Lecture 3

Compact generation of quasi-coherent sheaves

Let $\mathcal{X}$ be a derived stack. In Lecture 1 we discussed various finiteness properties for quasi-coherent sheaves: perfectness, dualizability, and compactness; we saw that all these notions agree when $\mathcal{X}$ is affine, and that the first two agree in general. The goal of this lecture is to prove that perfectness and compactness also agree for a very general class of derived *schemes*.

1. Semi-orthogonal decompositions.

Definition 1.1. *Let* $\mathbf{C}$ *be a stable presentable* ∞*-category. Let* $\mathbf{C}_+$ *and* $\mathbf{C}_-$ *be stable full subcategories. We say that* $\langle \mathbf{C}_+, \mathbf{C}_- \rangle$ *form a* semi-orthogonal decomposition *of* $\mathbf{C}$ *if the following hold:*

(i) *For any objects* $c_+ \in \mathbf{C}_+$ *and* $c_- \in \mathbf{C}_-$*, the mapping space* $\mathrm{Maps}(c_+, c_-)$ *is contractible.*

(ii) *There exists a right adjoint (resp. left adjoint) to the inclusion* $\mathbf{C}_+ \hookrightarrow \mathbf{C}$ *(resp. to the inclusion* $\mathbf{C}_- \hookrightarrow \mathbf{C}$*).*

1.2. It is relatively easy to construct semi-orthogonal decompositions using the following procedure.

Given a stable subcategory $\mathbf{D} \subset \mathbf{C}$, we define the *right orthogonal* of $\mathbf{D}$ to be the full subcategory of objects $c \in \mathbf{C}$ such that the mapping space $\mathrm{Maps}(d, c)$ is contractible for all $d \in \mathbf{D}$. We define the *left orthogonal* in a dual way.

We have (see [2, Prop. 7.2.1.4]):

Proposition 1.3. *Let* $\mathbf{C}$ *be a stable presentable* ∞*-category and* $\mathbf{D} \subset \mathbf{C}$ *a stable full subcategory. Then* $\mathbf{C}$ *admits a semi-orthogonal decomposition* $\langle \mathbf{C}_+, \mathbf{C}_- \rangle$ *with* $\mathbf{C}_+ = \mathbf{D}$ *iff the inclusion* $\mathbf{D} \hookrightarrow \mathbf{C}$ *admits a right adjoint. In this case,* $\mathbf{C}_-$ *is the right orthogonal of* $\mathbf{D}$*.*

Dually, it admits a semi-orthogonal decomposition $\langle \mathbf{C}_+, \mathbf{C}_- \rangle$ *with* $\mathbf{C}_- = \mathbf{D}$ *iff the inclusion* $\mathbf{D} \hookrightarrow \mathbf{C}$ *admits a left adjoint. In this case,* $\mathbf{C}_+$ *is the left orthogonal of* $\mathbf{D}$*.*

1.4. Note that any semi-orthogonal decomposition $\langle \mathbf{C}_+, \mathbf{C}_- \rangle$ gives rise to an exact sequence of stable presentable ∞-categories

$$\mathbf{C}_+ \hookrightarrow \mathbf{C} \to \mathbf{C}_-$$

where the second arrow is a left localization (i.e. its right adjoint is fully faithful).

2. The Thomason–Neeman localization theorem.

2.1. Let $u : \mathbf{C} \to \mathbf{D}$ fully faithful colimit-preserving functor of stable presentable ∞-categories. Let $\mathbf{D} \to \mathbf{D}/\mathbf{C}$ denote the cofibre of u (in the ∞-category of stable presentable ∞-categories and colimit-preserving functors). Equivalently, $\mathbf{D} \to \mathbf{D}/\mathbf{C}$ is the left localization of $\mathbf{D}$ with respect to the class of morphisms whose cofibre belongs to (the essential image of) $\mathbf{C}$.

Definition 2.2. *Let*

$$\mathbf{C}' \xrightarrow{u} \mathbf{C} \xrightarrow{v} \mathbf{C}'' \tag{2.1}$$

be a diagram of stable presentable ∞*-categories and colimit-preserving functors. We say that it is an* exact sequence *if it satisfies the following conditions:*

(i) *The composite* vu *is zero.*

(ii) *The functor* u *is fully faithful.*

(iii) *The canonical functor* $\mathbf{C}/\mathbf{C}' \to \mathbf{C}''$ *is an equivalence.*

Definition 2.3. *We say that* $\mathbf{C}$ *is* compactly generated *if there exists an essentially small set of objects which are compact and generate* $\mathbf{C}$ *under colimits.*

In the stable setting, this is equivalent to the following property. The *right orthogonal* of a set of objects $(c_i)_i$ in $\mathbf{C}$ is the full subcategory of objects $d \in \mathbf{C}$ such that each mapping space $\mathrm{Maps}(c_i[-n], d)$ is contractible for each i and all $n \geqslant 0$. Then a set of compact objects $(c_i)_i$ forms a set of compact generators iff their right orthogonal vanishes.

