Lecture 8

Pro-systems of K-theory spectra

In this lecture we will start looking at pro-systems of K-theory spectra, and begin to see how pro-systems help us pass from the derived world back to the classical world.

1. K-theory of projective bundles and derived blow-ups. We first tie up some loose ends from the previous two lectures. Proofs are omitted since they follow the same pattern as in classical algebraic geometry.

1.1. Let $\mathbf{C}$ be a stable presentable ∞-category. Earlier we considered two-term semi-orthogonal decompositions $\mathbf{C} = \langle \mathbf{C}_+, \mathbf{C}_- \rangle$. More generally, given a collection of full stable subcategories $\mathbf{C}_1, \ldots, \mathbf{C}_n$, we say that they form a *semi-orthogonal decomposition* if they generate $\mathbf{C}$ as a stable subcategory, and each $\mathbf{C}_j$ is right orthogonal to $\mathbf{C}_i$ for $j > i$ (i.e. $\mathrm{Maps}(c_i, c_j)$ is contractible for all $j > i$).

1.2. Let X be a derived scheme. Let $\mathcal{E}$ be a locally free sheaf of rank n, and let $\pi : \mathbf{P}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ denote the associated projective bundle. Then we have:

Theorem 1.3.

(i) *For each integer k, the assignment $\mathcal{F} \mapsto \pi^*(\mathcal{F}) \otimes \mathcal{O}(k)$ defines a fully faithful functor* $\mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\mathbf{P}_{\mathrm{X}}(\mathcal{E}))$.

(ii) *For each integer k, let $\mathrm{Qcoh}(\mathbf{P}_{\mathrm{X}}(\mathcal{E}))^{(k)}$ denote the essential image of the functor described in (i). Then there is a semi-orthogonal decomposition*

$$\mathrm{Qcoh}(\mathbf{P}_{\mathrm{X}}(\mathcal{E})) = \langle \mathrm{Qcoh}(\mathbf{P}_{\mathrm{X}}(\mathcal{E}))^{(k)}, \ldots, \mathrm{Qcoh}(\mathbf{P}_{\mathrm{X}}(\mathcal{E}))^{(k-n+1)} \rangle.$$

1.4. Let X be a derived scheme. Let $\mathrm{Z} \hookrightarrow \mathrm{X}$ be a regular closed immersion of codimension n and $p : \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \to \mathrm{X}$ the derived blow-up. Recall that we have a diagram

$$\begin{array}{ccc} \mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) & \xrightarrow{i_{\mathrm{E}}} & \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle p} \\ \mathrm{Z} & \xrightarrow{\quad i \quad} & \mathrm{X} \end{array}$$

We have:

Theorem 1.5.

(i) *The functor $p^* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})$ is fully faithful.*

(ii) *For each integer k, the assignment $\mathcal{F} \mapsto (i_{\mathrm{E}})_*(\pi^*(\mathcal{F}) \otimes \mathcal{O}(k))$ defines a fully faithful functor* $\mathrm{Qcoh}(\mathrm{Z}) \to \mathrm{Qcoh}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})$.

(iii) *For each integer k, let $\mathrm{Qcoh}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})^{(k)}$ denote the essential image of the functor described in (ii). Then there is a semi-orthogonal decomposition*

$$\mathrm{Qcoh}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}) = \langle p^* \mathrm{Qcoh}(\mathrm{X}), \mathrm{Qcoh}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})^{(1)} \ldots, \mathrm{Qcoh}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})^{(n)} \rangle.$$

1.6. Now suppose that X is quasi-compact and quasi-separated. By the compact generation results discussed in Lecture 3, we can pass to perfect complexes and we get exact sequences, whence exact triangles in K-theory. Since π_* preserves perfect complexes these exact sequences are split, so we also get splittings in K-theory:

Corollary 1.7 (Projective bundle formula)**.** *For any locally free sheaf* $\mathcal{E}$ *of rank* n *on* X*, there is a canonical isomorphism of spectra*

$$\bigoplus_{k=0}^{n-1} \mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathbf{P}_{\mathrm{X}}(\mathcal{E})).$$

Similarly we have:

Corollary 1.8 (Derived blow-up formula)**.** *For any regular closed immersion* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *of dimension* n*, there is a cartesian square*

$$\begin{array}{ccc} \mathrm{K}(\mathrm{X}) & \longrightarrow & \mathrm{K}(\mathrm{Z}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}) & \longrightarrow & \mathrm{K}(\mathbf{P}_{\mathrm{X}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})). \end{array}$$

