Lecture 5

The cotangent complex and Nisnevich descent

Last lecture we saw the proof of Zariski descent in algebraic K-theory. Once we define the notion of Nisnevich square, the proof for Nisnevich descent will be exactly the same. To this end, we will begin this lecture by introducing the notion of the cotangent complex and étale morphisms in derived algebraic geometry. Since we will need to work with the cotangent complex later in the course, we will take this opportunity to cover it in some detail.

1. Derived derivations. Recall that the module of (algebraic) Kähler differentials admits a universal property in terms of derivations. The cotangent complex can be viewed as a derived version of Kähler differentials, and admits a universal property in terms of a derived version of derivations.

1.1. Recall that for an ordinary commutative ring R and an ordinary R-module M, one can define a new commutative ring $\mathrm{R} \oplus \mathrm{M}$, the trivial square-zero extension of R by M, where the multiplication is defined by the formula

$$(r, m) \cdot (r', m') = (rr', r'm + rm').$$

Now let $\mathrm{R} \in \mathrm{SCRing}$ and $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}$ be a connective R-module. One can then construct a new simplicial commutative ring whose underlying R-module is the direct sum $\mathrm{R} \oplus \mathrm{M}$, and such that the ring structure on $\pi_0(\mathrm{R}) \oplus \pi_0(\mathrm{M})$ is given by the formula above.

Remark 1.2. There are various possible ways to actually construct $\mathrm{R} \oplus \mathrm{M} \in \mathrm{SCRing}$. Of course if we choose strict models for the ∞-category SCRing then this is trivial. To do this directly with the model of SCRing we gave, one could instead use Lurie's "straightening/unstraightening" correspondence: it is relatively easy to define a cartesian fibration over Poly and then "straighten" it into a presheaf $(\mathrm{Poly})^{\mathrm{op}} \to \mathrm{Spc}$. Alternatively, one could prove the equivalence

$$\mathrm{Mod}_{\mathrm{R}} \xrightarrow{\sim} \mathrm{Stab}(\mathrm{SCRing}_{\mathrm{R}\backslash/\mathrm{R}}),$$

where the right-hand side is the ∞-category of spectrum objects in the ∞-category of augmented simplicial commutative R-algebras; then $\mathrm{R} \oplus \mathrm{M}$ can be defined as the image of M by the functor

$$\mathrm{Mod}_{\mathrm{R}} \xrightarrow{\sim} \mathrm{Stab}(\mathrm{SCRing}_{\mathrm{R}\backslash/\mathrm{R}}) \xrightarrow{\Omega^\infty} \mathrm{SCRing}_{\mathrm{R}\backslash/\mathrm{R}}.$$

The latter approach is taken by Lurie in the setting of $\mathcal{E}_\infty$-ring spectra [1, § 7.3.4].

Definition 1.3. *Let* $\mathrm{R} \in \mathrm{SCRing}$ *and* $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}$. *The simplicial commutative ring* $\mathrm{R} \oplus \mathrm{M}$ *is called the* trivial infinitesimal extension *of* R *by* M.

1.4. Note that there is a canonical homomorphism

$$\mathrm{R} \oplus \mathrm{M} \to \mathrm{R} \tag{1.1}$$

given informally by the projection $(r, m) \mapsto r$. It admits a canonical section

$$\mathrm{R} \to \mathrm{R} \oplus \mathrm{M} \tag{1.2}$$

given informally by $r \mapsto (r, 0)$.

Let R be a simplicial commutative ring, A a simplicial commutative R-algebra, and M a connective A-module. For any A-module M, the homomorphism $\mathrm{A} \oplus \mathrm{M} \to \mathrm{A}$ (1.1) induces a canonical map

$$\mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}, \mathrm{A} \oplus \mathrm{M}) \to \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}, \mathrm{A}). \tag{1.3}$$

The space $\mathrm{Der}_{\mathrm{R}}(\mathrm{A}, \mathrm{M})$ of R-*linear derivations of* A *with values in* M is the homotopy fibre of this map at the point id_{A}.

In other words, a derivation is a morphism $\mathrm{A} \to \mathrm{A} \oplus \mathrm{M}$ together with a commutative diagram

$$\begin{tikzcd} & \mathrm{R} \arrow[dl] \arrow[dr] & \\ \mathrm{A} \arrow[rr, dashed] \arrow[dr, equal] & & \mathrm{A} \oplus \mathrm{M} \arrow[dl] \\ & \mathrm{A} & \end{tikzcd}$$

Example 1.5. The *trivial derivation* is defined by the canonical section (1.2). We denote this by $d_{\mathrm{triv}} : \mathrm{A} \to \mathrm{A} \oplus \mathrm{M}$.

