R be a connective $\mathcal{E}_\infty$-ring and $E : \mathrm{Stab}_R \to \mathrm{Spt}$ a functor. For any R-algebra A, resp. spectral algebraic space X over $\mathrm{Spec}(R)$, we set

$$E(A) \coloneqq E(\mathrm{Perf}_A), \quad \text{resp. } E(X) \coloneqq E(\mathrm{Perf}(X)),$$

where Perf_A is the stable ∞-category of A-modules and $\mathrm{Perf}(X)$ is the stable ∞-category of perfect complexes on X.

Remark 4.1.3. Any R-linear stable ∞-category $\mathbf{A}$ corepresents a functor $h_{\mathbf{A}} : \mathrm{Stab}_R \to \mathrm{Spt}$ given by the assignment

$$h_{\mathbf{A}}(\mathbf{A}') = \mathrm{Maps}_R(\mathbf{A}, \mathbf{A}'),$$

where Maps_R here denotes the mapping spectrum in the ∞-category of functors $\mathrm{Stab}_R \to \mathrm{Spt}$.

Remark 4.1.4. The Day convolution product endows the ∞-category of functors $\mathrm{Stab}_R \to \mathrm{Spt}$ with a closed symmetric monoidal structure for which the Yoneda embedding is symmetric monoidal. Given a functor $E : \mathrm{Stab}_R \to \mathrm{Spt}$, we write $E^{\mathbf{A}}$ for the internal hom object $\underline{\mathrm{Hom}}_R(h_{\mathbf{A}}, E)$, for any $\mathbf{A} \in \mathrm{Stab}_R$. Note that the assignment $\mathbf{A} \mapsto h_{\mathbf{A}}$ is *contravariant*, while $\mathbf{A} \mapsto E^{\mathbf{A}}$ is *covariant*. By the Yoneda lemma, the functor $E^{\mathbf{A}} : \mathrm{Stab}_R \to \mathrm{Spt}$ is given by

$$E^{\mathbf{A}}(\mathbf{A}') = E(\mathbf{A} \otimes \mathbf{A}')$$

for every $\mathbf{A}' \in \mathrm{Stab}_R$.

Definition 4.1.5. We say that E is *additive* if it sends split exact sequences of stable ∞-categories [BGT13, Def. 5.18] to split exact triangles of spectra. We say E is *localizing* if it sends short exact sequences of stable ∞-categories [BGT13, Def. 5.12] to exact triangles of spectra. Note that, unlike [BGT13], we do not require that E commutes with filtered colimits.

Remark 4.1.6. Note that if E is additive (resp. localizing), then the same holds for $E^{\mathbf{A}}$ for any $\mathbf{A} \in \mathrm{Stab}_R$.

Example 4.1.7. Let $\mathrm{K} : \mathrm{Stab} \to \mathrm{Spt}$ denote the algebraic K-theory functor. Recall that this is defined using the Waldhausen $S_\bullet$-construction (see [HA, Rmk. 1.2.2.5], [BGT13, Def. 7.1], or [Bar16, Sect. 10]) and takes values in connective spectra. Then K is additive by Waldhausen's additivity theorem ([BGT13, Prop. 7.10]).

Let $\mathbb{K} : \mathrm{Stab} \to \mathrm{Spt}$ denote the nonconnective algebraic K-theory functor, defined e.g. as in [BGT13, Def. 9.6] following Schlichting. This is a localizing invariant such that $\mathbb{K}_{\geq 0} \simeq \mathrm{K}$.

4.2. **The projective bundle formula.** In this subsection we prove a projective bundle formula computing $E(\mathbf{P}^{1,\flat}_R)$ for the *flat* projective line over any connective $\mathcal{E}_\infty$-ring R, and any R-linear additive invariant E. This essentially follows from a result of Lurie [SAG, Thm. 7.2.2.1].

4.2.1. Consider the following subsets of $\mathbf{Z} \times \mathbf{Z}$:

- M^+ consists of pairs (m, n) with $m + n = 0$ and $m \geqslant 0$.
- M^- consists of pairs (m, n) with $m + n = 0$ and $n \geqslant 0$.
- $M^\pm$ consists of pairs (m, n) with $m + n = 0$.

We view each of these as (additive) discrete commutative monoids. For any connective $\mathcal{E}_\infty$-ring R, we write $R[T]$, $R[T^{-1}]$ and $R[T^\pm]$ for the monoid algebras $R \otimes \Sigma^\infty_+(M^+)$, $R \otimes \Sigma^\infty_+(M^-)$, and $R \otimes \Sigma^\infty_+(M^\pm)$, respectively.

4.2.2. We write $p_+ : \mathrm{Spec}(R[T]) \to \mathrm{Spec}(R)$, $p_- : \mathrm{Spec}(R[T^{-1}]) \to \mathrm{Spec}(R)$, and $p_\pm : \mathrm{Spec}(R[T^\pm]) \to \mathrm{Spec}(R)$ for the respective projections. Note that under the obvious isomorphisms $M^+ \simeq \mathbf{N} \simeq M^-$, both $\mathrm{Spec}(R[T])$ and $\mathrm{Spec}(R[T^{-1}])$ are canonically identified with the *flat affine line* $\mathbf{A}^{1,\flat}_R$ (Definition 3.1.2). Similarly, under the isomorphism $M^\pm \simeq \mathbf{Z}$, $R[T^\pm]$ is identified with the monoid algebra $R \otimes \Sigma^\infty_+(\mathbf{Z})$. It can also be identified with the localization of $R[T]$ away from $T \in \pi_0(R[T]) \simeq \pi_0(R)[T]$, or the localization of $R[T^{-1}]$ away from T^{-1}. In particular, the projections p_+, p_- and $p_\pm$ are fibre-smooth in the sense of [SAG, Def. 11.2.3.1], and the affine spectral scheme $\mathbf{G}^\flat_{m,R} = \mathrm{Spec}(R[T^\pm])$ is equipped with open immersions

$$j_+ : \mathrm{Spec}(R[T^\pm]) \to \mathrm{Spec}(R[T]), \quad j_- : \mathrm{Spec}(R[T^-]) \to \mathrm{Spec}(R[T]).$$

The construction of the zero and unit sections of $\mathbf{A}^{1,\flat}$ described in Remark 3.1.3 can be adapted as follows. We recall the constructions of the zero and unit sections of $\mathrm{Spec}(R[T])$. Consider the set $\{0, 1\}$, viewed as a pointed multiplicative monoid with base point 0 and identity element 1. Since M^+ is freely generated as a (discrete) commutative monoid by the element $(1, 0) \in M^+$, either choice of element $i \in \{0, 1\}$ determines a unique homomorphism $M^+ \to \{0, 1\}$ sending $(1, 0) \mapsto i$. Each of these gives rise to $\mathcal{E}_\infty$-ring homomorphisms

$$\sigma_i : R[T] \to R \otimes \Sigma^\infty(\{0, 1\}) \simeq R,$$

where we identify $\{0, 1\}$ with the pointed 0-sphere S^0. We let s_i denote the induced morphisms

$$s_i : \mathrm{Spec}(R) \to \mathrm{Spec}(R[T])$$

for each $i \in \{0, 1\}$. The obvious analogous construction gives sections $s_i : \mathrm{Spec}(R) \to \mathrm{Spec}(R[T^{-1}])$. Similarly it is clear that the unit section s_1 factors through a morphism $s_1 : \mathrm{Spec}(R) \to \mathrm{Spec}(R[T^\pm])$.