In the compactly generated case, we can characterize exact sequences in terms of the full subcategories of compact objects:

Proposition 2.4. *Suppose we have a diagram (2.1), and assume that the categories* $\mathbf{C}$*,* $\mathbf{C}'$ *and* $\mathbf{C}''$ *are compactly generated. Suppose also that* u *and* v *preserve compact objects (equivalently, their right adjoints preserve colimits) and consider the induced diagram*

$$(\mathbf{C}')^{\mathrm{comp}} \xrightarrow{u^{\mathrm{comp}}} (\mathbf{C})^{\mathrm{comp}} \xrightarrow{v^{\mathrm{comp}}} (\mathbf{C}'')^{\mathrm{comp}},$$

on the full subcategories of compact objects, of small stable ∞*-categories and finite-colimit-preserving functors. Then (2.1) is exact iff the following conditions are satisfied:*

(i) *The composite* $v^{\mathrm{comp}} \circ u^{\mathrm{comp}}$ *is zero.*

(ii) *The functor* u^{comp} *is fully faithful.*

(iii) *The canonical functor* $(\mathbf{C})^{\mathrm{comp}}/(\mathbf{C}')^{\mathrm{comp}} \to (\mathbf{C}'')^{\mathrm{comp}}$ *is an equivalence* up to idempotent completion*, i.e. the functor*

$$((\mathbf{C})^{\mathrm{comp}}/(\mathbf{C}')^{\mathrm{comp}})^{\mathrm{idem}} \to ((\mathbf{C}'')^{\mathrm{comp}})^{\mathrm{idem}}$$

is an equivalence.

Recall that, if $\mathbf{C}$ is a small ∞-category, then its idempotent completion $\mathbf{C} \to (\mathbf{C})^{\mathrm{idem}}$ is the full subcategory of presheaves on $\mathbf{C}$ generated by the representables under direct summands. In the stable setting it can also be computed as $\mathrm{Ind}(\mathbf{C})^{\mathrm{comp}}$, i.e. the full subcategory of compact objects in the formal completion $\mathrm{Ind}(\mathbf{C})$ by filtered colimits.

Remark 2.5. The main content of Proposition 2.4 is that if the sequence (2.1) is exact, then any compact object $c'' \in \mathbf{C}''$ can be lifted to a compact object $c \in \mathbf{C}$, such that $v(c) \approx c''$ *at least up to direct summands* (i.e. $v(c)$ will have c'' as a direct summand). In fact, Neeman showed [3] (following Thomason) that c can be taken such that $v(c) \approx c'' \oplus c''[1]$.

2.6. Let $\mathbf{C}$ be a small stable ∞-category. We write $\mathrm{K}_0(\mathbf{C})$ for the free abelian group on isomorphism classes of objects of $\mathbf{C}$, modulo the subgroup generated by $[c] - [c'] - [c'']$ for all exact triangles $c' \to c \to c''$ in $\mathbf{C}$.

Proposition 2.7. *Let* $u : \mathbf{C} \to \mathbf{D}$ *be an exact fully faithful functor between stable* ∞*-categories. Suppose that every object* $d \in \mathbf{D}$ *is a direct summand of an object in the essential image of* u*. Then we have:*

(i) *The induced homomorphism of abelian groups*

$$\mathrm{K}_0(\mathbf{C}) \to \mathrm{K}_0(\mathbf{D}) \tag{2.2}$$

is injective.

(ii) *An object* $d \in \mathbf{D}$ *belongs to the essential image of* u *iff its class* $[d] \in \mathrm{K}_0(\mathbf{D})$ *belongs to the image of the homomorphism (2.2).*

In particular we deduce:

Corollary 2.8. *Suppose that the condition of Proposition 2.7 holds. Then for any object* $d \in \mathbf{D}$*, the object* $d \oplus d[1]$ *belongs to the essential image of* u*.*

To prove Proposition 2.7, it will be convenient to introduce a variant of $\mathrm{K}_0(\mathbf{C})$ for which Proposition 2.7 is trivially true. For this, we modify the definition of $\mathrm{K}_0(\mathbf{C})$ to only consider *split* exact triangles. Thus, let $\mathrm{K}_0^{\oplus}(\mathbf{C})$ denote the free abelian group on isomorphism classes of objects of $\mathbf{C}$, modulo the subgroup generated by elements $[c]-[c']-[c'']$ for all objects satisfying $c = c' \oplus c''$ in $\mathbf{C}$. The following property is easy to check:

Lemma 2.9. *Let c_1 and c_2 be objects of $\mathbf{C}$. Then we have $[c_1] = [c_2]$ in $\mathrm{K}_0^{\oplus}(\mathbf{C})$ iff there exists an object $c_3 \in \mathbf{C}$ such that $c_1 \oplus c_3 = c_2 \oplus c_3$.*

Using Lemma 2.9 one verifies easily that the analogue of Proposition 2.7 holds for $\mathrm{K}_0^{\oplus}$. To prove Proposition 2.7, we note that there is a canonical surjection $\mathrm{K}_0^{\oplus}(\mathbf{C}) \to \mathrm{K}_0(\mathbf{C})$ for any $\mathbf{C}$. We therefore have a diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{I}(\mathbf{C}) & \longrightarrow & \mathrm{K}_0^{\oplus}(\mathbf{C}) & \longrightarrow & \mathrm{K}_0(\mathbf{C}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{I}(\mathbf{D}) & \longrightarrow & \mathrm{K}_0^{\oplus}(\mathbf{D}) & \longrightarrow & \mathrm{K}_0(\mathbf{D}) & \longrightarrow & 0 \end{array} \tag{2.3}$$

Proposition 2.7 now immediately follows from the following lemma and some diagram chasing.

Lemma 2.10. *Under the assumptions of Proposition 2.7, the left-hand map*

$$\mathrm{I}(\mathbf{C}) \to \mathrm{I}(\mathbf{D})$$

is surjective.

