Lecture 4

Zariski descent for algebraic K-theory

Our goal for this lecture is to demonstrate Zariski descent for algebraic K-theory.

1. Algebraic K-theory.

1.1. Let $\mathbf{C}$ be a small stable ∞-category. We say that $\mathbf{C}$ is *idempotent-complete* if the canonical functor

$$\mathbf{C} \to (\mathbf{C})^{\mathrm{idem}}$$

is an equivalence. Recall that $(\mathbf{C})^{\mathrm{idem}}$, the idempotent-completion, can be defined as the full subcategory of presheaves on $\mathbf{C}$ generated by the representables under direct summands; equivalently it is the full subcategory of compact objects in the ind-completion: $(\mathbf{C})^{\mathrm{idem}} = \mathrm{Ind}(\mathbf{C})^{\mathrm{comp}}$.

In particular, for any stable presentable ∞-category $\mathbf{C}$, the full subcategory $(\mathbf{C})^{\mathrm{comp}}$ is idempotent-complete.

1.2. Nonconnective algebraic K-theory defines a functor

$$\mathrm{K} : \mathrm{Stab}^{\mathrm{idem}} \to \mathrm{Spt} \tag{1.1}$$

from the ∞-category of small stable idempotent-complete ∞-categories (and exact functors) to the ∞-category of spectra.

This functor has two especially important properties:

(i) K preserves filtered colimits.

(ii) For any exact sequence of small stable idempotent-complete ∞-categories

$$\mathbf{C}' \to \mathbf{C} \to \mathbf{C}'',$$

the induced sequence of spectra

$$\mathrm{K}(\mathbf{C}') \to \mathrm{K}(\mathbf{C}) \to \mathrm{K}(\mathbf{C}'')$$

is an exact triangle.

Instead of recalling the construction of the functor K, we will instead take an axiomatic approach. The idea is that the arguments we make will actually apply to a large class of interesting functors.

Definition 1.3. *Let* $\mathrm{E} : \mathrm{Stab} \to \mathrm{Spt}$ *be a functor. We say that it is* continuous *if it satisfies (i) and* localizing *if it satisfies (iii).*

Example 1.4. Nonconnective K-theory is continuous and localizing; see [2]. The reader can also find a construction of the functor K in *loc. cit.*

Remark 1.5. By applying the "connective cover" functor to $\mathrm{K} : \mathrm{Stab}^{\mathrm{idem}} \to \mathrm{Spt}$, we obtain the connective K-theory functor $\mathrm{K}^{\mathrm{cn}} : \mathrm{Stab}^{\mathrm{idem}} \to \mathrm{Spt}^{\mathrm{cn}}$. Connective K-theory fails to be localizing; instead it satisfies a weaker property called *additivity*: it sends any *split* exact sequence in $\mathrm{Stab}^{\mathrm{idem}}$ to a split exact sequence of spectra.

On the other hand, $\mathrm{Spt} \to \mathrm{Spt}^{\mathrm{cn}}$ commutes with limits, so any descent property we show for nonconnective K-theory will give us a descent result for connective K-theory (as a presheaf of *connective* spectra).

1.6.

2. The localization sequence in K-theory. The discussion in this section will apply to any localizing functor E instead of K.

2.1. Given a quasi-compact open immersion $j : \mathrm{U} \hookrightarrow \mathrm{X}$, we write

$$\mathrm{K}(\mathrm{X})_{\mathrm{U}} = \mathrm{K}(\mathrm{Perf}(\mathrm{X})_{\mathrm{U}}).$$

Recall from Lecture 3 that $\mathrm{Perf}(\mathrm{X})_{\mathrm{U}}$ denotes the full subcategory of perfect complexes on X which vanish on U.

Theorem 2.2 (Thomason)**.** *Let $j : \mathrm{U} \hookrightarrow \mathrm{X}$ be a quasi-compact open immersion of qcqs derived schemes. Then there is a canonical exact triangle*

$$\mathrm{K}(\mathrm{X})_{\mathrm{U}} \to \mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{U})$$

of spectra.

Proof. This follows from the exactness of the sequence

$$\mathrm{Perf}(\mathrm{X})_{\mathrm{U}} \hookrightarrow \mathrm{Perf}(\mathrm{X}) \to \mathrm{Perf}(\mathrm{U}),$$

which we saw in Lecture 3, in view of the localizing property of the functor $\mathrm{K} : \mathrm{Stab}^{\mathrm{idem}} \to \mathrm{Spt}$. □

Remark 2.3. When the schemes are classical and regular (nonsingular), we can identify the fibre term $\mathrm{K}(\mathrm{X})_{\mathrm{U}}$ more explicitly; we will come back to this later this lecture.

3. Zariski descent in K-theory. As in the previous section, we can replace K by any localizing functor E in this section as well.