1.9. Using Zariski descent for the standard affine cover of $\mathbf{P}^1$ (Lecture 4) and the projective bundle formula, one derives:

Theorem 1.10 (Bass fundamental theorem)**.** *Let* X *be a quasi-compact quasi-separated derived scheme. Then for each integer* n *we have a split exact sequence of abelian groups*

$$0 \to \mathrm{K}_n(\mathrm{X}) \to \mathrm{K}_n(\mathbf{A}^1_{\mathrm{X}}) \oplus \mathrm{K}_n(\mathbf{A}^1_{\mathrm{X}}) \to \mathrm{K}_n(\mathbf{A}^1_{\mathrm{X}} - s(\mathrm{X})) \to \mathrm{K}_{n-1}(\mathrm{X}) \to 0,$$

where $s : \mathrm{X} \hookrightarrow \mathbf{A}^1_{\mathrm{X}}$ *is the zero section.*

2. Pro-systems. We now briefly discuss pro-objects in the ∞-categorical setting (see [2, § A.8.1] for details).

2.1. Let $\mathbf{C}$ be an accessible ∞-category admitting finite limits. A *pro-object* of $\mathbf{C}$ is a cofiltered diagram $\{x_i\}_{i\in\mathbf{I}}$, i.e. a functor $\mathbf{I} \to \mathbf{C}$ with $\mathbf{I}$ cofiltered (and essentially small). Pro-objects in $\mathbf{C}$ form an ∞-category $\mathrm{Pro}(\mathbf{C})$, where mapping spaces are given by the formula

$$\mathrm{Maps}(\{x_i\}_i, \{y_j\}_j) = \varprojlim_j \varinjlim_i \mathrm{Maps}(x_i, y_j).$$

This ∞-category $\mathrm{Pro}(\mathbf{C})$ is the free completion of $\mathbf{C}$ by cofiltered limits ([2, Prop. A.8.1.6]). It can be realized alternatively as the full subcategory of $\mathrm{Funct}(\mathbf{C}, \mathrm{Spc})^{\mathrm{op}}$ spanned by accessible left-exact functors.

Any object $x \in \mathbf{C}$ can be viewed as a constant pro-system $\{x\}$ (indexed by the terminal category); the assignment $x \mapsto \{x\}$ defines a fully faithful functor $\mathbf{C} \hookrightarrow \mathrm{Pro}(\mathbf{C})$. If $\mathbf{C}$ is presentable, then this functor admits a right adjoint which is given by $\{x_i\}_i \mapsto \varprojlim_i x_i$ (where the limit is computed in $\mathbf{C}$).

2.2. Consider the ∞-category $\mathrm{Pro}(\mathrm{Spt})$ of pro-spectra. This is stable and admits a t-structure where truncations are given by $\tau_{\leqslant k}\{\mathrm{X}_i\}_i = \{\tau_{\leqslant k}\mathrm{X}_i\}_i$ and $\tau_{\geqslant k}\{\mathrm{X}_i\}_i = \{\tau_{\geqslant k}\mathrm{X}_i\}_i$; homotopy groups $\pi_k\{\mathrm{X}_i\}_i = \{\pi_k(\mathrm{X}_i)\}_i$ live in the heart, the category of pro-abelian groups.

2.3. Let $\mathrm{Pro}(\mathrm{Spt})_\pi$ denote the full subcategory of *Postnikov-complete* pro-spectra, i.e. pro-spectra $\{\mathrm{X}_i\}_i$ such the canonical morphism

$$\{\mathrm{X}_i\}_i \to \varprojlim_k \tau_{\leqslant k}\{\mathrm{X}_i\}_i$$

is invertible. The inclusion $\mathrm{Pro}(\mathrm{Spt})_\pi \hookrightarrow \mathrm{Pro}(\mathrm{Spt})$ admits a left adjoint L_π given by

$$\mathrm{L}_\pi(\{\mathrm{X}_i\}_i) = \varprojlim_k \tau_{\leqslant k}\{\mathrm{X}_i\}_i = \{\tau_{\leqslant k}\mathrm{X}_i\}_{i,k},$$

where the latter is a pro-object indexed by pairs (i,k). This exhibits $\mathrm{Pro(Spt)}_\pi$ as a left localization at the class of morphisms $\{X_i\}_i \to \{Y_j\}_j$ such that

$$\{\tau_{\leqslant k} X_i\}_i \to \{\tau_{\leqslant k} Y_j\}_j$$

is invertible for each integer k. We refer to such morphisms as *quasi-isomorphisms*. If $\{X_i\}_i$ and $\{Y_j\}_j$ are eventually connective, then this is equivalent to the condition that the morphisms of pro-abelian groups

$$\{\pi_k(X_i)\}_i \to \{\pi_k(Y_j)\}_j$$

are invertible for each integer k.