Proof. By construction, $\mathrm{I}(\mathbf{D})$ is generated by elements of the form $[d]-[d']-[d'']$ where $d' \to d \to d''$ is an exact triangle in $\mathbf{D}$. It suffices to construct, for any such triangle, another triangle $c' \to c \to c''$ which is in the essential image of u and is such that $[c]-[c']-[c''] = [d]-[d']-[d'']$. By assumption, there exist objects $e', e'' \in \mathbf{D}$ such that $d' \oplus e'$ and $d'' \oplus e''$ belong to the essential image of u. Then the desired triangle is

$$d' \oplus e' \to d \oplus e' \oplus e'' \to d'' \oplus e'',$$

where the middle term also belongs to the essential image of u because u is exact. □

Putting everything together, we have:

Theorem 2.11 (Thomason–Neeman localization theorem)**.** *Let*

$$\mathbf{C}' \xrightarrow{u} \mathbf{C} \xrightarrow{v} \mathbf{C}''$$

be an exact sequence of stable presentable ∞-categories. Suppose that $\mathbf{C}$, $\mathbf{C}'$ and $\mathbf{C}''$ are compactly generated, and that u and v preserve colimits and have colimit-preserving right adjoints. Then for any compact object $x \in \mathbf{C}''$, the object $x \oplus x[1]$ belongs to the essential image of v.

3. Perfect complexes on qcqs schemes. We begin working towards the compact generation theorem by showing that, under quasi-compact quasi-separated hypotheses, all perfect complexes are compact.

3.1. Let us take a minute to introduce some basic finiteness properties of derived schemes:

Definition 3.2.

(i) *A derived scheme* X *is* quasi-compact *if for any Zariski cover $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{X})_{\alpha \in \Lambda}$, there exists a finite subset $\Lambda_0 \subset \Lambda$ such that the family $(j_\alpha)_{\alpha \in \Lambda_0}$ is still a Zariski cover.*

(ii) *A morphism of derived schemes $f : \mathrm{Y} \to \mathrm{X}$ is* quasi-compact *if for any affine derived scheme* S *and any morphism $\mathrm{S} \to \mathrm{X}$, the spectral scheme $\mathrm{S} \times_{\mathrm{X}} \mathrm{Y}$ is quasi-compact.*

(iii) *A morphism of derived schemes* $f : \mathrm{Y} \to \mathrm{X}$ *is* quasi-separated *if the diagonal* $\mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ *is quasi-compact.*

(iv) *A derived scheme* X *is* quasi-separated *if it the morphism* $\mathrm{X} \to \mathrm{Spec}(\mathbf{Z})$ *is quasi-separated.*

(v) *A morphism of derived schemes* $f : \mathrm{Y} \to \mathrm{X}$ *is* separated *if the diagonal* $\mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ *is a closed immersion, i.e. it induces a closed immersion on underlying classical schemes. Equivalently,* $f_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ *is separated.*

(vi) *A derived scheme* X *is* separated *if for any open immersions* $\mathrm{U} \hookrightarrow \mathrm{X}$ *and* $\mathrm{V} \hookrightarrow \mathrm{X}$*, with* U *and* V *affine, the intersection* $\mathrm{U} \times_{\mathrm{X}} \mathrm{V}$ *is quasi-compact.*

Exercise 3.3. *Let* X *be a derived scheme. Then* X *is quasi-separated iff for any open immersions* $\mathrm{U} \hookrightarrow \mathrm{X}$ *and* $\mathrm{V} \hookrightarrow \mathrm{X}$*, with* U *and* V *affine, the intersection* $\mathrm{U} \times_{\mathrm{X}} \mathrm{V}$ *is quasi-compact.*

3.4. We have:

Proposition 3.5. *Let* X *be a quasi-compact derived scheme. If a quasi-coherent sheaf* $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$ *is compact, then it is a perfect complex.*

Proof. By definition, it suffices to show that $f^*\mathcal{F}$ is perfect for each morphism $f : \mathrm{S} \to \mathrm{X}$ where S is affine. We already know that the compact objects of Qcoh(S) are precisely the perfect complexes when S is affine. Therefore it suffices to show that f^* preserves compact objects, or equivalently that its right adjoint f_* preserves colimits. We saw that in Lecture 2 that this is true whenever f is quasi-compact, which holds in this case. □

3.6. Next we would like to prove a converse to Proposition 3.5.

We begin with a formal observation about compactness and limits. Let $(\mathbf{C}_\alpha)_\alpha$ be a *finite* diagram of presentable ∞-categories (and colimit-preserving functors) with limit **C**. Then we have:

Lemma 3.7. *Let* $c \in \mathbf{C}$ *be an object and write* $c_\alpha \in \mathbf{C}_\alpha$ *for its image for each* α*. If* c_α *is compact for each* α*, then* c *is compact.*

Proof. Recall that c is compact iff the functor $\mathrm{Maps}_{\mathbf{C}}(c, -)$ commutes with filtered colimits. Thus the claim follows from the fact that the operations of taking mapping spaces and forming limits of ∞-categories commute, and filtered colimits of spaces commute with finite limits. □

3.8. Recall that for any derived stack $\mathcal{X}$ we know (by definition) that $\mathrm{Qcoh}(\mathcal{X})$ can be written as a limit of ∞-categories Qcoh(S) with S affine. In general, it is not possible however to write it as a *finite* limit (in order to apply Lemma 3.7).

On the other hand, suppose that X is a derived scheme which admits an affine Zariski cover $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$. As discussed in Lecture 2, the Zariski excision property says that we have a cartesian square

$$\begin{array}{ccc} \mathrm{Qcoh}(\mathrm{X}) & \xrightarrow{j_{\mathrm{U}}^*} & \mathrm{Qcoh}(\mathrm{U}) \\ \big\downarrow{\scriptstyle j_{\mathrm{V}}^*} & & \big\downarrow{\scriptstyle (j'_{\mathrm{V}})^*} \\ \mathrm{Qcoh}(\mathrm{V}) & \xrightarrow{(j'_{\mathrm{U}})^*} & \mathrm{Qcoh}(\mathrm{U} \cap \mathrm{V}) \end{array} \tag{3.1}$$

In this case we can apply Lemma 3.7 and conclude that a quasi-coherent sheaf $\mathcal{F}_{\mathrm{X}} \in \mathrm{Qcoh}(\mathrm{X})$ is compact iff its restrictions to U and V are both compact. Since U and V are affine, this is equivalent to the condition that $\mathcal{F}_{\mathrm{X}}|_{\mathrm{U}}$ and $\mathcal{F}_{\mathrm{X}}|_{\mathrm{V}}$ are perfect. For example, this holds if $\mathcal{F}_{\mathrm{X}}$ is perfect, so we see that any perfect complex on X is a compact object.