3.1. We have:

Theorem 3.2 (Thomason)**.** *Let X be a qcqs derived scheme and let $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$ be a Zariski open cover. Then the induced square of spectra*

$$\begin{array}{ccc} \mathrm{K}(\mathrm{X}) & \longrightarrow & \mathrm{K}(\mathrm{U}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathrm{V}) & \longrightarrow & \mathrm{K}(\mathrm{U} \cap \mathrm{V}) \end{array}$$

is cartesian.

Remark 3.3. We will not make this explicit here, but it follows from a theorem of Voevodsky [9] that on the full subcategory $\mathrm{DSch}_{\mathrm{qcqs}}$ of quasi-compact quasi-separated derived schemes, the above condition is equivalent to Čech descent with respect to the Zariski topology for the presheaf of spectra $\mathrm{K} : (\mathrm{DSch}_{\mathrm{qcqs}})^{\mathrm{op}} \to \mathrm{Spt}$.

Proof. The claim is that the canonical map

$$\delta : \mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{U}) \underset{\mathrm{K}(\mathrm{U} \cap \mathrm{V})}{\times} \mathrm{K}(\mathrm{V})$$

is invertible. It suffices to show that the map induced on the fibres,

$$\varepsilon : \mathrm{Fib}(\mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{U})) \to \mathrm{Fib}(\mathrm{K}(\mathrm{V}) \to \mathrm{K}(\mathrm{U} \cap \mathrm{V})),$$

is invertible. Indeed, write $F := K(U) \times_{K(U \cap V)} K(V)$ and consider the diagram of cartesian squares

$$\begin{array}{ccccc}
\mathrm{Fib}(K(X) \to K(U)) & \longrightarrow & K(X) & & \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \delta} & & \\
\mathrm{Fib}(K(V) \to K(U \cap V)) & \longrightarrow & F & \longrightarrow & K(V) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K(U) & \longrightarrow & K(U \cap V).
\end{array}$$

By stability of the ∞-category of spectra, each of these squares, in particular the upper one, is also cocartesian.

By the localization sequence the map ε is identified with the canonical map

$$K(X)_U \to K(V)_{U \cap V}$$

which is induced by the canonical functor

$$\mathrm{Perf}(X)_U \to \mathrm{Perf}(V)_{U \cap V},$$

which is an equivalence by Zariski excision for the presheaf of ∞-categories $X \mapsto \mathrm{Perf}(X)$ (Lecture 3). □

4. Coherent sheaves and G-theory. When the schemes are classical and *regular* (nonsingular), one can identify the fibre term in the localization sequence much more explicitly.

4.1. First we define coherent sheaves in the derived setting. The definition is simpler when we impose a finiteness condition on the schemes.

Definition 4.2. *Let* R *be a simplicial commutative ring. We say that* R *is* coherent *if the following hold:*

(i) $\pi_0(R)$ *is coherent in the ordinary sense, i.e. every finitely generated ideal is finitely presented.*

(ii) *For each i, the $\pi_0(R)$-module $\pi_i(R)$ is of finite presentation.*

We say that R is *noetherian* if it is coherent and $\pi_0(R)$ is noetherian in the ordinary sense (i.e. every ideal is finitely generated).

The following is a generalization of the notion of "pseudocoherence" from SGA 6.

Definition 4.3. *Let* R *be a coherent simplicial commutative ring. An* R*-module* M *is* almost perfect *if the following hold:*

(i) M *is eventually connective, i.e. there exists some integer i such that $\pi_n(M) = 0$ for all $n < i$.*

(ii) *For each i, the $\pi_0(R)$-module $\pi_i(M)$ is of finite presentation.*

Exercise 4.4. *The property of almost perfectness is stable under finite (co)limits and direct summands.*

Corollary 4.5. *Let $R \in \mathrm{SCRing}$ be coherent. Then any perfect* R*-module is almost perfect.*

Proof. Since R itself is almost perfect as an R-module, this follows from Exercise 4.4. □

Remark 4.6. One can show that an R-module M is perfect iff it is almost perfect and of finite tor-amplitude.

Remark 4.7. One can define almost perfectness without the coherence assumption on R; see [4, § 7.2.4].

Definition 4.8. *Let* R *be a coherent simplicial commutative ring. An* R*-module* M *is* coherent *if it is almost perfect and eventually coconnective, i.e. there exists some integer* i *such that* $\pi_n(\mathrm{M}) = 0$ *for all* $n > 0$*.*

We let $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{coh}}$ denote the full subcategory of $\mathrm{Mod}_{\mathrm{R}}$ spanned by coherent R-modules. This is a stable idempotent-complete subcategory.

