Lecture 1

Derived algebraic geometry

1. Simplicial commutative rings. We begin with a brief review of the theory of simplicial commutative rings. Our aim will be to derive the theory of simplicial commutative rings by taking higher algebra as a starting point.

The ∞-category of simplicial commutative rings can be presented by a model structure on the category of simplicial objects in the category of ordinary commutative rings. In this lecture we take a different approach, based on the idea that a simplicial commutative ring is "a space equipped with a commutative ring structure"; here, as always in these lectures, we use the term *space* agnostically to mean "an object of the ∞-category of spaces" (as opposed to any particular presentation of the latter, e.g. by CW or Kan complexes).

There is some subtlety in the phrase "commutative ring structure". Recall that a commutative ring structure on a set is a structure of abelian group and a compatible structure of commutative monoid. For spaces, a natural analogue would be an $\mathcal{E}_\infty$-group space with a compatible $\mathcal{E}_\infty$-monoid structure, or equivalently, a connective $\mathcal{E}_\infty$-ring spectrum; the term $\mathcal{E}_\infty$ refers to commutativity up to a homotopy coherent system of compatibilities. There is a theory of derived algebraic geometry based on connective $\mathcal{E}_\infty$-ring spectra, but it is not what we will be using here.

The theory of simplicial commutative rings is instead obtained by imposing a stricter notion of commutativity; the resulting theory is much closer to classical commutative algebra. Though simplicial commutative rings are more classical and arguably more elementary than $\mathcal{E}_\infty$-ring spectra, we insist on viewing the latter as more fundamental since they form part of the basic language of higher algebra.

1.1. Let us begin by recalling a description of the category CRing of ordinary commutative rings in the style of Lawvere's thesis.

Any commutative ring R represents a presheaf $\tilde{\mathrm{R}}$ on CRing, and by restriction a presheaf on the full subcategory Poly spanned by the polynomial rings $\mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ for $n \geqslant 0$.

Exercise 1.2.

(i) *The presheaf* $\tilde{\mathrm{R}}$ *sends finite coproducts in* Poly *to products of sets.*

(ii) *The assignment* $\mathrm{R} \mapsto \tilde{\mathrm{R}}$ *defines an equivalence between* CRing *and the category of presheaves of sets on* Poly *which send finite coproducts to products.*

More generally, we can replace the category of sets by any category $\mathbf{C}$ which admits finite products, to obtain a description of the category of "commutative ring objects in $\mathbf{C}$" (with respect to the cartesian product).

Replacing the category of sets with the ∞-*category* of spaces, we arrive at the following definition:

Definition 1.3. *A* simplicial commutative ring *is a presheaf*

$$(\mathrm{Poly})^{\mathrm{op}} \to \mathrm{Spc},$$

valued in the ∞*-category of spaces, which sends finite coproducts in* Poly *to products of spaces.*

Let SCRing denote the ∞-category of simplicial commutative rings; this is the "non-abelian derived category" of Poly in the sense of Lurie, and by the general theory it is freely generated by Poly under sifted homotopy colimits.

Proposition 1.4. *Let* $\mathbf{C}$ *be an* ∞*-category which admits sifted colimits. Then the Yoneda embedding* $\mathrm{Poly} \hookrightarrow \mathrm{SCRing}$ *induces an equivalence of* ∞*-categories*

$$\mathrm{Funct}_{\mathrm{sift}}(\mathrm{SCRing}, \mathbf{C}) \to \mathrm{Funct}(\mathrm{Poly}, \mathbf{C}), \tag{1.1}$$

where the decoration $_{\mathrm{sift}}$ *denotes the full subcategory spanned by functors that commute with sifted colimits.*