3.9. More generally, suppose that X is quasi-compact, so that it admits a *finite* affine Zariski cover; if it is further *quasi-separated*, then we know the pairwise intersections are again quasi-compact. We can therefore argue in this case by induction on the size of the affine cover to reduce to the case $X = U \cup V$ as in Paragraph 3.8. We get:

Proposition 3.10. *Let* X *be a qcqs derived scheme. Then any perfect complex* $\mathcal{F} \in \mathrm{Perf}(X)$ *is a compact object of* Qcoh(X)*.*

4. Interlude: the small Zariski site. We make a brief digression to discuss the basic structure theory of open immersions of derived schemes.

In particular we will show that the small Zariski site of an affine derived scheme $S = \mathrm{Spec}(R)$ is equivalent to that of its underlying classical scheme $S_{\mathrm{cl}} = \mathrm{Spec}(\pi_0 R)$. This justifies the idea that elements of the higher homotopy groups $\pi_i(R)$ should be thought of as "higher order nilpotents": like usual nilpotents, they are invisible to the underlying topological space.

4.1. We begin with the most important example of an open immersion.

Let $X = \mathrm{Spec}(R)$ be an affine derived scheme. For any point $f \in R_{\mathrm{Spc}}$ in the underlying space of R, let

$$R \to R[f^{-1}]$$

denote the R-algebra defined by attaching a 1-cell to the polynomial algebra $R[x]$ which identifies $f \cdot x \simeq 1$. That is, we have a cocartesian square

$$\begin{array}{ccc} \mathbf{Z}[t] & \xrightarrow{t \mapsto fx-1} & R[x] \\ {\scriptstyle t\mapsto 0}\downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & R[f^{-1}] \end{array}$$

in SCRing. In particular we have $\pi_0(R[f^{-1}]) = \pi_0(R)[f^{-1}]$.

Lemma 4.2. *The morphism* $\mathrm{Spec}(R[f^{-1}]) \to \mathrm{Spec}(R)$ *is an open immersion.*

Proof. By construction, $\varphi : R \to R[f^{-1}]$ is of finite presentation.

The universal property of the fibred coproduct shows that for any simplicial commutative ring R', the mapping space

$$\mathrm{Maps}_{\mathrm{SCRing}}(R[f^{-1}], R')$$

is identified with a direct summand of $\mathrm{Maps}_{\mathrm{SCRing}}(R, R')$: it is the union of the connected components of homomorphisms $\varphi : R \to R'$ which send f to a unit in $\pi_0(R')$. In particular we see that φ is an epimorphism.

This universal property also shows that $\pi_*(R[f^{-1}]) = \pi_*(R)[f^{-1}]$, which implies that φ is flat. □

4.3. Let $i : Z \hookrightarrow X$ be a closed immersion of derived schemes. We define the *complementary open immersion* to i as follows.

Let U be the prestack defined as follows: for an affine derived scheme $S = \mathrm{Spec}(A)$, we define U(S) to be the full sub-∞-groupoid of X(S) spanned by morphisms $S \to X$ such that the square

$$\begin{array}{ccc} \varnothing & \hookrightarrow & S \\ \downarrow & & \downarrow \\ Z & \xrightarrow{i} & X \end{array}$$

is cartesian, where $\varnothing$ is the empty scheme.

Remark 4.4. Note that U only depends on Z_{cl}. That is, $Z \hookrightarrow X$ and $Z_{cl} \hookrightarrow X$ have the same open complement.

We will prove:

Proposition 4.5. *The prestack* U *is a derived scheme, and the canonical morphism* $j : U \to X$ *is an open immersion.*

4.6. We first make an simple observation.

Lemma 4.7. *Let* $j : U \hookrightarrow X$ *be an open immersion of derived schemes. Then there exists a closed immersion* $i : Z \hookrightarrow X$ *such that* j *is the complementary open immersion to* i.

Indeed let $i_0 : Z \hookrightarrow X_{cl}$ be a closed immersion which is complement to $j_{cl} : U_{cl} \hookrightarrow X_{cl}$. Then $i : Z \xrightarrow{i_0} X_{cl} \hookrightarrow X$ is a closed immersion which is complement to j.

4.8. We now show that any open subscheme of an affine derived scheme $S = \mathrm{Spec}(R)$ is Zariski-locally of the form $\mathrm{Spec}(R[f^{-1}])$ for some element $f \in R_{\mathrm{Spc}}$ in the underlying space.

Proposition 4.9. *Let* $X = \mathrm{Spec}(R)$ *be an affine derived scheme. For any open immersion* $j : U \hookrightarrow X$*, there exists an affine Zariski cover of* U *of the form* $(\mathrm{Spec}(R[f_\alpha^{-1}]) \hookrightarrow U)_\alpha$*, for some elements* $f_\alpha \in R_{\mathrm{Spc}}$.