4.2.3. We denote by $\mathbf{P}^{1,\flat}_R$ the *flat projective line* over R, see [SAG, Constr. 5.4.1.3] (where it is denoted $\mathbf{P}^1_R$). This is equipped with a canonical morphism $q : \mathbf{P}^{1,\flat}_R \to \mathrm{Spec}(R)$ which is fibre-smooth. Moreover there is a cartesian and cocartesian square of spectral schemes

$$\begin{array}{ccc} \mathrm{Spec}(R[T^{\pm}]) & \xrightarrow{j_-} & \mathrm{Spec}(R[T^{-1}]) \\ \downarrow{\scriptstyle j_+} & & \downarrow{\scriptstyle k_-} \\ \mathrm{Spec}(R[T]) & \xrightarrow{k_+} & \mathbf{P}^{1,\flat}_R \end{array} \tag{4.2.a}$$

where every arrow is an open immersion. In particular, this is a Nisnevich square.

Notation 4.2.4. For a functor $E : \mathrm{Stab}_R \to \mathrm{Spt}$, we write:

$$E^+ := E^{\mathrm{Perf}(R[T])}, \quad E^- := E^{\mathrm{Perf}(R[T^{-1}])}, \quad E^{\pm} := E^{\mathrm{Perf}(R[T^{\pm}])}, \quad E^{\boxtimes} := E^{\mathrm{Perf}(\mathbf{P}^{1,\flat}_R)}.$$

Note that $E^+(\mathbf{A}) = E(\mathbf{A} \otimes \mathrm{Perf}(R[T]))$ for $\mathbf{A} \in \mathrm{Stab}_R$, and similarly for E^-, $E^{\pm}$ and $E^{\boxplus}$.

Theorem 4.2.5. *Let E be an additive invariant of R-linear stable ∞-categories. Then the two functors* $\mathrm{Perf}(\mathrm{Spec}(R)) \to \mathrm{Perf}(\mathbf{P}^{1,\flat}_R)$*, given by $\mathcal{F} \mapsto q^*(\mathcal{F})$ and $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(-1)$, induce a canonical isomorphism*

$$q^* \oplus (q^* \otimes \mathcal{O}(-1)) : E \oplus E \xrightarrow{\sim} E^{\boxplus}.$$

Proof. By Yoneda, it will suffice to show that the maps

$$E(\mathbf{A}) \oplus E(\mathbf{A}) \to E(\mathbf{A} \otimes \mathrm{Perf}(\mathbf{P}^{1,\flat}_R))$$

are invertible for every $\mathbf{A} \in \mathrm{Stab}_R$. For this it will suffice to show that the map

$$E(R) \oplus E(R) \to E(\mathbf{P}^{1,\flat}_R)$$

is invertible for every additive invariant E (as we can then apply this to every $E^{\mathbf{A}}$).

By [SAG, Thm. 7.2.2.1] there is a semi-orthogonal decomposition on $\mathrm{Qcoh}(\mathbf{P}^{1,\flat}_R)$ into two full subcategories both canonically equivalent to $\mathrm{Qcoh}(\mathrm{Spec}(R)) \simeq \mathrm{Mod}_R$. Moreover, an inspection of the proof of *loc. cit.* shows that this restricts to a semi-orthogonal decomposition on $\mathrm{Perf}(\mathbf{P}^{1,\flat}_R)$ by two full subcategories both equivalent to $\mathrm{Perf}(\mathrm{Spec}(R)) \simeq \mathrm{Perf}_R$. Indeed, both functors q^* and q_* preserve perfect complexes (the latter by [SAG, Thm. 6.1.3.2]). The claim then follows by definition of additive invariants. $\square$

Remark 4.2.6. Let α denote the composite morphism

$$\alpha : E \oplus E \xrightarrow{\mu} E \oplus E \xrightarrow{q^* \oplus (q^* \otimes \mathcal{O}(-1))} E^{\boxplus},$$

where μ is the isomorphism induced by the invertible matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

By construction, α fits in the commutative diagram

$$\begin{array}{ccccc} & & E \oplus E & & \\ & \swarrow^{p_-^* \oplus 0} & \downarrow \alpha & \searrow^{p_+^* \oplus 0} & \\ E^- & \xleftarrow{k_-^*} & E^{\boxtimes} & \xrightarrow{k_+^*} & E^+. \end{array} \tag{4.2.b}$$

4.3. **The Bass fundamental sequence.**

Theorem 4.3.1. *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a localizing invariant. Then for any R-linear stable ∞-category $\mathbf{A}$, there is a natural exact sequence of abelian groups*

$$0 \to E_n^{\mathbf{A}}(R) \xrightarrow{(p_+^*, -p_-^*)} E_n^{\mathbf{A}}(R[T]) \oplus E_n^{\mathbf{A}}(R[T^{-1}]) \xrightarrow{(j_+^*, j_-^*)} E_n^{\mathbf{A}}(R[T^{\pm}]) \xrightarrow{\partial} E_{n-1}^{\mathbf{A}}(R) \to 0$$

for every integer n, where we write $E_n = \pi_n E$.

Proof. By replacing E with $E^{\mathbf{A}}$, we may assume that $\mathbf{A} = \mathrm{Perf}_R$. Since E satisfies Nisnevich descent (e.g. [CMNN, App. A]), the Nisnevich square (4.2.a) gives rise to a cartesian square

$$\begin{array}{ccc} E(\mathbf{P}_R^{1,\flat}) & \xrightarrow{k_-^*} & E(R[T^{-1}]) \\ \downarrow k_+^* & & \downarrow j_-^* \\ E(R[T]) & \xrightarrow{j_+^*} & E(R[T, T^{-1}]) \end{array}$$

and hence to a Mayer–Vietoris long exact sequence

$$\begin{aligned} \cdots \to E_{n+1}(R[T, T^{-1}]) \xrightarrow{\partial} E_n(\mathbf{P}_R^{1,\flat}) \xrightarrow{(k_+^*, -k_-^*)} \\ E_n(R[T]) \oplus E_n(R[T^{-1}]) \xrightarrow{j_+^* \oplus j_-^*} E_n(R[T, T^{-1}]) \xrightarrow{\partial} \cdots \end{aligned} \tag{4.3.a}$$

By the projective bundle formula (Theorem 4.2.5) and Remark 4.2.6, there is a canonical isomorphism $E(R) \oplus E(R) \simeq E(\mathbf{P}_R^{1,\flat})$ under which the map $(k_+^*, -k_-^*)$ in (4.3.a) is (p_+^*, p_-^*) on the first copy of $E_n(R)$ and $(0, 0)$ on the second. Here p_+ and p_- denote the respective projections $p_+ : \mathrm{Spec}(R[T]) \to \mathrm{Spec}(R)$ and $p_- : \mathrm{Spec}(R[T^{-1}]) \to \mathrm{Spec}(R)$. Since the boundary map is then surjective onto the second copy of $E_n(R)$, and the zero section induces canonical retractions of both p_+^* and p_-^*, we see that the long exact sequence (4.3.a) splits up into short exact sequences as in the claim. □