1.5. A simplicial commutative ring R amounts to the data of an "underlying space"

$$\mathrm{R}_{\mathrm{Spc}} := \mathrm{R}(\mathbf{Z}[\mathrm{T}]) \in \mathrm{Spc}$$

together with further structure which is encoded by the category Poly. For example, for each $n \geqslant 1$ we have a canonical map of spaces

$$\mathrm{mult} : (\mathrm{R}_{\mathrm{Spc}})^{\times n} \xleftarrow{\sim} \mathrm{R}(\mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]) \to \mathrm{R}(\mathbf{Z}[\mathrm{T}]) = \mathrm{R}_{\mathrm{Spc}}$$

induced by the homomorphism $\mathbf{Z}[\mathrm{T}] \to \mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ which sends $\mathrm{T} \mapsto \mathrm{T}_1 \cdots \mathrm{T}_n$. Similarly we have canonical maps

$$\mathrm{add} : (\mathrm{R}_{\mathrm{Spc}})^{\times n} \to \mathrm{R}_{\mathrm{Spc}}$$

determined by $\mathrm{T} \mapsto \mathrm{T}_1 + \cdots + \mathrm{T}_n$. There are two points of the space $\mathrm{R}_{\mathrm{Spc}}$, i.e. canonical maps

$$0 : \mathrm{pt} \to \mathrm{R}_{\mathrm{Spc}}, \qquad 1 : \mathrm{pt} \to \mathrm{R}_{\mathrm{Spc}} \tag{1.2}$$

induced by the two homomorphisms $\mathbf{Z}[\mathrm{T}] \to \mathbf{Z}$ given by $\mathrm{T} \mapsto 0$ and $\mathrm{T} \mapsto 1$, respectively.

Exercise 1.6. *Show that the homotopy groups* $\pi_*(\mathrm{R}_{\mathrm{Spc}})$ *admit a canonical graded commutative ring structure.*

1.7. We say that a simplicial commutative ring R is *discrete* if it is discrete as a presheaf, i.e. it takes values in sets. By Exercise 1.2, the full subcategory of discrete simplicial commutative rings is equivalent to CRing; we will take "discrete simplicial commutative ring" to be our definition of "ordinary commutative ring" from now on.

In particular, there is a 0-truncation functor $\mathrm{SCRing} \to \mathrm{CRing}$, left adjoint to the inclusion of discrete simplicial commutative rings, which we will denote by $\mathrm{R} \mapsto \pi_0(\mathrm{R})$.

1.8. For a simplicial commutative ring R, we write $\mathrm{SCRing}_{\mathrm{R}}$ for the ∞-category of R-algebras, i.e. simplicial commutative rings A equipped with a homomorphism $\mathrm{R} \to \mathrm{A}$.

The ∞-category $\mathrm{SCRing}_{\mathrm{R}}$ admits finite coproducts, so it can be endowed with the cocartesian monoidal structure; we denote the monoidal product by $\otimes_{\mathrm{R}}$, so that $\mathrm{A} \otimes_{\mathrm{R}} \mathrm{B} := \mathrm{A} \sqcup_{\mathrm{R}} \mathrm{B}$.

We write $\mathrm{R}[\mathrm{T}_1, \ldots, \mathrm{T}_n] := \mathrm{R} \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ for the polynomial R-algebras. One can show that $\mathrm{SCRing}_{\mathrm{R}}$ is freely generated under sifted colimits by the full subcategory $\mathrm{Poly}_{\mathrm{R}}$ spanned by $\mathrm{R}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ for $n \geqslant 0$.

1.9. Let $\mathcal{E}_\infty\text{-alg}^{\mathrm{cn}}$ denote the ∞-category of connective $\mathcal{E}_\infty$-ring spectra; this the full subcategory of $\mathcal{E}_\infty$-monoids R in the ∞-category of spectra with the property that $\pi_i(\mathrm{R}) = 0$ for $i < 0$.

There is a canonical fully faithful functor

$$\mathrm{CRing} \to \mathcal{E}_\infty\text{-alg}^{\mathrm{cn}} \tag{1.3}$$

which sends a commutative ring R to its Eilenberg–MacLane spectrum, and identifies ordinary commutative rings with discrete $\mathcal{E}_\infty$-ring spectra.

By Proposition 1.4 this functor extends in an essentially unique manner to a functor of ∞-categories

$$\mathrm{SCRing} \to \mathcal{E}_\infty\text{-alg}^{\mathrm{cn}} \tag{1.4}$$

which commutes with sifted colimits; we will denote it by $\mathrm{R} \mapsto \mathrm{R}_{\mathrm{Spt}}$. It is easy to see that its infinite loop space $\Omega^\infty(\mathrm{R}_{\mathrm{Spt}})$ coincides with the underlying space $\mathrm{R}_{\mathrm{Spc}}$ we already defined. Concretely speaking, $\mathrm{R}_{\mathrm{Spt}}$ is computed by taking a free simplicial resolution of R, applying the Eilenberg–MacLane functor degreewise, and then taking the geometric realization of the resulting simplicial spectrum.