2. The cotangent complex.

2.1. Let $\mathrm{R} \in \mathrm{SCRing}$ and $\mathrm{A} \in \mathrm{SCRing}_{\mathrm{R}}$ a simplicial commutative R-algebra.

Definition 2.2. *We say that a connective* A*-module* L *is a* cotangent complex *for* A *over* R *if it corepresents the functor* $\mathrm{M} \mapsto \mathrm{Der}_{\mathrm{R}}(\mathrm{A}, \mathrm{M})$*, i.e. if there are functorial isomorphisms*

$$\mathrm{Maps}_{\mathrm{Mod}_{\mathrm{A}}}(\mathrm{L}, \mathrm{M}) \approx \mathrm{Der}_{\mathrm{R}}(\mathrm{A}, \mathrm{M})$$

for any connective A*-module* M*.*

If such an L exists, then we say that A *admits a cotangent complex over* R and denote it by $\mathbf{L}_{\mathrm{A}/\mathrm{R}}$. Note that it is automatically unique (up to isomorphism in the ∞-category $\mathrm{Mod}_{\mathrm{A}}^{\mathrm{cn}}$) by the Yoneda lemma. We also write $\mathbf{L}_{\mathrm{R}} := \mathbf{L}_{\mathrm{R}/\mathbf{Z}}$ for the *absolute* cotangent complex.

We would like to prove that the cotangent complex always exists:

Theorem 2.3. *Let* $\mathrm{R} \in \mathrm{SCRing}$ *and* $\mathrm{A} \in \mathrm{SCRing}_{\mathrm{R}}$ *a simplicial commutative* R*-algebra. Then* A *admits a cotangent complex over* R*.*

2.4. We begin with a simple case.

Proposition 2.5. *Let* $\mathrm{R} \in \mathrm{SCRing}$*,* N *a connective* R*-module, and* $\mathrm{A} = \mathrm{Sym}_{\mathrm{R}}(\mathrm{N})$ *the free* R*-algebra on* N*. Then the* A*-module* $\mathrm{L} = \mathrm{N} \otimes_{\mathrm{R}} \mathrm{A}$ *is a cotangent complex for* A *over* R*.*

Proof. Follows immediately from the universal property of $\mathrm{A} = \mathrm{Sym}_{\mathrm{R}}(\mathrm{N})$. ☐

2.6. Recall that the ∞-category $\mathrm{SCRing}_{\mathrm{R}}$ is (freely) generated under sifted colimits by the polynomial R-algebras $\mathrm{R}[\mathrm{T}_1, \dots, \mathrm{T}_n] \approx \mathrm{Sym}_{\mathrm{R}}(\mathrm{R}^{\oplus n})$. This allows us to reduce Theorem 2.3 to the case of Proposition 2.5.

Proof of Theorem 2.3. We can write any $\mathrm{A} \in \mathrm{SCRing}_{\mathrm{R}}$ as a sifted colimit of polynomial R-algebras $\mathrm{A}_\alpha \approx \mathrm{R}[\mathrm{T}_1, \dots, \mathrm{T}_{n_\alpha}]$. Let L denote the connective A-module

$$\varinjlim_{\alpha} \mathbf{L}_{\mathrm{A}_\alpha/\mathrm{R}} \otimes_{\mathrm{A}_\alpha} \mathrm{A} \approx \varinjlim_{\alpha} \mathrm{A}_\alpha^{\oplus n_\alpha} \otimes_{\mathrm{A}_\alpha} \mathrm{A} = \varinjlim_{\alpha} \mathrm{A}^{\oplus n_\alpha}.$$

We claim that L is a cotangent complex for A over R, i.e. that for any connective A-module M we have functorial isomorphisms

$$\mathrm{Maps}_{\mathrm{Mod}_{\mathrm{A}}}(\mathrm{L}, \mathrm{M}) \approx \mathrm{Der}_{\mathrm{R}}(\mathrm{A}, \mathrm{M}).$$