Remark 4.3.2. In the case where the localizing invariant E is nonconnective algebraic K-theory $\mathbb{K}$ (Example 4.1.7), the map $\partial : \mathbb{K}_n(R[T^{\pm}]) \to \mathbb{K}_{n-1}(R)$ in the Bass fundamental sequence admits a natural *splitting*, up to an automorphism of $\mathbb{K}_{n-1}(R)$. Indeed, consider the automorphism of $R[T, T^{-1}]$ given by multiplication by T. This induces a point $b \in \mathbb{K}(R[T, T^{-1}])[-1]$ which we call the *Bott class*. Now cup product with b induces a canonical map

$$\mathbb{K}(R) \xrightarrow{p_{\pm}^*} \mathbb{K}(R[T^{\pm}]) \xrightarrow{b\cup} \mathbb{K}(R[T^{\pm}])[-1] \xrightarrow{\partial} \mathbb{K}(R)$$

which we claim is invertible. By $\mathbb{K}(R)$-linearity, this is equivalent to the assertion that ∂ sends $b \in \mathbb{K}_1(R[T^{\pm}]) \simeq \mathrm{K}_1(R[T^{\pm}])$ to a unit in $\mathbb{K}_0(R) \simeq \mathrm{K}_0(R)$. Since the 1-truncation

$\tau_{\leqslant 1}(\mathrm{K})$ is insensitive to positive homotopy groups [Lur14, Lect. 20, Cor. 4], we may replace R by $\pi_0(R)$. Now the claim is classical, see e.g. the proof of [TT90, Thm. 6.1(b)].

4.4. **Delooping localizing invariants of connective spectra.**

Notation 4.4.1. We let $\mathrm{Spt}_{\geqslant 0}$ denote the full subcategory of Spt spanned by connective spectra. A *connective fibre sequence* of spectra is a diagram

$$F \to X \to Y,$$

together with a null-homotopy of the composite $F \to Y$, such that the induced map $F \to \mathrm{Fib}(X \to Y)$ induces an isomorphism

$$\tau_{\geqslant 0}(F) \simeq \tau_{\geqslant 0}(\mathrm{Fib}(X \to Y))$$

of connective spectra.

Definition 4.4.2. We say that a functor $\mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ is *localizing* if it sends short exact sequences to connective fibre sequences.

Theorem 4.4.3. *Let R be a connective $\mathcal{E}_\infty$-ring. The assignment $E \mapsto \tau_{\geqslant 0}(E)$ determines an equivalence*

$$\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0})$$

from the ∞-category of localizing invariants $\mathrm{Stab}_R \to \mathrm{Spt}$, to the ∞-category of localizing invariants $\mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$.

Example 4.4.4. Let $\mathrm{K} : \mathrm{Stab} \to \mathrm{Spt}_{\geqslant 0}$ and $\mathbb{K} : \mathrm{Stab} \to \mathrm{Spt}$ be as in Example 4.1.7. Since $\mathbb{K}$ is localizing and satisfies $\mathbb{K}_{\geq 0} \simeq \mathrm{K}$, it follows from Theorem 4.4.3 that there is a canonical isomorphism of localizing invariants

$$\mathbb{K} \simeq \mathrm{K}^{\mathrm{B}}.$$

4.5. **The Bass construction.** Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be an arbitrary functor.

4.5.1. Define $V(E)$ and $W(E)$ such that there are cocartesian squares

$$\begin{array}{ccccccc}
E & \xrightarrow{\mathrm{incl}_1} & E \oplus E & \xrightarrow{\alpha} & E^{\boxplus} & \xrightarrow{(k_+^*, -k_-^*)} & E^+ \oplus E^- \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow & & \downarrow \\
0 & \longrightarrow & E & \longrightarrow & W(E) & \longrightarrow & V(E)
\end{array} \tag{4.5.a}$$

in $\mathrm{Fun}(\mathrm{Stab}_R, \mathrm{Spt})$. Here α is as in Remark 4.2.6. We denote by ψ_E the composite $E \to W(E) \to V(E)$.

Remark 4.5.2. The commutative diagram (4.2.b) provides a null-homotopy of the composite

$$E \xrightarrow{\mathrm{incl}_2} E \oplus E \xrightarrow{\alpha} E^{\boxplus} \xrightarrow{(k_+^*, -k_-^*)} E^+ \oplus E^-.$$

Since the composite $\mathrm{pr}_2 \circ \mathrm{incl}_2$ is the identity, combining this with the commutative diagram (4.5.a) yields a canonical null-homotopy of the morphism $\psi_E : E \to V(E)$.

Remark 4.5.3. The commutative diagram (4.2.b) identifies the upper horizontal composite with the morphism $(p_+^*, -p_-^*) : E \to E^+ \oplus E^-$. It follows that $V(E)$ fits in an exact triangle

$$E \xrightarrow{(p_+^*,-p_-^*)} E^+ \oplus E^- \to V(E).$$

Moreover, since the morphisms p_+^* and p_-^* admit splittings induced by the homomorphisms $R[T] \to R$ and $R[T^{-1}] \to R$, respectively, it follows that the associated long exact sequence splits into short exact sequences

$$0 \to \pi_n(E) \xrightarrow{(p_+^*,-p_-^*)} \pi_n(E^+) \oplus \pi_n(E^-) \to \pi_n(V(E)) \to 0 \tag{4.5.b}$$

for every n.

Remark 4.5.4. Note that we have commutative squares

$$\begin{array}{ccc} E^{\boxplus} & \xrightarrow{k_+^*} & E^+ \\ \downarrow{\scriptstyle k_-^*} & & \downarrow{\scriptstyle j_+^*} \\ E^- & \xrightarrow{j_-^*} & E^{\pm}, \end{array} \qquad \begin{array}{ccc} E^{\boxplus} & \xrightarrow{(k_+^*,-k_-^*)} & E^+ \oplus E^- \\ \downarrow & & \downarrow{\scriptstyle j_+^* \oplus j_-^*} \\ 0 & \longrightarrow & E^{\pm}, \end{array}$$

where the right-hand square is induced from the left-hand one. This gives rise to a canonical morphism $\theta_E : V(E) \to E^{\pm}$ fitting into the commutative diagram:

$$\begin{array}{ccc} E^{\boxplus} & \xrightarrow{(k_+^*,-k_-^*)} & E^+ \oplus E^- \\ \downarrow & & \downarrow \\ W(E) & \longrightarrow & V(E) \\ \downarrow & & \downarrow{\scriptstyle \theta_E} \\ 0 & \longrightarrow & E^{\pm} \end{array} \qquad (E^+ \oplus E^- \xrightarrow{j_+^* \oplus j_-^*} E^{\pm})$$

Construction 4.5.5. Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. Denote by $U(E)$ the fibre of the morphism $\theta_E : V(E) \to E^{\pm}$, so that there is a fibre sequence

$$U(E) \to V(E) \xrightarrow{\theta_E} E^{\pm}.$$

The canonical null-homotopy of the composite $W(E) \to V(E) \xrightarrow{\theta_E} E^{\pm}$ defined above gives rise to a canonical morphism $W(E) \to U(E)$. In particular, we get a canonical morphism

$$\phi_E : E \to W(E) \to U(E).$$

This gives rise to a tower

$$E \xrightarrow{\phi_E} U(E) \xrightarrow{U(\phi_E)} U^2(E) \xrightarrow{U^2(\phi_E)} \cdots$$

whose colimit we denote E^{B}, and call the *Bass construction* on E.

Remark 4.5.6. Since the functors V and $(-)^{\pm}$ commute with colimits and with $(-)^{\mathbf{A}}$ for $\mathbf{A} \in \mathrm{Stab}_R$, the same holds for U. In particular, it follows that there are canonical identifications $U^k(\phi_E) \simeq \phi_{U^k(E)}$ for each $k \geqslant 0$.