Remark 4.9. Let R be an ordinary commutative ring. Then we can think of M as a cochain complex of (ordinary) R-modules, and coherence amounts to the condition that it is bounded (above and below), and its cohomologies $\mathrm{H}^i(\mathrm{M})$ are finitely presented $\mathrm{H}^0(\mathrm{R})$-modules. Thus $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{coh}}$ is equivalent to the bounded derived category of coherent sheaves on Spec(R) in the usual sense.

Remark 4.10. Unlike in the classical setting, there is no inclusion $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}} \subset \mathrm{Mod}_{\mathrm{R}}^{\mathrm{coh}}$ in general. Indeed, R itself may not be eventually coconnective.

If we suppose that R is eventually coconnective, then any perfect R-module M is eventually coconnective, since the latter property is stable under finite (co)limits and direct summands. In this case we do have an inclusion $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}} \subset \mathrm{Mod}_{\mathrm{R}}^{\mathrm{coh}}$.

4.11. We now globalize the above definitions.

Definition 4.12. *Let* X *be a derived scheme. We say that* X *is* locally coherent *if for any affine derived scheme* $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ *and any open immersion* $j : \mathrm{S} \hookrightarrow \mathrm{X}$*, the simplicial commutative ring* R *is coherent. We say that* X *is* coherent *if it is locally coherent and quasi-compact.*

Given a locally coherent derived scheme X, we say that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$ is *coherent* if for any affine derived scheme $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ and any open immersion $j : \mathrm{S} \hookrightarrow \mathrm{X}$, the inverse image $j^*\mathcal{F}$ is coherent. We let $\mathrm{Coh}(\mathrm{X}) \subset \mathrm{Qcoh}(\mathrm{X})$ denote the full subcategory spanned by coherent sheaves. By the discussion above, this is an idempotent-complete stable small ∞-category.

4.13. Let X be a locally coherent derived scheme.

Definition 4.14. *The* G-theory *of* X *is defined as the spectrum*

$$\mathrm{G}(\mathrm{X}) = \mathrm{K}(\mathrm{Coh}(\mathrm{X})).$$

For a classical noetherian scheme X, a theorem of Schlichting [6] implies:

Theorem 4.15. *Let* X *be a noetherian classical scheme. Then the spectrum* G(X) *is connective.*

If we suppose further that X is *regular* (nonsingular), then one can show that the inclusion $\mathrm{Perf}(\mathrm{X}) \subset \mathrm{Coh}(\mathrm{X})$ is an equivalence [3, Exp. I]. Therefore, we have:

Proposition 4.16. *Let* X *be a regular noetherian classical scheme. Then the canonical map of spectra*

$$\mathrm{K}(\mathrm{X}) \to \mathrm{G}(\mathrm{X})$$

is an equivalence.

Corollary 4.17. *Let* X *be a regular noetherian classical scheme. Then* K(X) *is connective, i.e. the canonical map of spectra*

$$\mathrm{K}^{\mathrm{cn}}(\mathrm{X}) \to \mathrm{K}(\mathrm{X})$$

is an equivalence.

4.18. Quillen's dévissage shows:

Theorem 4.19 (Quillen)**.** *Let* X *be a noetherian classical scheme,* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *an open immersion, and* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *a complementary closed immersion. Then we have an exact triangle of spectra*

$$\mathrm{G}(\mathrm{Z}) \to \mathrm{G}(\mathrm{X}) \to \mathrm{G}(\mathrm{U}).$$

In particular, if X is regular, then we get $\mathrm{K}(\mathrm{X})_{\mathrm{U}} = \mathrm{G}(\mathrm{Z})$ in this situation.

4.20. For derived schemes, the relation between regularity and an isomorphism $\mathrm{K}(\mathrm{X}) \approx \mathrm{G}(\mathrm{X})$ is more subtle. As pointed out above, there is not even an inclusion $\mathrm{Perf}(\mathrm{X}) \subset \mathrm{Coh}(\mathrm{X})$ in general, and hence no canonical map $\mathrm{K}(\mathrm{X}) \to \mathrm{G}(\mathrm{X})$.

Let $\mathrm{R} \in \mathrm{SCRing}$ be coherent. Say that R is *almost regular* if any coherent R-module M is of finite tor-amplitude, hence perfect. Then by definition, if R is eventually coconnective and almost regular, we have an equivalence $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}} = \mathrm{Mod}_{\mathrm{R}}^{\mathrm{coh}}$, and in particular an isomorphism of spectra

$$\mathrm{K}(\mathrm{R}) \xrightarrow{\sim} \mathrm{G}(\mathrm{R}).$$

If $\pi_0(\mathrm{R})$ is regular, it is easy to see that R is almost regular iff $\pi_0(\mathrm{R})$ is of finite tor-amplitude as an R-module. See [1] for further discussion.

References.

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