1.10. For any simplicial commutative ring R, there is an induced functor

$$\mathrm{SCRing}_{\mathrm{R}} \to \mathcal{E}_\infty\text{-}\mathrm{alg}^{\mathrm{cn}}_{\mathrm{R}_{\mathrm{Spt}}}. \tag{1.5}$$

In general, this functor is neither fully faithful nor essentially surjective. Indeed the strictness of commutativity and associativity in the operations on polynomial rings makes the theory of simplicial commutative rings "stricter" than that of connective $\mathcal{E}_\infty$-ring spectra. More precisely, every simplicial commutative ring can be strictified in the sense that we have the following description of SCRing (which we will not actually use):

Proposition 1.11. *The ∞-category* SCRing *is canonically equivalent to the ∞-categorical localization of the ordinary category of simplicial commutative rings at the class of weak homotopy equivalences.*

If R is a $\mathbf{Q}$-algebra, then every $\mathcal{E}_\infty$-R-algebra can similarly be strictified, and therefore the functor (1.5) is actually an equivalence in this case.

1.12. For a simplicial commutative ring R, we define:

Definition 1.13. *An* R-module *is a module over the $\mathcal{E}_\infty$-ring spectrum* $\mathrm{R}_{\mathrm{Spt}}$.

We let $\mathrm{Mod}_{\mathrm{R}}$ denote the stable symmetric monoidal ∞-category of R-modules, and $\mathrm{Mod}^{\mathrm{cn}}_{\mathrm{R}}$ the full subcategory spanned by connective R-modules.

Note that if R is discrete, $\mathrm{Mod}_{\mathrm{R}}$ is canonically equivalent to the derived category of the abelian category of ordinary R-modules.

Remark 1.14. A more classical way to define $\mathrm{Mod}_{\mathrm{R}}$, using the presentation of Proposition 1.11, is as follows. Given a simplicial object in CRing, there is an associated normalized *cochain complex* which can be viewed as a dg-algebra with a (strictly) commutative multiplicative structure via the shuffle product; the localization of the ∞-category of dg-modules (with respect to quasi-isomorphisms) is then equivalent to $\mathrm{Mod}_{\mathrm{R}}$ as we defined it above.

Any R-algebra has an underlying connective R-module, given by the forgetful functor

$$\mathrm{SCRing}_{\mathrm{R}} \to \mathcal{E}_\infty\text{-}\mathrm{alg}^{\mathrm{cn}}_{\mathrm{R}_{\mathrm{Spt}}} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathrm{R}_{\mathrm{Spt}}} = \mathrm{Mod}^{\mathrm{cn}}_{\mathrm{R}}. \tag{1.6}$$

Proposition 1.15. *The functor (1.6) is conservative, commutes with limits, and admits a left adjoint.*

Let $\mathrm{M} \mapsto \mathrm{Sym}_{\mathrm{R}}(\mathrm{M})$ denote the left adjoint, so that for any connective R-module M, $\mathrm{Sym}_{\mathrm{R}}(\mathrm{M})$ denotes the free R-algebra generated by M; we have canonical functorial isomorphisms

$$\mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{Sym}_{\mathrm{R}}(\mathrm{M}), \mathrm{A}) = \mathrm{Maps}_{\mathrm{Mod}^{\mathrm{cn}}_{\mathrm{R}}}(\mathrm{M}, \mathrm{A})$$

for any R-algebra A. For example, we have $\mathrm{Sym}_{\mathrm{R}}(\mathrm{R}^{\oplus n}) \approx \mathrm{R}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ for $n \geqslant 0$.

2. Derived schemes. Any scheme S represents a presheaf

$$\mathrm{X} \mapsto \mathrm{Maps}(\mathrm{X}, \mathrm{S})$$

on the category of schemes, which satisfies fpqc descent by a theorem of Grothendieck. The fact that every scheme admits an affine Zariski cover implies that the inclusion of affine schemes into arbitary schemes induces an equivalence at the level of Zariski or fpqc sheaves. Therefore there is a fully faithful embedding of the category of schemes into the category of sheaves on the affine fpqc site. On the other hand, if we identify its essential image, we can take this as our *definition* of scheme. This is the philosophy we will take in our definition of derived scheme.