Theorem 4.5.7. *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ be a functor. The Bass construction E^{B} satisfies the following properties:*

(i) *If E satisfies the projective bundle formula and is $Q_{\pm}$-excisive, then the natural morphism $E \to E^{\mathrm{B}}$ induces an isomorphism $E \simeq \tau_{\geqslant 0}(E^{\mathrm{B}})$.*

(ii) *If $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ is a localizing invariant, then the functor $E^{\mathrm{B}} : \mathrm{Stab}_R \to \mathrm{Spt}$ is a localizing invariant.*

4.6. **Proof of Theorem 4.5.7.**

Lemma 4.6.1. *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ be a functor. If E satisfies the projective bundle formula and is $Q_{\pm}$-excisive, then we have:*

(i) *There is a connective fibre sequence*

$$E \xrightarrow{\psi_E} V(E) \xrightarrow{\theta_E} E^{\pm}$$

which is natural in E.

(ii) *There exist canonical morphisms*

$$\tau_{\geqslant 0}(\Omega(E^{\pm})) \to E, \quad \sigma_E : E \to \tau_{\geqslant 0}(\Omega(E^{\pm}))$$

which exhibit E as a retract of $\tau_{\geqslant 0}(\Omega(E^{\pm}))$, and are natural in E.

Proof. We first show the following weaker version of (i):

($*$) There is a connective fibre sequence

$$\tau_{\geqslant 0}(\Omega(E^{\pm})) \to E \xrightarrow{\psi_E} V(E).$$

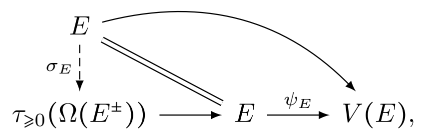
Since E satisfies the projective bundle formula, $\alpha : E \oplus E \to E^{\boxplus}$ is invertible. Considering the diagram (4.5.a), it follows that $E \to W(E)$ is also invertible and that it will suffice to show that the diagram

$$\tau_{\geqslant 0}(\Omega(E^{\pm})) \to W(E) \to V(E)$$

is a connective fibre sequence. Since the ∞-category $\mathrm{Spt}_{\geqslant 0}$ is prestable, the right-hand square in (4.5.a) is also cartesian [SAG, Cor. C.1.2.6], and in particular induces an isomorphism on fibres. The fact that E is $Q_{\pm}$-excisive implies that the fibre[6] of the upper arrow $(k_+^*, -k_-^*) : E^{\boxplus} \to E^+ \oplus E^-$ is $\tau_{\geqslant 0}(\Omega(E^{\pm}))$. This shows claim ($*$).

[6]Note that this fibre, computed in $\mathrm{Spt}_{\geqslant 0}$, is the same as the connective cover of the fibre computed in Spt.

For part (ii), the canonical null-homotopy of the morphism $\psi_E : E \to V(E)$ (Remark 4.5.2) gives rise to a morphism σ_E fitting in the commutative diagram



since the lower row is a fibre sequence by claim $(*)$.

Finally we prove (i). By (ii), the diagram $E \to V(E) \to E^{\pm}$ is a retract of the diagram

$$\tau_{\geqslant 0}(\Omega(E^{\pm})) \to \tau_{\geqslant 0}(\Omega(V(E)^{\pm})) \to \tau_{\geqslant 0}(\Omega((E^{\pm})^{\pm})).$$

It will suffice to show that the latter is a fibre sequence. Since the functor $E \mapsto E^{\pm}$ is left-exact and commutes with $\tau_{\geqslant 0}$, this follows from the fact that the diagram

$$\tau_{\geqslant 0}(\Omega(E)) \to \tau_{\geqslant 0}(\Omega(V(E))) \to \tau_{\geqslant 0}(\Omega(E^{\pm}))$$

is a fibre sequence, by $(*)$. □

Corollary 4.6.2. *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ be a localizing invariant. Then the morphism $\phi_E : E \to U(E)$ induces an isomorphism $E \simeq \tau_{\geqslant 0}(U(E))$.*

Proof. The claim is equivalent to the assertion that the diagram

$$E \to V(E) \to E^{\pm}$$

is a connective fibre sequence. This is Lemma 4.6.1(i), which applies since E is localizing. □

Lemma 4.6.3. *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. If E is localizing, then we have:*

(i) *There is an exact triangle*

$$E \xrightarrow{\psi_E} V(E) \xrightarrow{\theta_E} E^{\pm}$$

which is natural in E. In other words, the morphism $\phi_E : E \to U(E)$ is invertible.

(ii) *The canonical morphism $E \to E^{\mathrm{B}}$ is invertible.*

(iii) *There exist canonical morphisms*

$$\Omega(E^{\pm}) \to E, \quad \sigma_E : E \to \Omega(E^{\pm})$$

which exhibit E as a retract of $\Omega(E^{\pm})$, and are natural in E.

Proof. Parts (i) and (iii) follow by the same argument as in the proof of Lemma 4.6.1. The second follows from (i). □

Lemma 4.6.4. *The functor $E \mapsto U(E)$ preserves $\tau_{\geqslant 0}$-equivalences. That is, let $E \to E'$ be a morphism in $\mathrm{Fun}(\mathrm{Stab}_R, \mathrm{Spt})$ which induces an isomorphism $\tau_{\geqslant 0}(E) \simeq \tau_{\geqslant 0}(E')$. Then the induced map*

$$\tau_{\geqslant 0}(U(E)) \to \tau_{\geqslant 0}(U(E'))$$

is invertible.

Proof. Note that the analogous property holds for the functors $(-)^{\pm}$ and $V(-)$: in fact, they are even t-exact (the latter in view of the exact sequences (4.5.b)). As $U(-)$ is the fibre of the morphism $V(-) \to (-)^{\pm}$, the claim follows. □

Lemma 4.6.5. *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. We have:*

(i) *There is an exact triangle*

$$E^{\mathrm{B}} \xrightarrow{\psi_{E^{\mathrm{B}}}} V(E^{\mathrm{B}}) \xrightarrow{\theta_{E^{\mathrm{B}}}} (E^{\mathrm{B}})^{\pm}$$

which is natural in E.

(ii) *There exist canonical morphisms*

$$\Omega((E^{\mathrm{B}})^{\pm}) \to E^{\mathrm{B}}, \quad \sigma_{E^{\mathrm{B}}} : E^{\mathrm{B}} \to \Omega((E^{\mathrm{B}})^{\pm})$$

which exhibit E^{B} as a retract of $\Omega((E^{\mathrm{B}})^{\pm})$, and are natural in E.