Note that the left-hand side is isomorphic (functorially) to

$$\mathrm{Maps}_{\mathrm{Mod}_{\mathrm{A}}}(\mathrm{L}, \mathrm{M}) \approx \mathrm{Maps}_{\mathrm{Mod}_{\mathrm{A}}}(\varinjlim_{\alpha} \mathrm{A}^{\oplus n_\alpha}, \mathrm{M}) = \varprojlim_{\alpha} \mathrm{Maps}_{\mathrm{Mod}_{\mathrm{A}}}(\mathrm{A}^{\oplus n_\alpha}, \mathrm{M}) = \varprojlim_{\alpha} \Omega^\infty(\mathrm{M})^{\times n_\alpha}.$$

Let us now compute the space of derivations. Note that we have $\mathrm{A} \oplus \mathrm{M} = \varinjlim_{\alpha} \mathrm{A}_\alpha \oplus \mathrm{M}$; this formula is clearly true on the underlying spectra, and the forgetful functor $\mathrm{SCRing} \to \mathrm{Spt}^{\mathrm{cn}}$ is conservative and preserves sifted colimits. We therefore get functorial isomorphisms

$$\begin{aligned}
\mathrm{Der}_{\mathrm{R}}(\mathrm{A}, \mathrm{M}) &= \mathrm{Fib}(\mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}, \mathrm{A} \oplus \mathrm{M}) \to \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}, \mathrm{A})) \\
&= \mathrm{Fib}(\varprojlim_{\alpha} \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}_\alpha, \mathrm{A} \oplus \mathrm{M}) \to \varprojlim_{\alpha} \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}_\alpha, \mathrm{A})) \\
&= \varprojlim_{\alpha} \mathrm{Fib}(\mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}_\alpha, \mathrm{A} \oplus \mathrm{M}) \to \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}_\alpha, \mathrm{A})) \\
&= \varprojlim_{\alpha} \mathrm{Fib}((\mathrm{A} \oplus \mathrm{M})_{\mathrm{Spc}}^{\times n_\alpha} \to \mathrm{A}_{\mathrm{Spc}}^{\times n_\alpha}) \\
&= \varprojlim_{\alpha} \Omega^\infty(\mathrm{M})^{\times n_\alpha}.
\end{aligned}$$

Here we have used the fact that the A_α are free as R-algebras. □

2.7. The following observation will also be useful:

Proposition 2.8. *Let* R *be a simplicial commutative ring and let* A *be a simplicial commutative* R*-algebra which is locally of finite presentation. Then the cotangent complex* $\mathbf{L}_{\mathrm{A}/\mathrm{R}}$ *is a perfect* A*-module.*

Proof. It suffices to show that $\mathbf{L}_{\mathrm{A}/\mathrm{R}}$ is compact, i.e. that the functor

$$\mathrm{M} \mapsto \mathrm{Maps}_{\mathrm{Mod}_{\mathrm{R}}}(\mathbf{L}_{\mathrm{A}/\mathrm{R}}, \mathrm{M}) = \mathrm{Maps}_{\mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}}(\mathbf{L}_{\mathrm{A}/\mathrm{R}}, \mathrm{M}_{\geqslant 0}) \approx \mathrm{Der}_{\mathrm{R}}(\mathrm{A}, \mathrm{M}_{\geqslant 0})$$

preserves filtered colimits. This follows from the fact that each of the following constructions commute with filtered colimits: $\mathrm{M} \mapsto \mathrm{M}_{\geqslant 0}$, $\mathrm{M} \mapsto \mathrm{A} \oplus \mathrm{M}$, $\mathrm{B} \mapsto \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{A}, \mathrm{B})$ (since A is locally of finite presentation), and $(\mathrm{X} \to \mathrm{Y}) \mapsto \mathrm{Fib}(\mathrm{X} \to \mathrm{Y})$ for maps of spaces $\mathrm{X} \to \mathrm{Y}$. □

2.9. The proof of Theorem 2.3 actually shows that the cotangent complex is the “nonabelian derived functor” of the sheaf of Kähler differentials. This has the following immediate consequence:

Proposition 2.10. *Let* R *be a simplicial commutative ring and let* A *be a simplicial commutative* R*-algebra. Then we have*

$$\pi_0(\mathbf{L}_{\mathrm{A}/\mathrm{R}}) \approx \Omega_{\pi_0(\mathrm{A})/\pi_0(\mathrm{R})}.$$

2.11. The following fundamental properties follow immediately from the definitions.

Proposition 2.12.