2.1. The *fpqc pretopology* on $(\mathrm{SCRing})^{\mathrm{op}}$ is defined as follows.

Definition 2.2. *A family of homomorphisms* $(\mathrm{R} \to \mathrm{R}_\alpha)_{\alpha\in\Lambda}$ *is* fpqc covering *if the following conditions hold:*

(i) The set Λ *is finite.*

(ii) For each $\alpha \in \Lambda$*, the homomorphism* $\mathrm{R} \to \mathrm{R}_\alpha$ *is flat (i.e. the underlying* R*-module of* R_α *is flat).*

(iii) The induced homomorphism $\mathrm{R} \to \prod_\alpha \mathrm{R}_\alpha$ *is faithfully flat.*

Recall that a connective R-module M is *flat* if for any discrete R-module N, the tensor product $\mathrm{R} \otimes_{\mathrm{M}} \mathrm{N}$ is discrete. It is *faithfully flat* if it is flat, and a connective R-module N is zero iff $\mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}$ is zero.

Definition 2.3.

(i) *A* derived prestack *is a presheaf of spaces on* $(\mathrm{SCRing})^{\mathrm{op}}$.

(ii) *A* derived stack *is an fpqc sheaf of spaces on* $(\mathrm{SCRing})^{\mathrm{op}}$*, i.e. a derived prestack which satisfies fpqc descent.*

Let us recall the descent condition in this setting. Let $(\mathrm{R} \to \mathrm{R}_\alpha)_\alpha$ be an fpqc covering family, and write $\tilde{\mathrm{R}} = \prod_\alpha \mathrm{R}_\alpha$. Let $\check{\mathrm{C}}(\mathrm{R}/\tilde{\mathrm{R}})_\bullet$ denote the Čech nerve of $\mathrm{R} \to \tilde{\mathrm{R}}$, a cosimplicial object given degree-wise by the $(n+1)$-fold tensor product

$$\check{\mathrm{C}}(\mathrm{R}/\tilde{\mathrm{R}})^n = \tilde{\mathrm{R}} \otimes_{\mathrm{R}} \cdots \otimes_{\mathrm{R}} \tilde{\mathrm{R}}.$$

Now, a derived prestack $\mathcal{X}$ satisfies fpqc descent if for all such fpqc covering families, the canonical morphism

$$\mathcal{X}(\mathrm{R}) \to \varprojlim_{n\in\boldsymbol{\Delta}} \mathcal{X}(\check{\mathrm{C}}(\mathrm{R}/\tilde{\mathrm{R}})^n) \tag{2.1}$$

is invertible.

2.4. Given a simplicial commutative ring R, we let $\mathrm{Spec}(\mathrm{R})$ denote the derived prestack represented by R.

Proposition 2.5. *For any simplicial commutative ring* R*, the presheaf* $\mathrm{Spec}(\mathrm{R})$ *is an fpqc sheaf. In particular, the fpqc topology is subcanonical.*

This follows from the fact that, for any fpqc covering morphism $\mathrm{A} \to \mathrm{B}$ in SCRing, the canonical morphism $\mathrm{A} \to \varprojlim_{n\in\boldsymbol{\Delta}} \check{\mathrm{C}}(\mathrm{A}/\mathrm{B})^n$ is invertible. This can be shown using the associated Bousfield–Kan spectral sequence, which degenerates on the second page. Alternatively it follows immediately from some general machinery developed by Lurie, see [2, Thm. D.6.3.5].

Definition 2.6. *An* affine derived scheme *is a derived stack which is isomorphic to* $\mathrm{Spec}(\mathrm{R})$ *for some simplicial commutative ring* R.

We let $\mathrm{DSch}^{\mathrm{aff}}$ denote the ∞-category of derived affine schemes, which is equivalent to $(\mathrm{SCRing})^{\mathrm{op}}$ by construction.

2.7. In order to give the definition of derived scheme, we need to define the notion of *open immersion* between derived stacks.