Proof. Part (ii) will follow from (i) as in the proof of Lemma 4.6.1. For (i) it will suffice to show that the morphism $\phi_{E^{\mathrm{B}}} : E^{\mathrm{B}} \to U(E^{\mathrm{B}})$ is invertible. Since the functors U, V and W each commute with colimits, it is clear from the construction that this morphism is the colimit of the morphisms $\phi_{U^k(E)} \simeq U^k(\phi_E) : U^k(E) \to U^{k+1}(E)$, $k \geqslant 0$ (Remark 4.5.6). That is, $\phi_{E^{\mathrm{B}}}$ is identified with the canonical morphism fitting in the diagram

$$\begin{array}{ccccccccc}
E & \xrightarrow{\phi_E} & U(E) & \xrightarrow{U(\phi_E)} & U^2(E) & \xrightarrow{U^2(\phi_E)} & \cdots & \longrightarrow & E^{\mathrm{B}} \\
\downarrow{\scriptstyle \phi_E} & & \downarrow{\scriptstyle U(\phi_E)} & & \downarrow{\scriptstyle U^2(\phi_E)} & & & & \downarrow{\scriptstyle \phi_{E^{\mathrm{B}}}} \\
U(E) & \xrightarrow[U(\phi_E)]{} & U^2(E) & \xrightarrow[U^2(\phi_E)]{} & U^3(E) & \xrightarrow[U^3(\phi_E)]{} & \cdots & \longrightarrow & U(E^{\mathrm{B}}),
\end{array}$$

and is clearly invertible. □

Corollary 4.6.6. *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. For every integer $n \geqslant 0$, denote by $(E^{\mathrm{B}})^{\pm n} : \mathrm{Stab}_R \to \mathrm{Spt}$ the functor*

$$((((E^{\mathrm{B}})^{\pm})^{\pm})^{\cdots})^{\pm}$$

obtained from the Bass construction E^{B} by an n-fold iteration of the functor $(-)^{\pm}$ (e.g. $(E^{\mathrm{B}})^{\pm 0} = E^{\mathrm{B}}$). Then for each $n \geqslant 0$, the functor

$$\Sigma^n(E^{\mathrm{B}}) : \mathrm{Stab}_R \to \mathrm{Spt}$$

is a retract of $(E^{\mathrm{B}})^{\pm n} : \mathrm{Stab}_R \to \mathrm{Spt}$.

Proof. By induction, it will suffice to consider $n = 1$ and, by adjunction, to show that E^{B} is a retract of $\Omega(E^{\mathrm{B}})^{\pm}$. This is Lemma 4.6.5(ii). □

4.6.7. *Proof of Theorem 4.5.7(i).* By Corollary 4.6.2, $\phi_E : E \to U(E)$ is a $\tau_{\geqslant 0}$-equivalence. It follows then from Lemma 4.6.4 that $U^k(\phi_E) : U^k(E) \to U^{k+1}(E)$ is a $\tau_{\geqslant 0}$-equivalence for every $k \geqslant 0$. It follows that the transfinite composite $E \to E^{\mathrm{B}}$ is also a $\tau_{\geqslant 0}$-equivalence.

4.6.8. *Proof of Theorem 4.5.7(ii).* Let $\mathbf{A}' \to \mathbf{A} \to \mathbf{A}''$ be an exact sequence of small stable ∞-categories and consider the induced diagram of spectra

$$E^{\mathrm{B}}(\mathbf{A}') \to E^{\mathrm{B}}(\mathbf{A}) \to E^{\mathrm{B}}(\mathbf{A}'').$$

To show that this is an exact triangle, it will suffice to show that each of the induced diagrams

$$\tau_{\geqslant n}(E^{\mathrm{B}}(\mathbf{A}')) \to \tau_{\geqslant n}(E^{\mathrm{B}}(\mathbf{A})) \to \tau_{\geqslant n}(E^{\mathrm{B}}(\mathbf{A}''))$$

is a fibre sequence in $\mathrm{Spt}_{\geqslant n}$, for every $n \leqslant 0$. In other words, it will suffice to show that each functor $\tau_{\geqslant n}(E^{\mathrm{B}}) : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant n}$ is a localizing invariant.

By Corollary 4.6.6 we know that $\tau_{\geqslant n}(E^{\mathrm{B}})$ is a retract of $\tau_{\geqslant 0}((E^{\mathrm{B}})^{\pm n})$. Since $(-)^{\pm}$ is left-exact, the latter is isomorphic to $\tau_{\geqslant 0}(E^{\mathrm{B}})^{\pm n} \simeq E^{\pm n}$ (Theorem 4.5.7(i)), which is localizing because E is.

4.7. **Proof of Theorem 4.4.3.**

We are now ready to prove Theorem 4.4.3, which asserts that the canonical functor

$$\tau_{\geqslant 0} : \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0}),$$

given by the assignment $E \mapsto \tau_{\geqslant 0}(E)$, is an equivalence.

4.7.1. Note that the Bass construction (Construction 4.5.5) defines a functor $E \mapsto E^{\mathrm{B}}$ from $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0})$ to $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt})$. We claim first that this is a fully faithful left adjoint to $\tau_{\geqslant 0}$. For $E \in \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0})$, the unit map

$$\eta_E : E \xrightarrow{\sim} \tau_{\geqslant 0}(E^{\mathrm{B}})$$

comes from the natural isomorphisms of Theorem 4.5.7(i). For $E \in \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt})$, the co-unit map

$$\varepsilon_E : (\tau_{\geqslant 0}(E))^{\mathrm{B}} \to E^{\mathrm{B}}$$

is induced from $\tau_{\geqslant 0}(E) \to E$ in view of the fact that the natural map $E \to E^{\mathrm{B}}$ is invertible (Lemma 4.6.3(ii)). One easily verifies the triangle identities.

4.7.2. In order to conclude that the functor $\tau_{\geqslant 0}$ is an equivalence, it will suffice to show that it is conservative (so that the co-unit maps are necessarily isomorphisms). Let $E \to E'$ be a morphism of localizing invariants in $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt})$, and suppose that the induced morphism $\tau_{\geqslant 0}(E) \to \tau_{\geqslant 0}(E')$ is invertible. We argue by decreasing induction on n that the map $\pi_n(E(\mathbf{A})) \to \pi_n(E'(\mathbf{A}))$ is invertible for every $n \leqslant 0$ and every $\mathbf{A} \in \mathrm{Stab}_R$. For $n = 0$ this holds by assumption. The induction step follows from the Bass fundamental sequence (Theorem 4.3.1). The claim follows.

5. Homotopy invariant K-theory

5.1. Homotopy invariant K-theory.

Notation 5.1.1. Given a spectral scheme S, let $\mathrm{Perf}(S)$ denote the stable ∞-category of perfect complexes on S. Denote by $\mathrm{K}(S)$ and $\mathrm{K}^{\mathrm{B}}(S)$ the spectra

$$\mathrm{K}(\mathrm{Perf}(S)), \quad \mathrm{K}^{\mathrm{B}}(\mathrm{Perf}(S)),$$

respectively. Here K is algebraic K-theory (Example 4.1.7) and K^{B} is the Bass construction, equivalent to nonconnective algebraic K-theory $\mathbb{K}$ (Example 4.4.4).

Construction 5.1.2. For any spectral scheme S, consider the spectrum

$$\mathrm{KH}(S) \coloneqq \varinjlim \mathrm{K}(S \times \mathbf{A}^n),$$

where $\mathbf{A}^n$ is the n-dimensional spectral affine space (Example 2.1.3) and the colimit is indexed by the opposite of the (cosifted) full subcategory $\mathbf{A}_S \subseteq \mathrm{Aff}_{/S}$ whose objects are spectral affine spaces $S \times \mathbf{A}^n$ $(n \geqslant 0)$. We also write $\mathrm{KH}(R) = \mathrm{KH}(\mathrm{Spec}(R))$ for any connective $\mathcal{E}_\infty$-ring R.