Proof. Let $i : Z \hookrightarrow X$ be a complementary closed immersion and take f_α to be (lifts of) generators of the ideal cutting out Z_{cl} in X_{cl}. Then we have open immersions $U_\alpha = \mathrm{Spec}(R[f_\alpha^{-1}]) \hookrightarrow X$. It is clear that each $U_\alpha \to X$ factors through U and will show that the map $\coprod_\alpha \mathrm{Spec}(R[f_\alpha^{-1}]) \to U$ is an effective epimorphism. It suffices to show that for any $S = \mathrm{Spec}(A) \to U$ there exists a Zariski-covering family $(\mathrm{Spec}(A_\beta) \hookrightarrow A)_\beta$ such that each $\mathrm{Spec}(A_\beta) \to U$ lifts to a morphism $\mathrm{Spec}(A_\beta) \to U_\alpha$ for some α (which depends on β).

Let $\varphi : R \to A$ be the homomorphism corresponding to $S \to U \hookrightarrow X$. Since it factors through U, the image of the ideal $I \subset \pi_0(R)$ generates $\pi_0(A)$. We can therefore write $1 = \sum_\alpha a_\alpha \cdot \varphi(f_\alpha)$ for some elements $a_\alpha \in \pi_0(R)$ (only finitely many of which are nonzero). Now consider the family $(A \to A[\varphi(f_\beta)^{-1}])_j$, indexed by the j's such that $a_\beta \neq 0$. Then we have lifts $A_\beta \to R[f_\beta^{-1}]$ for each β, and it suffices to show that $A \to \prod_\beta A_\beta$ is faithfully flat, so that $(A \to A[\varphi(f_\beta)^{-1}])_j$ is indeed Zariski-covering.

Since it is flat, it suffices to show that the induced map $\pi_0(A) \to \prod_j \pi_0(A_j)$ is faithfully flat (in the usual sense); see Lemma 4.10 below. Let M be a discrete module over $\pi_0(A)$ such that $M \otimes_{\pi_0(A)} \pi_0(A_\beta) = 0$ for all β (ordinary tensor product). It suffices to show that $M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m} \subset \pi_0(A)$. For a given $\mathfrak{m}$ we can choose an index γ such that $\varphi(f_\gamma) \notin \mathfrak{m}$, so that the map $A \to A_{\mathfrak{m}}$ factors through A_γ; then we have

$$M_{\mathfrak{m}} = M \otimes_{\pi_0(A)} \pi_0(A)_{\mathfrak{m}} = M \otimes_{\pi_0(A)} \pi_0(A_\gamma) \otimes_{\pi_0(A_\gamma)} \pi_0(A)_{\mathfrak{m}} = 0,$$

whence the desired conclusion. □

Here we used the following lemma:

Lemma 4.10. *A morphism of simplicial commutative rings* $A \to B$ *is faithfully flat iff it is flat and* $\pi_0(A) \to \pi_0(B)$ *is faithfully flat (in the ordinary sense).*

Proof. We prove the condition is sufficient. Let M be a connective A-module such that $M \otimes_A B = 0$. We will show that $\pi_n(M) = 0$ for all n. Since $\pi_0(A) \to \pi_0(B)$ is faithfully flat it suffices to show that $\pi_n(M) \otimes_{\pi_0(A)} \pi_0(B) = 0$ for all n, where the tensor product is the usual tensor product (as opposed to the derived one). But by flatness this is identified with $\pi_n(M \otimes_A B)$, so the claim follows. □

We can now return to the proof of Proposition 4.5:

Proof of Proposition 4.5. It is clear that U is an fpqc sheaf. It suffices to construct an affine Zariski cover. Since the claim is local we can assume that X is affine, and conclude using Proposition 4.9. □

4.11. For a derived scheme X, let $\mathrm{Open}_{/\mathrm{X}}$ denote the ∞-category of derived schemes U equipped with open immersions $j : \mathrm{U} \hookrightarrow \mathrm{X}$.

Theorem 4.12. *Let* $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ *be an affine derived scheme. Then the base change functor*

$$\mathrm{Open}_{/\mathrm{X}} \to \mathrm{Open}_{/\mathrm{X}_{\mathrm{cl}}}$$

is an equivalence. In particular, $\mathrm{Open}_{/\mathrm{X}}$ *is a* 1*-category (a poset, in fact).*

Proof. Let us show that the functor is essentially surjective. Given an open immersion $j^0 : \mathrm{U}^0 \hookrightarrow \mathrm{X}_{\mathrm{cl}}$, we can find a Zariski cover by open subschemes of the form $\mathrm{U}_{0,\alpha} = \mathrm{Spec}(\pi_0(\mathrm{R})[f_\alpha^{-1}]) \hookrightarrow \mathrm{U}^0$ with $f_\alpha \in \pi_0(\mathrm{R})$. Choose lifts of f_α to R arbitrarily and let $\mathrm{U}_\alpha = \mathrm{Spec}(\mathrm{R}[f_\alpha^{-1}]) \hookrightarrow \mathrm{X}$. Then let $j : \mathrm{U} \hookrightarrow \mathrm{X}$ be the image of the map

$$\coprod_\alpha \mathrm{U}_\alpha \to \mathrm{X}.$$

It is immediate from the construction that this is an open immersion and that $\mathrm{U} \times_{\mathrm{X}} \mathrm{X}_{\mathrm{cl}} = \mathrm{U}_{\mathrm{cl}} = \mathrm{U}^0$.