(i) *Let* $\mathrm{A} \to \mathrm{B} \to \mathrm{C}$ *be homomorphisms of simplicial commutative rings. Then we have an exact triangle in* $\mathrm{Mod}_{\mathrm{C}}$*:*

$$\mathbf{L}_{\mathrm{B}/\mathrm{A}} \otimes_{\mathrm{B}} \mathrm{C} \to \mathbf{L}_{\mathrm{C}/\mathrm{A}} \to \mathbf{L}_{\mathrm{C}/\mathrm{B}}.$$

(ii) *Suppose we have a cocartesian square in* SCRing*:*

$$\begin{tikzcd} \mathrm{A} \arrow[r] \arrow[d] & \mathrm{B} \arrow[d] \\ \mathrm{A}' \arrow[r] & \mathrm{B}'. \end{tikzcd}$$

Then we have a canonical isomorphism

$$\mathbf{L}_{\mathrm{B}/\mathrm{A}} \otimes_{\mathrm{B}} \mathrm{B}' \approx \mathbf{L}_{\mathrm{B}'/\mathrm{A}'}$$

in $\mathrm{Mod}_{\mathrm{B}'}$.

3. Infinitesimal extensions. Next we briefly review how the cotangent complex controls deformation theory along infinitesimal extensions, and provides a derived version of Kodaira–Spencer deformation theory.

3.1. Let $\mathrm{R} \in \mathrm{SCRing}$, $\mathrm{A} \in \mathrm{SCRing}_{\mathrm{R}}$, and $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}$. Let $d \in \mathrm{Der}_{\mathrm{R}}(\mathrm{A}, \mathrm{M}[1])$ be an R-linear derivation of A valued in M[1], or equivalently a morphism $d : \mathbf{L}_{\mathrm{A}/\mathrm{R}} \to \mathrm{M}[1]$.

We define the *infinitesimal extension* of A along d, denoted $\varphi_d : \mathrm{A}^d \to \mathrm{A}$, by the cartesian square in $\mathrm{SCRing}_{\mathrm{R}}$

$$\begin{tikzcd}
\mathrm{A}^d \arrow[r] \arrow[d, "\varphi_d"] & \mathrm{A} \arrow[d, "d_{\mathrm{triv}}"] \\
\mathrm{A} \arrow[r, "d"] & \mathrm{A} \oplus \mathrm{M}[1].
\end{tikzcd}$$

3.2. Given an infinitesimal extension $\varphi_d : \mathrm{A}^d \to \mathrm{A}$, we can recover the module M as the homotopy fibre

$$\mathrm{M} \approx \mathrm{Fib}(\mathrm{A}^d \to \mathrm{A})$$

in $\mathrm{Mod}_{\mathrm{R}}$. Indeed, this fibre is isomorphic by construction to the fibre of $d_{\mathrm{triv}} : \mathrm{A} \to \mathrm{A} \oplus \mathrm{M}[1]$. Both squares in the diagram

$$\begin{tikzcd}
\mathrm{Fib}(d_{\mathrm{triv}}) \arrow[r] \arrow[d] & \mathrm{A} \arrow[r] \arrow[d, "d_{\mathrm{triv}}"] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{A} \oplus \mathrm{M}[1] \arrow[r] & \mathrm{M}[1]
\end{tikzcd}$$

are cartesian, whence the claim.

3.3. Let $\mathrm{R} \in \mathrm{SCRing}$ and $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}$ a connective R-module. Let $d \in \mathrm{Der}(\mathrm{R}, \mathrm{M}[1])$ be a ($\mathbf{Z}$-linear) derivation of R valued in M[1] and let $\varphi_d : \mathrm{R}^d \to \mathrm{R}$ the associated infinitesimal extension.

Definition 3.4. *Let* $\mathrm{A} \in \mathrm{SCRing}_{\mathrm{R}}$ *a simplicial commutative* R*-algebra. A* deformation *of* A *along the infinitesimal extension* $\mathrm{R}^d \to \mathrm{R}$ *is an* R^d*-algebra* $\tilde{\mathrm{A}}$ *fitting into a cocartesian square*

$$\begin{tikzcd}
\mathrm{R}^d \arrow[r, "\varphi^d"] \arrow[d, dashed] & \mathrm{R} \arrow[d] \\
\tilde{\mathrm{A}} \arrow[r, dashed] & \mathrm{A}.
\end{tikzcd}$$

3.5. Consider the exact triangle

$$\mathbf{L}_{\mathrm{R}/\mathrm{A}}[-1] \to \mathbf{L}_{\mathrm{R}} \otimes_{\mathrm{R}} \mathrm{A} \to \mathbf{L}_{\mathrm{A}}.$$

Here we write $\mathbf{L}_{\mathrm{R}} := \mathbf{L}_{\mathrm{R}/\mathbf{Z}}$, $\mathbf{L}_{\mathrm{A}} := \mathbf{L}_{\mathrm{A}/\mathbf{Z}}$.