This definition may appear ad-hoc. In the language of $\mathcal{C}$-fibred S^1-spectra (Example 2.8.2), we can give a more systematic definition:

Construction 5.1.3. Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. The assignments $X \mapsto \mathrm{K}(X)$ and $X \mapsto \mathrm{K}^{\mathrm{B}}(X)$ define presheaves of spectra

$$(\mathcal{C}_{/S})^{\mathrm{op}} \to \mathrm{Spt},$$

which we shall denote by $\mathrm{K}|_{\mathcal{C}}$ and $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$ and which we view as $\mathcal{C}$-fibred S^1-spectra over S. Then $\mathrm{KH}|_{\mathcal{C}}$ is the $\mathcal{C}$-fibred S^1-spectrum defined as the $\mathbf{A}^1$-localization of $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$:

$$\mathrm{KH}|_{\mathcal{C}} \coloneqq \mathrm{L}_{\mathbf{A}^1}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}).$$

The formula (2.1.b) shows that the spectrum of global sections recovers $\mathrm{KH}(S)$ as defined above:

$$\Gamma(S, \mathrm{KH}|_{\mathcal{C}}) \simeq \mathrm{KH}(S).$$

Proposition 5.1.4. *The $\mathcal{C}$-fibred S^1-spectrum $\mathrm{KH}|_{\mathcal{C}}$ satisfies Nisnevich excision and $\mathbf{A}^1$-homotopy invariance; that is, it is motivic. Moreover, the canonical morphism of motivic $\mathcal{C}$-fibred S^1-spectra*

$$\mathbf{L}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \to \mathrm{KH}|_{\mathcal{C}}$$

is invertible.

Proof. Recall that any localizing invariant satisfies Nisnevich excision, see e.g. [CMNN, Prop. A.13], so $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$ satisfies Nisnevich excision. Thus by Proposition 2.8.6 we have

$$\mathbf{L}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \simeq \mathrm{L}_{\mathbf{A}^1}\,\mathrm{L}_{\mathrm{Nis}}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \simeq \mathrm{L}_{\mathbf{A}^1}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}),$$

where the latter is $\mathrm{KH}|_{\mathcal{C}}$ by definition. □

Definition 5.1.5. We may repeat Construction 5.1.3 in the setting of $\mathcal{C}^{\mathrm{cl}}$-fibred spectra over S_{cl} (notation as in Example 2.4.2). The restriction of $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$ along $u : \mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \to \mathcal{C}_{/S}$ (Remark 2.5.5) is the $\mathcal{C}^{\mathrm{cl}}$-fibred S^1-spectrum $u^*(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \simeq v_!(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}})$ which we denote simply by $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}^{\mathrm{cl}}}$. We define the $\mathcal{C}^{\mathrm{cl}}$-fibred S^1-spectrum $\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{C}^{\mathrm{cl}}}$ as the $\mathbf{A}^1_{\mathrm{cl}}$-localization of $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}^{\mathrm{cl}}}$:

$$\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{C}^{\mathrm{cl}}} := \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}^{\mathrm{cl}}}).$$

We define $\mathrm{KH}^{\mathrm{cl}}(S_{\mathrm{cl}})$ as the spectrum of global sections $\Gamma(S_{\mathrm{cl}}, \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{C}^{\mathrm{cl}}})$. This is nothing else than Weibel's homotopy invariant K-theory spectrum (see [Cis13] for this point of view).

To formulate the main result of this subsection, we introduce the following notation:

Notation 5.1.6. Let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a narrow subcategory (e.g. $\mathcal{A}_{/S} = \mathrm{Sm}_{/S}$), let $\mathcal{A}^{\mathrm{cl}}_{/S} \subseteq \mathrm{AffCl}_{/S}$ be as in Example 2.4.2, and let $w : \mathcal{A}_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ be the classical truncation functor (Construction 2.4.6). Recall the equivalence

$$\mathbf{L}w_! : \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}} \to \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathrm{Spt}}, \quad w^* : \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathrm{Spt}} \to \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}}$$

from Theorem 2.8.5.

Then we have:

Theorem 5.1.7. *Let the notation be as in 5.1.6. Then there are canonical isomorphisms*

$$\mathbf{L}w_!(\mathrm{KH}|_{\mathcal{A}}) \simeq \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}},$$
$$\mathrm{KH}|_{\mathcal{A}} \simeq w^*(\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}})$$

of motivic $\mathcal{A}^{\mathrm{cl}}$-fibred S^1-spectra over S_{cl}, resp. of motivic $\mathcal{A}$-fibred S^1-spectra over S.

See Subsect. 5.4 for the proof. Note that this immediately implies Theorem B.

Corollary 5.1.8. *For every quasi-compact quasi-separated spectral algebraic space S, there is a canonical isomorphism of spectra $\mathrm{KH}(S) \simeq \mathrm{KH}^{\mathrm{cl}}(S_{\mathrm{cl}})$, functorial in S.*

Proof. By Nisnevich descent we may assume that S is affine. Passing to global sections in Theorem 5.1.7, we get an isomorphism of spectra

$$\mathrm{KH}(S) = \Gamma(S, \mathrm{KH}|_{\mathcal{A}}) \xrightarrow{\sim} \Gamma(S, w^*(\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}})) \simeq \Gamma(S_{\mathrm{cl}}, \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}}) = \mathrm{KH}^{\mathrm{cl}}(S_{\mathrm{cl}})$$

as claimed. □

Corollary 5.1.9. *For any connective $\mathcal{E}_\infty$-ring R, there is a canonical isomorphism of spectra $\mathrm{KH}(R) \simeq \mathrm{KH}^{\mathrm{cl}}(\pi_0(R))$, functorial in R.*

5.2. Connective comparison.

In this subsection our goal is to prove the following two statements:

Proposition 5.2.1. *Let the notation be as in 5.1.6. There is a canonical isomorphism of $\mathcal{A}^{\mathrm{cl}}$-fibred S^1-spectra*

$$\mathbf{L}w_!(\mathrm{K}|_{\mathcal{A}}) \to \mathbf{L}(\mathrm{K}|_{\mathcal{A}^{\mathrm{cl}}}).$$

Proposition 5.2.2. *Let the notation be as in 5.1.6. Let $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a broad subcategory containing $\mathcal{A}_{/S}$ and let $\iota : \mathcal{A}_{/S} \hookrightarrow \mathcal{B}_{/S}$ denote the inclusion. Then the canonical morphisms of $\mathcal{B}$-fibred S^1-spectra*

$$\mathrm{L}_{\mathrm{Nis}}\, \iota_!(\mathrm{K}|_{\mathcal{A}}) \to \mathrm{K}|_{\mathcal{B}}$$
$$\mathbf{L}\iota_!(\mathrm{K}|_{\mathcal{A}}) \to \mathbf{L}(\mathrm{K}|_{\mathcal{B}})$$

are invertible.

Corollary 5.2.3. *The motivic $\mathcal{B}$-fibred S^1-spectrum $\mathbf{L}(\mathrm{K}|_{\mathcal{B}})$ is nil-local (Definition 2.5.2).*

Proof. By Proposition 5.2.2, $\mathbf{L}(\mathrm{K}|_{\mathcal{B}})$ belongs to the essential image of $\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}} \to \mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}}$. Hence it is nil-local by Theorem 2.6.2. □

We will deduce Propositions 5.2.1 and 5.2.2 from a representability statement, Proposition 5.2.5 below.