We begin with the following preliminary definition:

Definition 2.8. *A homomorphism of simplicial commutative rings* $\mathrm{R} \to \mathrm{R}'$ *is* locally of finite presentation *if it exhibits* R' *as a compact object of* $\mathrm{SCRing}_{\mathrm{R}}$*, i.e. if the functor*

$$\mathrm{A} \mapsto \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{R}', \mathrm{A})$$

commutes with filtered colimits.

Now let $j : \mathcal{U} \to \mathcal{X}$ be a morphism of derived stacks. First suppose that $\mathcal{X} = \mathrm{Spec}(\mathrm{R})$ and $\mathcal{U} = \mathrm{Spec}(\mathrm{A})$ are both affine. In this case we say that j is an *open immersion* if the corresponding homomorphism $\mathrm{R} \to \mathrm{A}$ is locally of finite presentation, flat, and an epimorphism, i.e. the co-diagonal homomorphism $\mathrm{A} \otimes_{\mathrm{R}} \mathrm{A} \to \mathrm{A}$ is invertible.

Next suppose that $\mathcal{U}$ is possibly non-affine. Then we say that j is an open immersion if it is a monomorphism, and there exists a family $(\mathcal{U}_\alpha \to \mathcal{U})_\alpha$ which induces an effective epimorphism[1] $\coprod_\alpha \mathcal{U}_\alpha \to \mathcal{U}$, such that each $\mathcal{U}_\alpha$ is affine, and each composite $\mathcal{U}_\alpha \to \mathcal{X}$ is an open immersion of affine derived schemes.

$$\begin{tikzcd} & \mathcal{U}_\alpha \arrow[dl] \arrow[d, hook] \\ \mathcal{U} \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

Finally, we define j to be an open immersion in the general case if, for any affine derived scheme $\mathrm{Spec}(\mathrm{R})$ and any morphism $\mathrm{Spec}(\mathrm{R}) \to \mathcal{X}$, the base change $\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}(\mathrm{R}) \to \mathrm{Spec}(\mathrm{R})$ is an open immersion in the above sense.

2.9. We are now ready to give the definition of derived scheme.

Definition 2.10.

(i) *A* Zariski cover *of a derived stack* $\mathcal{X}$ *is a family* $(j_\alpha : \mathcal{U}_\alpha \hookrightarrow \mathcal{X})_\alpha$ *where each* j_α *is an open immersion, and the induced morphism*

$$\coprod_\alpha \mathcal{U}_\alpha \to \mathcal{X}$$

is an effective epimorphism.

(ii) *An* affine Zariski cover *of a derived stack* $\mathcal{X}$ *is a Zariski cover* $(\mathcal{U}_\alpha \hookrightarrow \mathcal{X})_\alpha$ *where each* $\mathcal{U}_\alpha$ *is an affine derived scheme.*

(iii) *A derived stack* $\mathcal{X}$ *is* schematic *if it admits an affine Zariski cover. A* derived scheme *is a schematic derived stack.*

[1]Recall that a morphism of sheaves $\mathcal{X} \to \mathcal{Y}$ is an *effective epimorphism* if the canonical morphism of sheaves

$$\varinjlim_{n \in \boldsymbol{\Delta}^{\mathrm{op}}} \check{\mathrm{C}}(\mathcal{X}/\mathcal{Y})_n \to \mathcal{Y}$$

is invertible. Here $\check{\mathrm{C}}(\mathcal{X}/\mathcal{Y})_\bullet$ is the Čech nerve, a simplicial object with $\check{\mathrm{C}}(\mathcal{X}/\mathcal{Y})_n = \mathcal{X} \times_{\mathcal{Y}} \cdots \times_{\mathcal{Y}} \mathcal{X}$ (the $(n+1)$-fold fibred product).

2.11. We say that a *classical scheme* X is a derived stack which admits an affine Zariski cover $(\mathrm{Spec}(\mathrm{R}_\alpha) \hookrightarrow \mathrm{X})_\alpha$ by *classical* affine schemes, i.e. where R_α are discrete. In this case X is discrete as a presheaf (i.e. it takes values in sets), and therefore this does indeed recover the classical notion of scheme.