Example 2.4. Let X be a spectrum and consider the constant pro-spectrum $\{X\}$. This is generally not Postnikov-complete. Indeed the canonical morphism $\{X\} \to L_\pi\{X\}$ is invertible in Pro(Spt) iff X is eventually coconnective, because we have $L_\pi\{X\} = \{\tau_{\leqslant i} X\}_i$ and therefore

$$\mathrm{Maps}_{\mathrm{Pro(Spt)}}(L_\pi\{X\}, \{X\}) = \varinjlim_i \mathrm{Maps}_{\mathrm{Spt}}(\tau_{\leqslant i} X, X).$$

2.5. We will also make use of the ∞-category Pro(SCRing) of pro-simplicial commutative rings. We define Postnikov-complete objects and quasi-isomorphisms in Pro(SCRing) just as above. Note that a morphism $\{A_i\}_i \to \{B_j\}_j$ is a quasi-isomorphism iff the induced morphism of pro-spectra $\{(A_i)_{\mathrm{Spt}}\}_i \to \{(B_j)_{\mathrm{Spt}}\}_j$ is a quasi-isomorphism. This is also equivalent to the condition that the morphisms of pro-abelian groups $\{\pi_k(A_i)\}_i \to \{\pi_k(B_j)\}_j$ are isomorphisms for all k (since simplicial commutative rings have connective underlying spectra).

3. A model for connective K-theory. To go further we will finally need a model for connective K-theory. Using Zariski descent we will be able to reduce many questions of interest to the affine case, where we can give a model for (connective) K-theory that is much more naive than the Waldhausen $S_\bullet$-construction.

3.1. Let R be a simplicial commutative ring. Let $\mathrm{Mod}_R^{\mathrm{proj}}$ denote the full subcategory of Mod_R spanned by finitely generated projective R-modules, i.e. direct summands of free modules $R^{\oplus n}$. The tensor product of two finitely generated projective R-modules is again finitely generated projective, so the ∞-category $\mathrm{Mod}_R^{\mathrm{proj}}$ inherits a symmetric monoidal structure. This induces a structure of $\mathcal{E}_\infty$-monoid on the underlying ∞-groupoid $(\mathrm{Mod}_R^{\mathrm{proj}})^{\approx}$ (obtained by discarding non-invertible 1-morphisms). If $X \mapsto X^{\mathrm{gp}}$ denotes group completion of $\mathcal{E}_\infty$-monoids, we have:

Theorem 3.2. *There is an isomorphism of group-like $\mathcal{E}_\infty$-spaces*

$$\Omega^\infty K(\mathrm{Spec}(R)) \approx ((\mathrm{Mod}_R^{\mathrm{proj}})^{\approx})^{\mathrm{gp}}.$$

Furthermore, this isomorphism is functorial in R.

4. K-theory and truncations. Next we will explain why K-theory behaves well with respect to quasi-isomorphisms of pro-systems.

4.1. Let R be a simplicial commutative ring. For $k \geqslant 0$, let $\tau_{\leqslant k}R$ denote the kth Postnikov truncation of R. For $k < 0$ set $\tau_{\leqslant k}R = \tau_{\leqslant 0}R$.

The following observation, due to Jacob Lurie, says that the $(k+1)$-truncation of the K-theory spectrum K(R) only depends on $\tau_{\leqslant k}R$.

Proposition 4.2. *For each integer k, the canonical morphism of simplicial commutative rings $R \to \tau_{\leqslant k}R$ induces an isomorphism of spectra*

$$\tau_{\leqslant k+1}K(R) \to \tau_{\leqslant k+1}K(\tau_{\leqslant k}R).$$

Using the Bass fundamental sequence (Theorem 1.10), it will suffice to show this for $k \geqslant 0$. This will follow from the following observation:

Claim 4.3. *For each $k \geqslant 0$, the canonical morphism of simplicial commutative rings* $\mathrm{R} \to \tau_{\leqslant k}\mathrm{R}$ *induces an equivalence of $(k+1)$-categories*

$$\tau^{\mathrm{cat}}_{\leqslant k+1}(\mathrm{Mod}^{\mathrm{proj}}_{\mathrm{R}}) \to \mathrm{Mod}^{\mathrm{proj}}_{\tau_{\leqslant k}\mathrm{R}}.$$

Here $\tau^{\mathrm{cat}}_{\leqslant k+1}$ denotes the "categorical" truncation that turns an ∞-category into a $(k+1)$-category (by truncating the mapping spaces).