Construction 5.2.4. Let R be a connective $\mathcal{E}_\infty$-ring. Denote by $\mathrm{Mod}_R^{\mathrm{proj}}$ the ∞-category of finitely generated projective R-modules, and by $\mathrm{Mod}_R^{\mathrm{free}}$ the full subcategory of free R-modules of finite rank. Let $X(R)$ denote the underlying ∞-groupoid $(\mathrm{Mod}_R^{\mathrm{free}})^{\simeq}$, obtained by discarding non-invertible morphisms. Note that $X(R)$ is nothing else than the coproduct of the classifying spaces $\mathrm{BGL}_n(R)$ over $n \geqslant 0$, where $\mathrm{GL}_n(R)$ is the space of automorphisms of the free R-module $R^{\oplus n}$. Note also that the symmetric monoidal structure on $\mathrm{Mod}_R^{\mathrm{proj}}$ induces a structure of $\mathcal{E}_\infty$-monoid on $X(R)$. Moreover, formation of $X(R)$ is functorial and we may regard the assignment $\mathrm{Spec}(R) \mapsto X(R)$ as a presheaf of $\mathcal{E}_\infty$-spaces on the site of affine spectral schemes.

Proposition 5.2.5. *Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be any admissible subcategory. Denote by X_S the presheaf on $\mathcal{C}_{/S}$ given by the assignment $\mathrm{Spec}(R) \mapsto X(R)$, and by $(X_S)^{\mathrm{gp}}$ its group completion. Then there is a canonical morphism of $\mathcal{C}$-fibred $\mathcal{E}_\infty$-groups*

$$(X_S)^{\mathrm{gp}} \to \Omega^\infty(\mathrm{K}|_{\mathcal{C}})$$

which induces an isomorphism $\mathrm{L}_{\mathrm{Zar}}(X_S)^{\mathrm{gp}} \simeq \Omega^\infty(\mathrm{K}|_{\mathcal{C}})$.

Proof. Let X'_S denote the presheaf $\mathrm{Spec}(R) \mapsto (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$. Then by [Lur14, Lect. 19, Thm. 5] there is a canonical isomorphism $(X'_S)^{\mathrm{gp}} \simeq \Omega^\infty(\mathrm{K}|_{\mathcal{C}})$. Therefore it will suffice to show that the monomorphism $X_S \hookrightarrow X'_S$ induces an effective epimorphism of Zariski sheaves $\mathrm{L}_{\mathrm{Zar}}(X_S) \to X'_S$ (see [HTT, Ex. 5.2.8.16]). This is clear since every finitely generated projective R-module is Zariski-locally free. □

Remark 5.2.6. Note that the functors $\mathrm{L}_{\mathrm{Nis}}\, \iota_!$ and $\mathbf{L}\iota_!$ preserve connective objects. Indeed, the essential images of the fully faithful functors

$$\mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}_{\geq 0}} \hookrightarrow \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}},$$
$$\mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}_{\geq 0}} \hookrightarrow \mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}},$$

are generated under colimits by objects of the form $\Sigma^\infty_+(X)$ with $X \in \mathcal{A}_{/S}$, resp. $X \in \mathcal{B}_{/S}$.

Proof of Proposition 5.2.1. Since both source and target are connective, it will suffice to show the claim for the underlying $\mathcal{A}$-fibred $\mathcal{E}_\infty$-group $\Omega^\infty(\mathrm{K}|_{\mathcal{A}})$. Since each of the functors $\mathrm{L}_{\mathrm{Nis}}$, $\iota_!$ and ι^* commutes with colimits and finite products, and hence with group completion (see e.g. [Hoy, Lem. 5.5]), we reduce using Proposition 5.2.5 to showing the analogous claim for the $\mathcal{A}$-fibred $\mathcal{E}_\infty$-monoid X_S, i.e., that the canonical morphism

$$\mathbf{L}w_!(X_S) \simeq X^{\mathrm{cl}}_{S_{\mathrm{cl}}}$$

is invertible, where the right-hand side is the construction analogous to Construction 5.2.4 in classical algebraic geometry. The claim now follows from the fact that $w : \mathcal{A}_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ sends the spectral affine schemes[7] $\mathrm{GL}_{n,S}$ to their classical counterparts. □

Proof of Proposition 5.2.2. As above, we reduce to the analogous claim for the $\mathcal{B}$-fibred $\mathcal{E}_\infty$-monoid X_S. This follows from the fact that the classifying spaces $\mathrm{BGL}_{n,S}$ are colimits of finite products of the smooth spectral schemes $\mathrm{GL}_{n,S}$. □

5.3. **Bott periodicity.** We keep the notation of 5.1.6. In this subsection we will show that $\mathrm{KH}|_{\mathcal{B}}$ can be described as the Bott periodization of the motivic localization of $\mathrm{K}|_{\mathcal{B}}$, and similarly for $\mathrm{KH}|_{\mathcal{A}}$. This will allow us to prove a nonconnective analogue of Proposition 5.2.2, see Corollary 5.3.7.

Construction 5.3.1. Let R be a connective $\mathcal{E}_\infty$-ring. Denote by $R\{T\}$ the free $\mathcal{E}_\infty$-R-algebra on one generator T and by $R\{T,T^{-1}\}$ the localization away from $T \in \pi_0(R\{T\}) \simeq \pi_0(R)[T]$. Just as in Remark 4.3.2, the automorphism of $R\{T,T^{-1}\}$ given by multiplication by T induces a canonical element $b \in \mathrm{K}_1(R\{T,T^{-1}\})$ which we also call the Bott class. Again by [Lur14, Lect. 20, Cor. 4] there is a canonical isomorphism $\mathrm{K}_1(R\{T,T^{-1}\}) \simeq \mathrm{K}_1(\pi_0(R)[T,T^{-1}])$ under which b corresponds to the usual Bott class. In particular, the canonical bijection $\mathrm{K}_1(R\{T,T^{-1}\}) \to \mathrm{K}_1(R[T,T^{-1}])$ (induced by ε, see Remark 3.3.1) sends b to b.

Remark 5.3.2. The Bott class may be regarded as a morphism of $\mathcal{B}$-fibred S^1-spectra

$$b : \Sigma^\infty(\mathbf{G}_{m,S}, 1)[1] \to \mathrm{K}|_{\mathcal{B}}.$$

To simplify notation, we set $\mathbf{T}_S := \Sigma^\infty(\mathbf{G}_{m,S}, 1)[1]$.

Definition 5.3.3. Recall that the $\mathcal{B}$-fibred S^1-spectrum $\mathrm{K}|_{\mathcal{B}}$ admits an $\mathcal{E}_\infty$-ring structure, induced by the symmetric monoidal structure on perfect complexes. We say that a $\mathrm{K}|_{\mathcal{B}}$-module $\mathcal{F}$ is *Bott-periodic* if the canonical morphism

$$b : \mathcal{F} \to \underline{\mathrm{Hom}}(\mathbf{T}_S, \mathcal{F})$$

induced by the Bott class (via the action of $\mathrm{K}|_{\mathcal{B}}$ on $\mathcal{F}$) is invertible. We define Bott-periodic $\mathrm{K}|_{\mathcal{A}}$-modules similarly. Note that the full subcategory spanned by Bott-periodic $\mathrm{K}|_{\mathcal{B}}$-modules (resp. $\mathrm{K}|_{\mathcal{A}}$-modules) is a left localization; we refer to the left adjoint Q as *Bott periodization.*

[7] Since $\mathrm{GL}_{n,S}$ are Zariski-open inside spectral affine spaces, they belong not only to $\mathrm{Sm}_{/S}$ but even to $\mathcal{A}^0_{/S}$ (Example 2.1.9) and hence to any narrow $\mathcal{A}_{/S}$.