The inclusion $\mathrm{CRing} \hookrightarrow \mathrm{SCRing}$ defines an inclusion $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DSch}^{\mathrm{aff}}$. Given a derived prestack $\mathcal{X}$, we let $\mathcal{X}_{\mathrm{cl}}$ denote its restriction to the classical site. For an affine derived scheme $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, we have $\mathrm{X}_{\mathrm{cl}} = \mathrm{Spec}(\pi_0(\mathrm{R}))$ by adjunction. More generally, X_{cl} is a classical scheme for any derived scheme X; we refer to it as the *underlying classical scheme*. The functor $\mathrm{X} \mapsto \mathrm{X}_{\mathrm{cl}}$ is right adjoint to the inclusion of classical schemes into derived schemes.

3. Quasi-coherent sheaves.

3.1. Let $\mathrm{Spec}(\mathrm{R})$ be an affine derived scheme. We define the stable ∞-category of quasi-coherent sheaves on S by

$$\mathrm{Qcoh}(\mathrm{Spec}(\mathrm{R})) = \mathrm{Mod}_{\mathrm{R}}.$$

This defines a presheaf of ∞-categories

$$(\mathrm{DSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}, \tag{3.1}$$

where for a morphism $\mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$ the contravariant functoriality f^* is induced by $\mathrm{M} \mapsto \mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}'$.

3.2. Let $\mathcal{X}$ be a derived stack. We define the stable ∞-category of quasi-coherent sheaves on $\mathcal{X}$ as the limit

$$\mathrm{Qcoh}(\mathcal{X}) = \varprojlim_{\mathrm{S} \to \mathcal{X}} \mathrm{Qcoh}(\mathrm{S})$$

over morphisms $\mathrm{S} \to \mathcal{X}$, where S is an affine derived scheme. That is, we extend the presheaf (3.1) to a presheaf

$$(\mathrm{DStk})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat} \tag{3.2}$$

by right Kan extension along the Yoneda embedding.

Thus a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ consists of the data of

(1) For every affine derived scheme $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ and every morphism $f : \mathrm{S} \to \mathcal{X}$, a quasi-coherent sheaf $f^*(\mathcal{F}) \in \mathrm{Qcoh}(\mathrm{S})$ (i.e. an R-module).

(2) For every commutative triangle

$$\begin{tikzcd} \mathrm{S}' = \mathrm{Spec}(\mathrm{R}') \arrow[r, "f'"] \arrow[d, "g"] & \mathcal{X} \\ \mathrm{S} = \mathrm{Spec}(\mathrm{R}) \arrow[ur, "f"'] & \end{tikzcd}$$

an isomorphism $g^*(f^*\mathcal{F}) \to (f')^*\mathcal{F}$ in $\mathrm{Qcoh}(\mathrm{S}')$.

(3) A homotopy coherent system of compatibilities between these isomorphisms.

3.3. Taking symmetric monoidal structures into account, the presheaf (3.1) actually takes values in symmetric monoidal ∞-categories, and its right Kan extension provides a lift of (3.2) (since the forgetful functor from symmetric monoidal ∞-categories to plain ∞-categories preserves and detects limits).

In particular, the ∞-category $\mathrm{Qcoh}(\mathcal{X})$ is canonically symmetric monoidal for each derived stack $\mathcal{X}$, as are the functors f^*.

3.4. In the schematic case, there is a simpler description of the ∞-category $\mathrm{Qcoh}(\mathrm{X})$:

Proposition 3.5. *Let* X *be a derived scheme. Then we have a canonical equivalence of* ∞-*categories*

$$\mathrm{Qcoh}(\mathrm{X}) \xrightarrow{\sim} \varprojlim_{\mathrm{U} \hookrightarrow \mathrm{X}} \mathrm{Qcoh}(\mathrm{U}), \tag{3.3}$$

where the limit is taken over open immersions $\mathrm{U} \to \mathrm{X}$ *with* U *an affine derived scheme.*