Proof. Fully faithfulness amounts to the claim that the canonical map

$$\tau_{\leqslant k} \operatorname{Maps}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}, \mathrm{N}) \to \operatorname{Maps}_{\mathrm{Mod}_{\tau_{\leqslant k}\mathrm{R}}}(\tau_{\leqslant k}\mathrm{M}, \tau_{\leqslant k}\mathrm{N})$$

is invertible for all $\mathrm{M}, \mathrm{N} \in \mathrm{Mod}^{\mathrm{proj}}_{\mathrm{R}}$. If $\mathrm{M} = \mathrm{R}^{\oplus n}$ is free, then this is identified with the canonical isomorphism

$$\tau_{\leqslant k}\Omega^{\infty}(\mathrm{N})^{\times n} \to \Omega^{\infty}(\tau_{\leqslant k}\mathrm{N})^{\times n}.$$

In general it follows that the map in question is a retract of an isomorphism, hence an isomorphism. (Note that this in fact holds for $\mathrm{N} \in \mathrm{Mod}_{\mathrm{R}}$ arbitrary.)

It remains to show that the functor

$$\mathrm{Mod}^{\mathrm{proj}}_{\mathrm{R}} \to \mathrm{Mod}^{\mathrm{proj}}_{\tau_{\leqslant k}\mathrm{R}}$$

is essentially surjective. Recall that $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}$ is finitely generated and projective iff it is locally free of finite rank. Therefore the claim follows from the fact that R and $\tau_{\leqslant k}\mathrm{R}$ have equivalent small Zariski sites (and we can use the fully faithfulness to lift gluing data). □

4.4. The functor $\mathrm{K} : \mathrm{SCRing} \to \mathrm{Spt}$ extends object-wise to a functor $\mathrm{K} : \mathrm{Pro}(\mathrm{SCRing}) \to \mathrm{Pro}(\mathrm{Spt})$. As a corollary of the previous observation, we deduce:

Corollary 4.5. *The functor* $\mathrm{K} : \mathrm{Pro}(\mathrm{SCRing}) \to \mathrm{Pro}(\mathrm{Spt})$ *preserves quasi-isomorphisms.*

Proof. Let $\{\mathrm{A}_i\}_i \to \{\mathrm{B}_j\}_j$ be a quasi-isomorphism of pro-simplicial rings and consider the induced morphism of pro-spectra

$$\{\mathrm{K}(\mathrm{A}_i)\}_i \to \{\mathrm{K}(\mathrm{B}_j)\}_j.$$

It suffices to show that it induces isomorphisms of pro-spectra

$$\{\tau_{\leqslant k}\mathrm{K}(\mathrm{A}_i)\}_i \to \{\tau_{\leqslant k}\mathrm{K}(\mathrm{B}_j)\}_j$$

for each k. By Proposition 4.2 this is levelwise isomorphic to the pro-spectrum

$$\{\tau_{\leqslant k}\mathrm{K}(\tau_{\leqslant k-1}\mathrm{A}_i)\}_i \to \{\tau_{\leqslant k}\mathrm{K}(\tau_{\leqslant k-1}\mathrm{B}_j)\}_j,$$

so this follows from the assumption that $\{\mathrm{A}_i\}_i \to \{\mathrm{B}_j\}_j$ is a quasi-isomorphism. □

5. Pro-systems of regular closed immersions. Let R be a commutative ring and $f \in \mathrm{R}$ an element. Recall that the construction $\mathrm{R}/\!\!/(f)$ is discrete iff f is regular, i.e. a non-zero divisor. Under noetherian hypotheses, the next proposition shows that, even when f is a zero divisor, we can consider this construction as discrete if we consider the pro-system of all infinitesimal neighbourhoods $\mathrm{R}/\!\!/(f^n)$.

Proposition 5.1. *Let* R *be a (discrete) noetherian commutative ring. Then for any sequence of elements $(f_1, \ldots, f_r)$, the pro-simplicial ring $\{\mathrm{R}/\!\!/(f_i^n)_i\}_n$ is* quasi-discrete*, i.e. the canonical morphism of pro-simplicial rings*

$$\{\mathrm{R}/\!\!/(f_i^n)_i\}_n \to \{\mathrm{R}/(f_i^n)_i\}_n$$

is a quasi-isomorphism. More generally, for any discrete finitely generated R*-module* M*, the canonical morphism*

$$\{\mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}/\!\!/(f_i^n)_i\}_n \to \{\mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}/(f_i^n)_i\}_n$$

is a quasi-isomorphism.