Theorem 5.3.4. *The canonical morphisms*

$$\mathrm{K}|_{\mathcal{B}} \to \mathrm{KH}|_{\mathcal{B}}$$
$$\mathrm{K}|_{\mathcal{A}} \to \mathrm{KH}|_{\mathcal{A}}$$

induce isomorphisms

$$\mathrm{Q}\left(\mathbf{L}(\mathrm{K}|_{\mathcal{B}})\right) \simeq \mathrm{KH}|_{\mathcal{B}},$$
$$\mathrm{Q}\left(\mathbf{L}(\mathrm{K}|_{\mathcal{A}})\right) \simeq \mathrm{KH}|_{\mathcal{A}}$$

of Bott-periodic motivic fibred S^1-spectra.

The following description of Bott periodization will be used in the proof of Theorem 5.3.4.

Remark 5.3.5. Explicitly, the Bott periodization of a *motivic* $\mathrm{K}|_{\mathcal{B}}$-module $\mathcal{F}$ can be computed as the colimit of the tower

$$\mathcal{F} \xrightarrow{b} \underline{\mathrm{Hom}}(\mathbf{T}_S, \mathcal{F}) \xrightarrow{b} \underline{\mathrm{Hom}}(\mathbf{T}_S^{\otimes 2}, \mathcal{F}) \xrightarrow{b} \cdots$$

according to Theorem 3.8 and the proof of Lemma 4.9 of [Hoy]. Moreover, if $\mathcal{F}$ is nil-local, then we may replace b by the Bott class in $\mathbf{G}_{m,S}^{\flat}$ (Remark 4.3.2).

Remark 5.3.6. Since the functors $\mathbf{L}\iota_!$ and ι^* are symmetric monoidal (Remarks 2.3.7 and 2.8.4), they extend to an adjunction

$$\mathbf{L}\iota_! : \mathrm{Mod}_{\mathbf{L}(\mathrm{K}|_{\mathcal{A}})}(\mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}}) \rightleftarrows \mathrm{Mod}_{\mathbf{L}(\mathrm{K}|_{\mathcal{B}})}(\mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}}) : \iota^*.$$

Since $\mathbf{T}_S$ belongs to the essential image of $\mathbf{L}\iota_!$ (as $\mathbf{G}_{m,S}$ belongs to $\mathcal{A}_{/S}$), it follows from Remark 2.3.7(iii) and the fact that ι^* preserves colimits that ι^* preserves Bott-periodic objects and commutes with the Bott periodization functor Q. Its left adjoint on Bott-periodic objects is given by $\mathcal{F} \mapsto \mathrm{Q}(\mathbf{L}\iota_!(\mathcal{F}))$ and preserves Q-equivalences.

Proof of Theorem 5.3.4. Since ι^* commutes with Q (Remark 5.3.5), it will suffice to consider the first map. Just as in the classical setting [Cis13, Prop. 2.10], one observes that up to $\mathbf{A}^{1,\flat}$-localization, hence also up to $\mathbf{A}^1$-localization by Lemma 3.3.3, the Bass construction (Construction 4.5.5) simplifies to give the formula

$$\mathrm{KH}|_{\mathcal{B}} \simeq \varinjlim\left(\mathbf{L}(\mathrm{K}|_{\mathcal{B}}) \xrightarrow{b} \underline{\mathrm{Hom}}\left(\Sigma^\infty(\mathbf{G}_{m,S}^{\flat}, 1)[1], \mathbf{L}(\mathrm{K}|_{\mathcal{B}})\right) \xrightarrow{b} \cdots\right),$$

where the maps are induced by the Bott class $b \in \mathrm{K}_1(\mathbf{G}_{m,S}^{\flat})$ (Remark 4.3.2). But by Remark 5.3.5 and Corollary 5.2.3, this is isomorphic to the Bott periodization $\mathrm{Q}(\mathbf{L}(\mathrm{K}|_{\mathcal{B}}))$ with respect to $b \in \mathrm{K}_1(\mathbf{G}_{m,S})$. □

Using Theorem 5.3.4 we may deduce the following S^1-stable analogue of Proposition 5.2.2, which holds up to $\mathbf{A}^1$-homotopy and Bott periodization:

Corollary 5.3.7. *The canonical morphism*

$$\mathbf{L}\iota_!(\mathrm{KH}|_{\mathcal{A}}) \simeq \mathbf{L}\iota_!\iota^*(\mathrm{KH}|_{\mathcal{B}}) \to \mathrm{KH}|_{\mathcal{B}}$$

induces an isomorphism

$$\mathrm{Q}\left(\mathbf{L}\iota_!(\mathrm{KH}|_{\mathcal{A}})\right) \simeq \mathrm{KH}|_{\mathcal{B}}$$

of Bott-periodic motivic $\mathcal{B}$-fibred S^1-spectra.

Proof. Follows immediately from Theorem 5.3.4 and Remark 5.3.6. □

Remark 5.3.8. In the statement of Corollary 5.3.7, we can replace the source with $\mathbf{L}\iota_!(\mathrm{K}^{\mathrm{B}}|_{\mathcal{A}})$. That is, the canonical map

$$\mathrm{Q}(\mathbf{L}\iota_!(\mathrm{K}^{\mathrm{B}}|_{\mathcal{A}})) \to \mathrm{KH}|_{\mathcal{B}}$$

is also invertible. This follows from Proposition 5.1.4 and the fact that $\iota_!$ preserves motivic equivalences (Lemma 2.2.5).

5.4. **Proof of Theorem 5.1.7.** The only remaining ingredient is the behaviour of Bott periodization under the equivalence

$$\mathbf{L}w_! : \mathbf{H}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$$

of Theorem A. But the fact that it commutes with internal homs (Remark 2.7.9(iv)) immediately implies that we have

$$\mathbf{L}w_!(\mathrm{Q}(\mathcal{F})) \simeq \mathrm{Q}^{\mathrm{cl}}(\mathbf{L}w_!(\mathcal{F}))$$

in $\mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ for every $\mathbf{L}(\mathrm{K})$-module $\mathcal{F}$ in $\mathbf{H}(\mathcal{A}_{/S})$. Here Q^{cl} denotes Bott periodization of a SmCl-fibred motivic space with the classical Bott element.

Now consider the commutative diagram in $\mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$

$$\begin{array}{ccc}
\mathbf{L}w_!\,(\mathrm{Q}(\mathbf{L}(\mathrm{K}|_{\mathcal{A}}))) & \longrightarrow & \mathbf{L}w_!(\mathrm{KH}|_{\mathcal{A}}) \\
\downarrow & & \downarrow \\
\mathrm{Q}(\mathbf{L}(\mathrm{K}|_{\mathcal{A}^{\mathrm{cl}}})) & \longrightarrow & \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}}.
\end{array}$$

The assertion of Theorem 5.1.7 is that the right-hand vertical arrow is invertible. Since the horizontal arrows are isomorphisms by Theorem 5.3.4 (and its classical analogue), it will suffice to demonstrate the invertibility of the left-hand vertical arrow. Since $\mathbf{L}w_!$ commutes with Q, this is identified with the Bott periodization of the canonical morphism

$$\mathbf{L}w_!(\mathrm{K}|_{\mathcal{A}}) \simeq \mathbf{L}w_!(\mathbf{L}(\mathrm{K}|_{\mathcal{A}})) \to \mathbf{L}(\mathrm{K}|_{\mathcal{A}^{\mathrm{cl}}}),$$

which is invertible by Proposition 5.2.1.

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: `denis-charles.cisinski@mathematik.uni-regensburg.de`

URL: `http://www.mathematik.uni-regensburg.de/cisinski/`

Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

E-mail address: `khan@ihes.fr`

URL: `https://www.preschema.com`