Proof. As a right Kan extension, the presheaf $\mathrm{Qcoh}(-)$ sends colimits of derived stacks to limits. Choosing an affine Zariski cover $(\mathrm{X}_\alpha \hookrightarrow \mathrm{X})_\alpha$, we have therefore a canonical equivalence

$$\mathrm{Qcoh}(\mathrm{X}) = \varprojlim_{n \in \mathbf{\Delta}} \mathrm{Qcoh}(\check{\mathrm{C}}(\mathrm{X}_\alpha/\mathrm{X})_n),$$

where $\check{\mathrm{C}}(\mathrm{X}_\alpha/\mathrm{X})_\bullet$ denotes the Čech nerve of $\coprod_\alpha \mathrm{X}_\alpha \to \mathrm{X}$. Using the induced Zariski cover $(\mathrm{U}_\alpha \hookrightarrow \mathrm{U})_\alpha$, where $\mathrm{U}_\alpha = \mathrm{X}_\alpha \times_{\mathrm{X}} \mathrm{U}$, we get an analogous description of $\mathrm{Qcoh}(\mathrm{U})$ for any open $\mathrm{U} \hookrightarrow \mathrm{X}$. This reduces us to showing that the canonical functor

$$\mathrm{Qcoh}(\mathrm{V}) \to \varprojlim_{\mathrm{U} \hookrightarrow \mathrm{X}} \mathrm{Qcoh}(\mathrm{V} \underset{\mathrm{X}}{\times} \mathrm{U})$$

is an equivalence, where $\mathrm{V} \hookrightarrow \mathrm{X}$ is any of the terms of the Čech nerve. By an easy cofinality argument, this is equivalent to showing the equivalence (3.3) where X is replaced by any of the terms of the Čech nerve.

Suppose that each of the pairwise intersections of the X_α's were affine; then we would have reduced to the affine case, which is obvious. We are not so far from that: since each of the X_α's themselves are affine, we know that their pairwise intersections are open subschemes in affine derived schemes, and hence are *separated*, which is equivalent to their underlying classical schemes being separated, and implies that they admit affine Zariski covers where each of the pairwise intersections are affine. Thus we can run the argument again to reduce from the separated case to the affine case, and conclude. $\square$

3.6. Next we wish to define the ∞-category of perfect complexes on a derived stack.

Definition 3.7. *An* R-*module* M *is* perfect *if it can be built from* R *using finite colimits and direct summands.*

Remark 3.8. Let R be an ordinary commutative ring. Then under the equivalence between $\mathrm{Mod}_{\mathrm{R}}$ and the derived category of (ordinary) R-modules, perfect R-modules correspond to bounded cochain complexes of finitely generated projective R-modules.

Exercise 3.9. *Let* M *be an* R-*module. Show that the following conditions are equivalent:*

(i) M *is perfect.*

(ii) M *is compact, i.e.* $\mathrm{Maps}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}, -)$ *commutes with filtered colimits.*

(iii) M *is dualizable, or equivalently,* $\mathrm{N} \mapsto \mathrm{M} \otimes \mathrm{N}$ *commutes with limits (as an endofunctor of* $\mathrm{Mod}_{\mathrm{R}}$*).*

We say that a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is *perfect*, or a *perfect complex*, if for every affine derived scheme S and every morphism $f : \mathrm{S} \to \mathcal{X}$, the quasi-coherent sheaf $f^*(\mathcal{F})$ is perfect. We let $\mathrm{Perf}(\mathcal{X})$ denote the full subcategory of $\mathrm{Qcoh}(\mathcal{X})$ spanned by perfect complexes.

Proposition 3.10. *Let* $\mathcal{X}$ *be a derived stack. Then a quasi-coherent sheaf* $\mathcal{F}$ *on* X *is perfect iff it is dualizable.*

The claim is that $\mathcal{F}$ is dualizable iff $f^*\mathcal{F}$ is dualizable for all morphisms $f : \mathrm{S} \to \mathcal{X}$ with S affine. This follows formally from the description of dualizable objects given in statement (iii) of Exercise 3.9 (which holds in any stable symmetric monoidal presentable ∞-category where the monoidal product commutes with colimits in each argument). Alternatively, given duals $(f^*\mathcal{F})^\vee$ for each $f : \mathrm{S} \to \mathcal{X}$, one can construct a global dual $\mathcal{F}^\vee$ by hand.

3.11. The conditions of perfectness and compactness are less closely related in general. However, we have the following result, which will be our goal for next time:

Theorem 3.12 (Toën)**.** *Let* X *be a quasi-compact quasi-separated derived scheme. Then a quasi-coherent sheaf on* X *is perfect iff it is compact.*

References.

[1] Jacob Lurie, *Higher algebra.*
[2] Jacob Lurie, *Spectral algebraic geometry.*
[3] Bertrand Toën, Gabriele Vezzosi, *Homotopical algebraic geometry II.*