It remains to show that it is fully faithful. Given open immersions $j_1 : \mathrm{U}_1 \hookrightarrow \mathrm{X}$ and $j_2 : \mathrm{U}_2 \hookrightarrow \mathrm{X}$, consider the map

$$\mathrm{Maps}_{\mathrm{Open}_{/\mathrm{X}}}(\mathrm{U}_1, \mathrm{U}_2) \to \mathrm{Maps}_{\mathrm{Open}_{/\mathrm{X}_{\mathrm{cl}}}}((\mathrm{U}_1)_{\mathrm{cl}}, (\mathrm{U}_2)_{\mathrm{cl}})$$

Suppose that $\mathrm{U}_1 = \mathrm{Spec}(\mathrm{R}[f_1^{-1})$ and $\mathrm{U}_2 = \mathrm{Spec}(\mathrm{R}[f_2^{-1}])$. In this case we are looking at

$$\mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{R}[f_1^{-1}], \mathrm{R}[f_2^{-1}]) \to \mathrm{Maps}_{\mathrm{CRing}_{\pi_0(\mathrm{R})}}(\pi_0\mathrm{R}[f_1^{-1}], \pi_0\mathrm{R}[f_2^{-1}]).$$

By the universal property of the localization $\mathrm{R}[f_1^{-1}]$, the source is either empty or contractible depending on whether the image of f_1 is invertible in $\mathrm{R}[f_2^{-1}]$; the same holds for the target using the universal property of the classical localization $\pi_0\mathrm{R}[f_1^{-1}]$.

In general, we reduce to this case using Proposition 4.9. □

5. Compact generation of affine schemes. We begin with the affine case. If $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, we already know that $\mathrm{Qcoh}(\mathrm{X}) = \mathrm{Mod}_{\mathrm{R}}$ is compactly generated by the perfect R-module R.

5.1. Given an open immersion $j : \mathrm{U} \hookrightarrow \mathrm{X}$, we will write $\mathrm{Qcoh}(\mathrm{X})_{\mathrm{U}}$ for the kernel of the restriction functor $j^* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\mathrm{U})$. We will show that $\mathrm{Qcoh}(\mathrm{X})_{\mathrm{U}}$ is compactly generated when j is quasi-compact.

Proposition 5.2. *Let* $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ *be an affine derived scheme and* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *be a quasi-compact open immersion. Then the following hold:*

(i) *The* ∞*-category* $\mathrm{Qcoh}_{\mathrm{U}}(\mathrm{X})$ *is compactly generated by a single perfect complex.*

(ii) *There is a semi-orthogonal decomposition*

$$\mathrm{Qcoh}(\mathrm{X}) = \langle \mathrm{Qcoh}(\mathrm{X})_{\mathrm{U}}, j_* \mathrm{Qcoh}(\mathrm{U}) \rangle.$$

Proof. By Proposition 4.9 there exists an affine Zariski cover $\mathrm{U} = \bigcup_i \mathrm{U}_i$ where $\mathrm{U}_i = \mathrm{Spec}(\mathrm{R}[f_i^{-1}])$ and $f_1, \ldots, f_n$ are points in the underlying space of R; since U is quasi-compact, this cover is finite. Consider the perfect complexes

$$\mathcal{K}_i = \mathrm{Cofib}(\mathcal{O}_\mathrm{X} \xrightarrow{f_i} \mathcal{O}_\mathrm{X}), \quad \mathcal{K} = \bigotimes_{1 \leqslant i \leqslant n} \mathcal{K}_i.$$

Note that we have $j^*\mathcal{K} = 0$. To show that $\mathcal{K}$ is a compact generator, it suffices to show that for any $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})_\mathrm{U}$ in the right orthogonal of $\mathcal{K}$, i.e. with $\mathrm{Maps}_{\mathrm{Qcoh}(\mathrm{X})_\mathrm{U}}(\mathcal{K}, \mathcal{F}) = \mathrm{pt}$, we have $\mathcal{F} = 0$. Write $\mathcal{K}_{\neq j} = \otimes_{i \neq j} \mathcal{K}_i$ for each j; by adjunction, we have

$$\mathrm{pt} = \mathrm{Maps}(\mathcal{K}, \mathcal{F}) = \mathrm{Maps}(\mathcal{K}_1, \underline{\mathrm{Hom}}(\mathcal{K}_{\neq 1}, \mathcal{F}))$$

which means that f_1 acts invertibly on $\underline{\mathrm{Hom}}(\mathcal{K}_{\neq 1}, \mathcal{F})$, i.e. that $\underline{\mathrm{Hom}}(\mathcal{K}_{\neq 1}, \mathcal{F})$ is an $\mathcal{O}_\mathrm{X}[f_1^{-1}]$-module. We therefore have

$$\underline{\mathrm{Hom}}(\mathcal{K}_{\neq 1}, \mathcal{F}) = \underline{\mathrm{Hom}}(\mathcal{K}_{\neq 1}, \mathcal{F}) \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{O}_\mathrm{X}[f_1^{-1}] = \underline{\mathrm{Hom}}(\mathcal{K}_{\neq 1}, \mathcal{F} \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{O}_\mathrm{X}[f_1^{-1}]) = 0,$$

using the fact that $j^*\mathcal{F} = 0$ and $\mathrm{Spec}(\mathrm{R}[f_1^{-1}]) \subset \mathrm{U}$ at the end. Arguing inductively we eventually get

$$\underline{\mathrm{Hom}}(\mathcal{K}_n, \mathcal{F}) = 0,$$

which means that f_n acts invertibly on $\mathcal{F}$, hence $\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{O}_\mathrm{X}[f_n^{-1}] = 0$. Thus $\mathrm{Qcoh}(\mathrm{X})_\mathrm{U}$ is compactly generated by $\mathcal{K}$.