The derivation d corresponds to a morphism $d : \mathbf{L}_{\mathrm{R}} \to \mathrm{M}[1]$. Suppose that for some reason the composite

$$\mathrm{obstr}_d^{\mathrm{A}} : \mathbf{L}_{\mathrm{A}/\mathrm{R}}[-1] \to \mathbf{L}_{\mathrm{R}} \otimes_{\mathrm{R}} \mathrm{A} \xrightarrow{d \otimes \mathrm{id}_{\mathrm{A}}} \mathrm{M}[1] \otimes_{\mathrm{R}} \mathrm{A}$$

is null-homotopic, so that we get an induced morphism

$$d' : \mathbf{L}_{\mathrm{A}} \to \mathrm{M}[1] \otimes_{\mathrm{R}} \mathrm{A}.$$

This defines a derivation of A with values in $\mathrm{M} \otimes_{\mathrm{R}} \mathrm{A}[1]$, and we let $\tilde{\mathrm{A}} := \mathrm{A}^{d'} \to \mathrm{A}$ denote the associated infinitesimal extension. This gives a deformation of A in the above sense.

In fact, we have:

Theorem 3.6 (Lurie, Toën–Vezzosi)**.** *The space of deformations of the* R*-algebra* A *along the infinitesimal extension* $\mathrm{R}^d \to \mathrm{R}$ *is equivalent to the space of null-homotopies of the composite*

$$\mathrm{obstr}_d^{\mathrm{A}} : \mathbf{L}_{\mathrm{A}/\mathrm{R}}[-1] \to \mathbf{L}_{\mathrm{R}} \otimes_{\mathrm{R}} \mathrm{A} \xrightarrow{d \otimes \mathrm{id}_{\mathrm{A}}} \mathrm{M}[1] \otimes_{\mathrm{R}} \mathrm{A}.$$

See [1, Prop. 7.4.2.5].

3.7. In other words, we have a class $\mathrm{obstr}_d^{\mathrm{A}} \in \mathrm{Ext}^2_{\mathrm{A}}(\mathbf{L}_{\mathrm{A}/\mathrm{R}}, \mathrm{M} \otimes_{\mathrm{R}} \mathrm{A})$ which measures the obstruction for the existence of a deformation along $\mathrm{R}^d \to \mathrm{R}$.

4. Smooth and étale morphisms.

4.1. Let $\mathrm{R} \in \mathrm{SCRing}$ and $\mathrm{A} \in \mathrm{SCRing}_{\mathrm{R}}$ a simplicial commutative R-algebra.

Definition 4.2.

(i) *We say that* A *is* formally smooth *over* R *if the cotangent complex* $\mathbf{L}_{\mathrm{A}/\mathrm{R}}$ *is a finitely generated projective* A*-module (i.e. it is a direct summand of a free* A*-module* $\mathrm{A}^{\oplus n}$ *for some* $n \geqslant 0$*).*

(ii) *We say that* A *is* formally étale *over* R *if the cotangent complex* $\mathbf{L}_{\mathrm{A}/\mathrm{R}}$ *is zero.*

(iii) *We say that* A *is smooth (resp. étale) over* R *if it is locally of finite presentation and formally smooth (resp. formally étale).*

Theorem 4.3 (Lurie, Toën–Vezzosi)**.** *An* R*-algebra* A *is smooth (resp. étale) iff it is flat, and* $\pi_0(\mathrm{A})$ *is smooth (resp. étale) as a* $\pi_0(\mathrm{R})$*-algebra.*

See [2, § B.1.1] or [3, Thm. 2.2.2.6].