Proof. It is clear that it is a levelwise isomorphism on π_0, so it suffices to show that the pro-system $\{\pi_k(\mathrm{M} \otimes \mathrm{R}/\!\!/(f_i^n)_i)\}_n$ vanishes for each $k > 0$. We argue by induction on the number of elements in the sequence $(f_1, \ldots, f_r)$. If $r = 0$ then the claim is clear since M is discrete.

For $r > 0$ we make use of the cofibre sequences

$$\mathrm{M}_{r-1}(n) \xrightarrow{f_r^n} \mathrm{M}_{r-1}(n) \to \mathrm{M}_r(n),$$

where $\mathrm{M}_r(n) := \mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}/\!\!/(f_1^n, \ldots, f_r^n)$, and $\mathrm{M}_{r-1}(n) := \mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}/\!\!/(f_1^n, \ldots, f_{r-1}^n)$. Looking at the long exact sequence on homotopy groups,

$$\cdots \to \{\pi_k(\mathrm{M}_r(n))\} \to \{\pi_{k-1}(\mathrm{M}_{r-1}(n))\} \xrightarrow{f_n^r} \{\pi_{k-1}(\mathrm{M}_{r-1}(n))\} \to \cdots$$

we reduce by induction to showing that the kernel of the morphism of pro-abelian groups

$$\mathrm{Ker}(\{\pi_0\mathrm{M}_{r-1}(n)\}_n \xrightarrow{f_r^n} \{\pi_0\mathrm{M}_{r-1}(n)\}_n)$$

vanishes. Note that $\pi_0\mathrm{M}_{r-1}(n) = \mathrm{M}/(f_1^n, \ldots, f_{r-1}^n)$. For each pair $m, n > 0$, write

$$\mathrm{K}(n, m) := \mathrm{Ker}(\{\mathrm{M}/(f_1^n, \ldots, f_{r-1}^n)\}_n \xrightarrow{f_r^m} \{\mathrm{M}/(f_1^n, \ldots, f_{r-1}^n)\}_n)$$

so that we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{K}(n,n) & \longrightarrow & \mathrm{K}(n-1,n) & \longrightarrow & \mathrm{K}(n-2,n) \\
\downarrow f_r & \searrow & \downarrow f_r & & \downarrow f_r \\
\mathrm{K}(n,n-1) & \longrightarrow & \mathrm{K}(n-1,n-1) & \longrightarrow & \mathrm{K}(n-2,n-1) \\
\downarrow f_r & & \downarrow f_r & \searrow & \downarrow f_r \\
\mathrm{K}(n,n-2) & \longrightarrow & \mathrm{K}(n-1,n-2) & \longrightarrow & \mathrm{K}(n-2,n-2)
\end{array}$$

The claim is that the "diagonal" tower vanishes as a pro-system, i.e. for each n, the transition morphism $\mathrm{K}(n', n') \to \mathrm{K}(n, n)$ is zero for some $n' > n$. By the commutativity it suffices to show that for each fixed n, the "vertical" pro-systems $\{\mathrm{K}(n, m)\}_m$ vanish. Note that there are canonical inclusions for each n,

$$\mathrm{K}(n, 1) \subset \mathrm{K}(n, 2) \subset \cdots$$

of submodules of $\mathrm{M}/(f_1^n, \ldots, f_{r-1}^n)$; the latter is a finitely generated R-module so the noetherian assumption implies that this chain stabilizes, whence the claim. □

5.2. Applying Corollary 4.5 we deduce:

Corollary 5.3. *Let* R *be a (discrete) noetherian commutative ring. Then for any sequence of elements* $(f_1, \ldots, f_r)$*, the morphism of pro-spectra*

$$\{\mathrm{K}(\mathrm{R}/\!\!/(f_i^n)_i)\}_n \to \{\mathrm{K}(\mathrm{R}/(f_i^n)_i)\}_n$$

is a quasi-isomorphism.

This observation will be instrumental in passing from descent for derived blow-ups to pro-descent for classical blow-ups.

References.

[1] M. Kerz, F. Strunk, G. Tamme, *Algebraic K-theory and descent for blow-ups.*

[2] Jacob Lurie, *Spectral algebraic geometry*, version of 2017-10-13, available at `http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf`.