We now consider (iii). By Proposition 1.3 and the fact that j_* is fully faithful and admits a left adjoint j^*, there exists a semi-orthogonal decomposition

$$\mathrm{Qcoh}(\mathrm{X}) = \langle {}^\perp(j_* \,\mathrm{Qcoh}(\mathrm{U})), j_* \,\mathrm{Qcoh}(\mathrm{U}) \rangle,$$

where ${}^\perp j_* \,\mathrm{Qcoh}(\mathrm{U})$ is the left orthogonal to $j_* \,\mathrm{Qcoh}(\mathrm{U})$. It suffices to show that ${}^\perp j_* \,\mathrm{Qcoh}(\mathrm{U}) = \mathrm{Qcoh}(\mathrm{X})_\mathrm{U}$. This follows by adjunction: $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$ is left orthogonal to $j_* \,\mathrm{Qcoh}(\mathrm{U})$ iff

$$\mathrm{Maps}(\mathcal{F}, j_*\mathcal{G}) = \mathrm{Maps}(j^*\mathcal{F}, \mathcal{G}) = \mathrm{pt}$$

for all $\mathcal{G} \in \mathrm{Qcoh}(\mathrm{U})$, or equivalently if $j^*\mathcal{F} = 0$. □

6. Compact generation of qcqs schemes. Let X be a derived scheme and $j : \mathrm{U} \hookrightarrow \mathrm{X}$ an open immersion. We will write $\mathrm{Qcoh}(\mathrm{X})_\mathrm{U}$ for the kernel of the restriction functor $j^* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\mathrm{U})$, and similarly $\mathrm{Perf}(\mathrm{X})_\mathrm{U}$ for the kernel of $j^* : \mathrm{Perf}(\mathrm{X}) \to \mathrm{Perf}(\mathrm{U})$.

6.1. We now prove Thomason's compact generation theorem, as generalized to qcqs schemes by Bondal–Van den Bergh, and to the derived setting by Toën.

Theorem 6.2. *Let* X *be a qcqs derived scheme and* $j : \mathrm{U} \hookrightarrow \mathrm{X}$ *a quasi-compact open immersion. Then the following hold:*

(i) *The* ∞*-category* $\mathrm{Qcoh}_\mathrm{U}(\mathrm{X})$ *is compactly generated by a single perfect complex.*

(ii) *An object of* $\mathrm{Qcoh}_\mathrm{U}(\mathrm{X})$ *is compact iff it is a perfect complex.*

(iii) *There is a semi-orthogonal decomposition*

$$\mathrm{Qcoh}(\mathrm{X}) = \langle \mathrm{Qcoh}(\mathrm{X})_\mathrm{U}, j_* \,\mathrm{Qcoh}(\mathrm{U}) \rangle.$$

Of course taking U to be empty, we find that Qcoh(X) is compactly generated for any qcqs derived scheme X. The statement for general U will be needed to get descent for K-theory, but for simplicity of exposition we will only prove the case $\mathrm{U} = \varnothing$ as the general case follows the same idea.

Proof. We have already seen (ii) (Proposition 3.5 and Proposition 3.10).

The proof of (iii) is the same as in the affine case (Proposition 5.2). The key point is that j_* is fully faithful and right adjoint to j^*.

We now consider statement (i) (in the case $\mathrm{U} = \varnothing$). By quasi-compactness, X admits a *finite* affine Zariski cover $\mathrm{U}_1, \ldots, \mathrm{U}_n$. By quasi-separatedness, the pairwise intersections $\mathrm{U}_i \cap \mathrm{U}_j$ are again quasi-compact.

We will show that Qcoh(X) is compactly generated by a single object, by using induction to reduce to the affine case. Let $\mathrm{U} = \mathrm{U}_1 \cup \mathrm{U}_2 \cup \cdots \cup \mathrm{U}_{n-1}$ and $\mathrm{V} = \mathrm{U}_n$. We have a cartesian square

$$\begin{array}{ccc} \mathrm{U} \cap \mathrm{V} & \xhookrightarrow{j'_{\mathrm{U}}} & \mathrm{V} \\ \big\downarrow j'_{\mathrm{V}} & & \big\downarrow j_{\mathrm{V}} \\ \mathrm{U} & \xhookrightarrow{j_{\mathrm{U}}} & \mathrm{X} \end{array}$$

The claim holds for Qcoh(V) since V is affine, and by induction we can assume that it holds also for Qcoh(U); let $\mathcal{Q}_{\mathrm{U}} \in \mathrm{Qcoh}(\mathrm{U})$ be a compact generator. Since V is affine, we have by Proposition 5.2 an exact sequence

$$\mathrm{Qcoh}(\mathrm{V})_{\mathrm{U}\cap\mathrm{V}} \to \mathrm{Qcoh}(\mathrm{V}) \to \mathrm{Qcoh}(\mathrm{U} \cap \mathrm{V}) \tag{6.1}$$

with the Koszul complex $\mathcal{K}_{\mathrm{V}} \in \mathrm{Qcoh}(\mathrm{V})_{\mathrm{U}\cap\mathrm{V}}$ a compact generator. The conditions of Theorem 2.11 are satisfied and we find that the compact object $\mathcal{Q}_{\mathrm{U}}|_{\mathrm{U}\cap\mathrm{V}} \in \mathrm{Qcoh}(\mathrm{U} \cap \mathrm{V})$ lifts to a compact object $\mathcal{Q}_{\mathrm{V}} \in \mathrm{Qcoh}(\mathrm{V})$ such that $\mathcal{Q}_{\mathrm{V}}|_{\mathrm{U}\cap\mathrm{V}} = (\mathcal{Q}_{\mathrm{U}} \oplus \mathcal{Q}_{\mathrm{U}}[1])|_{\mathrm{U}\cap\mathrm{V}}$.