4.4. Theorem 4.3 immediately gives the following useful characterization of open immersions:

Corollary 4.5. *Let* A *be an* R*-algebra. Then the following conditions are equivalent:*

(i) *The homomorphism* $\mathrm{R} \to \mathrm{A}$ *is locally of finite presentation, flat, and an epimorphism. That is,* $\mathrm{Spec}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{R})$ *is an open immersion of derived schemes.*

(ii) *The homomorphism* $\mathrm{R} \to \mathrm{A}$ *is étale and an epimorphism.*

5. The cotangent complex for schemes.

5.1. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of derived schemes. Let $y : \mathrm{Spec}(\mathrm{R}) \to \mathrm{Y}$ be an R-point (with $\mathrm{R} \in \mathrm{SCRing}$).

For any connective R-module $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}$ we have a commutative square

$$\begin{tikzcd}
\mathrm{Y}(\mathrm{R} \oplus \mathrm{M}) \arrow[r, "\mathrm{pr}_*"] \arrow[d, "f"] & \mathrm{Y}(\mathrm{R}) \arrow[d, "f"] \\
\mathrm{X}(\mathrm{R} \oplus \mathrm{M}) \arrow[r, "\mathrm{pr}_*"] & \mathrm{X}(\mathrm{R})
\end{tikzcd}$$

where the map $\mathrm{pr} : \mathrm{R} \oplus \mathrm{M} \to \mathrm{R}$ is the canonical projection (1.1), which determines a canonical map

$$\mathrm{Y}(\mathrm{R} \oplus \mathrm{M}) \to \mathrm{Y}(\mathrm{R}) \underset{\mathrm{X}(\mathrm{R})}{\times} \mathrm{X}(\mathrm{R} \oplus \mathrm{M}). \tag{5.1}$$

The point $y \in \mathrm{Y}(\mathrm{R})$ and the point $(d_{\mathrm{triv}})_*(f(y)) \in \mathrm{X}(\mathrm{R} \oplus \mathrm{M})$,

$$(d_{\mathrm{triv}})_*(f(y)) : \mathrm{Spec}(\mathrm{R} \oplus \mathrm{M}) \xrightarrow{d_{\mathrm{triv}}} \mathrm{Spec}(\mathrm{R}) \xrightarrow{y} \mathrm{Y} \xrightarrow{f} \mathrm{X},$$

together with the canonical isomorphism

$$(\mathrm{pr})_*(d_{\mathrm{triv}})_*(f(y)) \approx f(y) \in \mathrm{X}(\mathrm{R}),$$

determine a point in the target of (5.1).

We define the space of M-valued derivations of p (at the point y) as the homotopy fibre at this point:

$$\mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathrm{M}) := \mathrm{Fib}(\mathrm{Y}(\mathrm{R} \oplus \mathrm{M}) \to \mathrm{Y}(\mathrm{R}) \underset{\mathrm{X}(\mathrm{R})}{\times} \mathrm{X}(\mathrm{R} \oplus \mathrm{M})).$$

Definition 5.2. *We say that* $\mathrm{L}_y \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}$ *is a cotangent complex for* f *at the point* y*, if* L *corepresents the functor* $\mathrm{M} \mapsto \mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathrm{M})$*, i.e. there are functorial isomorphisms of spaces*

$$\mathrm{Maps}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{L}_y, \mathrm{M}) \approx \mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathrm{M}).$$

When such L_y exists, we say that f admits a cotangent complex at the point y. We write it as $y^*\mathcal{L}_f$ or $y^*\mathcal{L}_{\mathrm{Y}/\mathrm{X}}$, and view it as a quasi-coherent sheaf on Spec(R).

Example 5.3. Suppose that X = Spec(A) and Y = Spec(B) are affine. Then any morphism $f : \mathrm{Y} \to \mathrm{X}$ admits a cotangent complex at any point $y : \mathrm{Spec}(\mathrm{R}) \to \mathrm{Y}$, which is given by

$$y^*\mathcal{L}_{\mathrm{Spec}(\mathrm{B})/\,\mathrm{Spec}(\mathrm{A})} = \mathbf{L}_{\mathrm{B}/\mathrm{A}} \otimes_{\mathrm{B}} \mathrm{R}.$$

Definition 5.4. *Let* $\mathcal{L}$ *be a connective quasi-coherent sheaf on* Y*. We say that* $\mathcal{L}$ *is a (global) cotangent complex for* $f : \mathrm{Y} \to \mathrm{X}$ *if for any point* $y \in \mathrm{Y}(\mathrm{R})$ *with* $\mathrm{R} \in \mathrm{SCRing}$*, the inverse image* $y^*\mathcal{L}$ *is a cotangent complex for* f *at* y*.*

If $\mathcal{L}$ exists, we say that f admits a (global) cotangent complex, and we write it as $\mathcal{L}_f$ or $\mathcal{L}_{\mathrm{Y}/\mathrm{X}}$.