By the Zariski excision property (Lecture 2) we have the cartesian square

$$\begin{array}{ccc} \mathrm{Qcoh}(\mathrm{X}) & \xrightarrow{j^*_{\mathrm{U}}} & \mathrm{Qcoh}(\mathrm{U}) \\ \big\downarrow j^*_{\mathrm{V}} & & \big\downarrow (j'_{\mathrm{V}})^* \\ \mathrm{Qcoh}(\mathrm{V}) & \xrightarrow{(j'_{\mathrm{U}})^*} & \mathrm{Qcoh}(\mathrm{U} \cap \mathrm{V}). \end{array} \tag{6.2}$$

We can therefore define two quasi-coherent sheaves $\mathcal{Q}^1_{\mathrm{X}}$, $\mathcal{Q}^2_{\mathrm{X}}$ on X as follows. The first $\mathcal{Q}^1_{\mathrm{X}} \in \mathrm{Qcoh}(\mathrm{X})$ is glued from $0 \in \mathrm{Qcoh}(\mathrm{U})$ and $\mathcal{K}_{\mathrm{V}} \in \mathrm{Qcoh}(\mathrm{V})$ via the canonical isomorphisms

$$0|_{\mathrm{U}\cap\mathrm{V}} \xrightarrow{\alpha} 0 \xleftarrow{\beta} \mathcal{K}_{\mathrm{V}}|_{\mathrm{U}\cap\mathrm{V}}.$$

The second $\mathcal{Q}^2_{\mathrm{X}} \in \mathrm{Qcoh}(\mathrm{X})$ is glued from $\mathcal{Q}_{\mathrm{U}} \oplus \mathcal{Q}_{\mathrm{U}}[1] \in \mathrm{Qcoh}(\mathrm{U})$ and $\mathcal{Q}_{\mathrm{V}} \in \mathrm{Qcoh}(\mathrm{V})$, via the canonical isomorphisms

$$(\mathcal{Q}_{\mathrm{U}} \oplus \mathcal{Q}_{\mathrm{U}}[1])|_{\mathrm{U}\cap\mathrm{V}} \xrightarrow{\alpha} (\mathcal{Q}_{\mathrm{U}} \oplus \mathcal{Q}_{\mathrm{U}}[1])|_{\mathrm{U}\cap\mathrm{V}} \xleftarrow{\beta} (\mathcal{Q}_{\mathrm{V}})|_{\mathrm{U}\cap\mathrm{V}}.$$

By Lemma 3.7, both $\mathcal{Q}^1_{\mathrm{X}}$ and $\mathcal{Q}^2_{\mathrm{X}}$ are compact in Qcoh(X).

Now we claim that $\mathcal{Q}_{\mathrm{X}} := \mathcal{Q}^1_{\mathrm{X}} \oplus \mathcal{Q}^2_{\mathrm{X}}$ is a compact generator of X. Let $\mathcal{F}_{\mathrm{X}} \in \mathrm{Qcoh}(\mathrm{X})$ be right orthogonal to $\mathcal{Q}_{\mathrm{X}}$ (hence to both $\mathcal{Q}^i_{\mathrm{X}}$'s); it suffices to show that $\mathcal{F}_{\mathrm{X}} = 0$. Using the square (6.2), it suffices to show that $\mathcal{F}_{\mathrm{X}}|_{\mathrm{U}} = 0$ and $\mathcal{F}_{\mathrm{X}}|_{\mathrm{V}} = 0$.

First we show that $\mathcal{F}_{\mathrm{X}}|_{\mathrm{V}}$ is in the essential image of the fully faithful functor $(j'_{\mathrm{U}})_* : \mathrm{Qcoh}(\mathrm{U} \cap \mathrm{V}) \hookrightarrow \mathrm{Qcoh}(\mathrm{V})$, i.e. that $\mathcal{F}_{\mathrm{X}}|_{\mathrm{V}} = (j'_{\mathrm{U}})_*(\mathcal{F}_{\mathrm{X}}|_{\mathrm{U}\cap\mathrm{V}})$. This will imply that it suffices to show that $\mathcal{F}_{\mathrm{X}}|_{\mathrm{U}} = 0$ (as then $\mathcal{F}_{\mathrm{X}}|_{\mathrm{V}} = 0$ as well). Indeed, by the exact sequence (6.1) (which is a semi-orthogonal decomposition) this claim is equivalent to the assertion that $\mathcal{F}_{\mathrm{X}}|_{\mathrm{V}}$ is right orthogonal to $\mathrm{Qcoh}(\mathrm{V})_{\mathrm{U}\cap\mathrm{V}}$, or equivalently to its generator $\mathcal{K}_{\mathrm{V}}$. Since $\mathcal{Q}^1_{\mathrm{X}}|_{\mathrm{U}} = 0$, the cartesian square (6.2) shows that for each $n \geqslant 0$, we have

$$\mathrm{Maps}(\mathcal{K}_{\mathrm{V}}[-n], \mathcal{F}_{\mathrm{X}}|_{\mathrm{V}}) = \mathrm{Maps}(\mathcal{Q}^1_{\mathrm{X}}[-n], \mathcal{F}_{\mathrm{X}})$$

which is contractible since $\mathcal{F}_{\mathrm{X}}$ is right orthogonal to $\mathcal{Q}^1_{\mathrm{X}}$.

It remains to show that $\mathcal{F}_X|_U = 0$. Since $\mathcal{Q}_U$ is a compact generator of Qcoh(U), it will suffice to show that the mapping spaces $\mathrm{Maps}(\mathcal{Q}_U[-n], \mathcal{F}_X|_U)$ are contractible for $n \geqslant 0$. In fact, we have

$$\mathrm{Maps}(\mathcal{Q}_U[-n], \mathcal{F}_X|_U) = \mathrm{Maps}(\mathcal{Q}^2_X[-n], \mathcal{F}_X)$$

which is contractible since $\mathcal{F}_X$ is right orthogonal to $\mathcal{Q}^2_X$. The isomorphism of mapping spaces follows from the cartesian square (6.2) and the isomorphism $\mathcal{F}_X|_V = (j'_U)_*(\mathcal{F}_X|_{U\cap V})$. □

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