Remark 5.5. In the non-schematic case, i.e. when X and Y are derived stacks, it is not appropriate to require that the cotangent complex be connective.

5.6. The following property follows from the definitions:

Proposition 5.7. *Let* $\mathrm{Z} \xrightarrow{g} \mathrm{Y} \xrightarrow{f} \mathrm{X}$ *be morphisms of derived schemes. Suppose that* f *admits a cotangent complex. Then* g *admits a cotangent complex iff* $f \circ g$ *admits a cotangent complex. In either of these cases we moreover have an exact triangle*

$$g^*\mathcal{L}_{\mathrm{Y}/\mathrm{X}} \to \mathcal{L}_{\mathrm{Z}/\mathrm{X}} \to \mathcal{L}_{\mathrm{Z}/\mathrm{Y}}$$

in Qcoh(Z)*.*

5.8. As we saw already (Corollary 4.5), any open immersion of affine schemes is étale and has vanishing cotangent complex. This also holds in the non-affine case:

Proposition 5.9. *Let* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *be an open immersion of derived schemes. Then* j *admits a cotangent complex, and moreover* $\mathcal{L}_{\mathrm{U}/\mathrm{X}} = 0$*.*

Proof. It suffices to show that $u^*\mathcal{L}_{\mathrm{U}/\mathrm{X}} = 0$ for all points $u \in \mathrm{U}(\mathrm{R})$ and all $\mathrm{R} \in \mathrm{SCRing}$. This amounts to the claim that for any $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{cn}}$, the canonical map

$$\mathrm{U}(\mathrm{R} \oplus \mathrm{M}) \to \mathrm{U}(\mathrm{R}) \underset{\mathrm{X}(\mathrm{R})}{\times} \mathrm{X}(\mathrm{R} \oplus \mathrm{M})$$

has contractible fibre at the point $(u, (d_{\mathrm{triv}})_*(j(u)))$. Since j is a monomorphism, it is easy to see that this map is also a monomorphism, i.e. has empty or contractible fibres. Note that the point $(d_{\mathrm{triv}})_*(u) \in \mathrm{U}(\mathrm{R} \oplus \mathrm{M})$ lives in the fibre, so the claim follows. □

Remark 5.10. The proof of Proposition 5.9 only used the fact that j is an monomorphism.

5.11. We now show that any morphism of derived schemes admits a cotangent complex.

Theorem 5.12. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of derived schemes. Then f admits a cotangent complex $\mathcal{L}_{\mathrm{Y}/\mathrm{X}} \in \mathrm{Qcoh}(\mathrm{Y})$.*

Proof. If X and Y both admit *absolute* cotangent complexes $\mathcal{L}_{\mathrm{X}}$ and $\mathcal{L}_{\mathrm{Y}}$, respectively (i.e. cotangent complexes over $\mathrm{Spec}(\mathbf{Z})$), then we can set

$$\mathcal{L}_{\mathrm{Y}/\mathrm{X}} := \mathrm{Cofib}(f^*\mathcal{L}_{\mathrm{X}} \to \mathcal{L}_{\mathrm{Y}})$$

in view of Proposition 5.7. Therefore we can assume that $\mathrm{X} = \mathrm{Spec}(\mathbf{Z})$.

Recall that, since Y is a (derived) scheme, we have an equivalence (Lecture 1)

$$\mathrm{Qcoh}(\mathrm{Y}) = \varprojlim_{\mathrm{U} \xrightarrow{y} \mathrm{Y}} \mathrm{Qcoh}(\mathrm{U}), \tag{5.2}$$

where the limit is taken over pairs (U, y) with $\mathrm{U} = \mathrm{Spec}(\mathrm{R})$ affine, and y an *open immersion*. Therefore it suffices to construct a compatible system of quasi-coherent sheaves $(y^*\mathcal{L}_{\mathrm{Y}/\mathrm{X}})_y$ for all such pairs (U, y). According to the exact triangle

$$y^*\mathcal{L}_{\mathrm{Y}} \to \mathcal{L}_{\mathrm{U}} \to \mathcal{L}_{\mathrm{U}/\mathrm{Y}}$$

from Proposition 5.7, and the fact that $\mathcal{L}_{\mathrm{U}/\mathrm{Y}} = 0$ (Proposition 5.9), we must have

$$y^*\mathcal{L}_{\mathrm{Y}} = \mathcal{L}_{\mathrm{U}} \tag{5.3}$$

if $\mathcal{L}_{\mathrm{Y}}$ and $\mathcal{L}_{\mathrm{U}}$ exist. Note that $\mathcal{L}_{\mathrm{U}}$ exists and is just the R-module $\mathbf{L}_{\mathrm{R}} := \mathbf{L}_{\mathrm{R}/\mathbf{Z}}$, viewed as a quasi-coherent sheaf on $\mathrm{U} = \mathrm{Spec}(\mathrm{R})$. It is easy to show that the formula (5.3) defines an object in the limit (5.2) (which is indexed by a poset), and one verifies that it is indeed a cotangent complex for Y. □

6. Smooth and étale morphisms of schemes.

6.1. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of derived schemes. We can extend "local properties" to the schematic case as follows:

Definition 6.2. *We say that p is étale (resp. smooth, flat, locally of finite presentation) if there exist affine Zariski covers $(\mathrm{Y}_\alpha \hookrightarrow \mathrm{Y})_\alpha$ and $(\mathrm{X}_\beta \hookrightarrow \mathrm{X})_\beta$ such that the following holds: for each α, there exists an index β and a morphism of affine derived schemes $\mathrm{Y}_\alpha \to \mathrm{X}_\beta$ which is étale (resp. smooth, flat, locally of finite presentation) and fits in a commutative square*

$$\begin{tikzcd} \mathrm{Y}_\alpha \arrow[r] \arrow[d, hook] & \mathrm{X}_\beta \arrow[d, hook] \\ \mathrm{Y} \arrow[r] & \mathrm{X}. \end{tikzcd}$$

6.3. It follows easily from the definition that étaleness and smoothness can be detected using the cotangent complex:

Proposition 6.4. *Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of derived schemes which is locally of finite presentation. Then p is étale (resp. smooth) iff the cotangent complex $\mathcal{L}_{\mathrm{Y}/\mathrm{X}}$ is zero (resp. is locally free of finite rank).*

7. Nisnevich descent. Finally, we now state Thomason's Nisnevich descent theorem.

7.1. Suppose we have a cartesian square of qcqs derived schemes:

$$\begin{tikzcd} \mathrm{W} \arrow[r, hook] \arrow[d] & \mathrm{V} \arrow[d, "p"] \\ \mathrm{U} \arrow[r, hook, "j"] & \mathrm{X}. \end{tikzcd} \tag{7.1}$$

We say that it is a *Nisnevich square* if it satisfies the following conditions:

(i) The morphism j is a quasi-compact open immersion.

(ii) The morphism p is quasi-compact and étale.

(iii) There exists a closed immersion $\mathrm{Z} \hookrightarrow \mathrm{X}$ complementary to U such that the induced morphism

$$p^{-1}(\mathrm{Z}) \to \mathrm{Z}$$

is invertible.

7.2. The proof of Zariski descent in K-theory (Lecture 4) immediately extends, essentially word-for-word, to Nisnevich squares.

Theorem 7.3 (Thomason)**.** *Let* X *be a qcqs derived scheme and suppose we have a Nisnevich square (7.1). Then the induced square of spectra*

$$\begin{tikzcd} \mathrm{K}(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{K}(\mathrm{U}) \arrow[d] \\ \mathrm{K}(\mathrm{V}) \arrow[r] & \mathrm{K}(\mathrm{U} \cap \mathrm{V}) \end{tikzcd}$$

is cartesian.

Remark 7.4. One can define a Grothendieck topology on the site of qcqs (derived) schemes, associated to the pretopology generated by covering families $\{j, p\}$ for all Nisnevich squares (7.1). Hoyois has shown that over qcqs schemes, this topology agrees with all known definitions of the Nisnevich topology. As we remarked in Lecture 4, it is then a theorem of Voevodsky that the above condition is equivalent to Čech descent with respect to the Nisnevich topology for the presheaf of spectra $\mathrm{K} : (\mathrm{DSch}_{\mathrm{qcqs}})^{\mathrm{op}} \to \mathrm{Spt}$.

References.

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[3] Bertrand Toën, Gabriele Vezzosi, *Homotopical algebraic geometry II.*