Lecture 0
Overview

1. . Let X be a quasi-compact quasi-separated scheme and let $\mathrm{K}(\mathrm{X})$ denote the (nonconnective) algebraic K-theory of perfect complexes on X. This is a spectrum such that $\pi_0\mathrm{K}(\mathrm{X}) = \mathrm{K}_0(\mathrm{X})$ is the Grothendieck group of perfect complexes, as defined in SGA 6.

Our goal in this course is to study various properties of the assignment

$$\mathrm{X} \mapsto \mathrm{K}(\mathrm{X}). \tag{1.1}$$

1.1. A first observation is that it has contravariant functoriality: given a morphism of qcqs schemes $f : \mathrm{Y} \to \mathrm{X}$, there is a canonical map $f^* : \mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{Y})$ induced by the derived inverse image of perfect complexes. In fact, (1.1) defines a presheaf of spectra.

We would therefore like to think of it as a cohomology theory for schemes. Recall that in topology, a generalized cohomology theory on smooth manifolds is roughly a presheaf of spectra satisfying an excision or Mayer–Vietoris property and homotopy invariance (with respect to the real line). We could define a *generalized motivic cohomology theory* on smooth schemes (over a fixed base S) as a presheaf of spectra satisfying Mayer–Vietoris for Zariski or Nisnevich squares, and homotopy invariance with respect to the affine line $\mathbf{A}^1$.

Is $\mathrm{X} \mapsto \mathrm{K}(\mathrm{X})$ as a generalized cohomology theory in this sense? An important theorem of Thomason says that the Mayer–Vietoris condition holds:

Theorem 1.2 (Thomason)**.** *Let* X *be a quasi-compact quasi-separated scheme and let* $\mathrm{U} \subset \mathrm{X}$ *and* $\mathrm{V} \subset \mathrm{X}$ *be quasi-compact open subsets such that* $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$*. Then the presheaf* $\mathrm{X} \mapsto \mathrm{K}(\mathrm{X})$ *sends the square*

$$\begin{tikzcd} \mathrm{U} \cap \mathrm{V} \arrow[r, hook] \arrow[d, hook] & \mathrm{V} \arrow[d, hook] \\ \mathrm{U} \arrow[r, hook] & \mathrm{X} \end{tikzcd}$$

to a homotopy cartesian square of spectra

$$\begin{tikzcd} \mathrm{K}(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{K}(\mathrm{U}) \arrow[d] \\ \mathrm{K}(\mathrm{V}) \arrow[r] & \mathrm{K}(\mathrm{U} \cap \mathrm{V}). \end{tikzcd}$$

In fact, this holds more generally for Nisnevich squares, where $\mathrm{V} \to \mathrm{X}$ *is an étale morphism which induces an isomorphism* $p^{-1}(\mathrm{X} - \mathrm{U})_{\mathrm{red}} \to (\mathrm{X} - \mathrm{U})_{\mathrm{red}}$ *on underlying topological spaces.*

Unfortunately, algebraic K-theory fails to be $\mathbf{A}^1$-invariant, but the situation is not that bad; if we restrict to nonsingular schemes, then we have:

Theorem 1.3 (Quillen)**.** *Let* X *be a regular noetherian scheme. Then the canonical map* $\mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{X} \times \mathbf{A}^1)$ *is invertible.*

Thus K-theory defines a generalized motivic cohomology theory in the above sense, at least when the base S is regular (for example $\mathrm{S} = \mathrm{Spec}(k)$ with k a field, or $\mathrm{S} = \mathrm{Spec}(\mathbf{Z})$).

2. .

2.1. Theorem 1.3 comes from an identification

$$\mathrm{K}(\mathrm{X}) \xrightarrow{\sim} \mathrm{G}(\mathrm{X}) \tag{2.1}$$

for X regular and noetherian. Here G(X) denotes the G-theory of X, defined as K(Coh(X)), the algebraic K-theory of coherent sheaves on X. If X is not regular, we can still make sense of G(X), and the isomorphism

$$\mathrm{G}(\mathrm{X}) \xrightarrow{\sim} \mathrm{G}(\mathrm{X} \times \mathbf{A}^1)$$

holds for any noetherian X.

However, G-theory does not behave quite like a cohomology theory, but rather as a Borel–Moore homology theory. Indeed, it does not have contravariant functoriality for arbitrary morphisms; instead it has covariance with respect to proper morphisms, since (higher) direct image along proper morphisms preserves coherence. From this perspective the isomorphism (2.1) can be viewed as an instance of Poincaré duality.

2.2. Given a presheaf of spectra $\mathcal{F}$ on the category of smooth S-schemes, there is a general construction that imposes $\mathbf{A}^1$-homotopy invariance in a universal way:

$$\mathcal{F} \mapsto \mathrm{L}_{\mathbf{A}^1}(\mathcal{F}).$$

This is a certain localization of the ∞-category of presheaves of spectra.

Let $\mathrm{KH} = \mathrm{L}_{\mathbf{A}^1}(\mathrm{K})$ be the $\mathbf{A}^1$-localization of K-theory.

Theorem 2.3 (Thomason–Weibel)**.**

(i) KH *satisfies Mayer–Vietoris for Nisnevich squares.*

(ii) KH *satisfies* $\mathbf{A}^1$*-homotopy invariance.*

(iii) *If* X *is regular, then the canonical map of spectra* $\mathrm{K}(\mathrm{X}) \to \mathrm{KH}(\mathrm{X})$ *is invertible.*

Therefore KH defines a generalized motivic cohomology theory over *arbitrary* (qcqs) base schemes.

Remark 2.4. We warn the reader that it is important in this definition to consider *nonconnective* K-theory as opposed to its connective cover K^{cn}. The presheaf $\mathrm{L}_{\mathbf{A}^1}(\mathrm{K}^{\mathrm{cn}})$ is not at all well-behaved.

2.5. To sum up, we have a commutative triangle

$$\begin{tikzcd} \mathrm{K} \arrow[r] \arrow[dr] & \mathrm{KH} \arrow[d] \\ & \mathrm{G} \end{tikzcd}$$

of presheaves[1] of spectra on smooth S-schemes, when S is noetherian.

The right-hand vertical arrow comes from the fact that G-theory satisfies $\mathbf{A}^1$-homotopy invariance. When S is regular, all three arrows are invertible. In particular, in this case K actually takes values in connective spectra.

3. . A main theme of these lectures will be trying to understand the spectrum K(X) when X is singular.

[1] Any morphism of smooth S-schemes is of finite tor-amplitude and G-theory admits contravariant functoriality along such morphisms.

3.1. Using Mayer–Vietoris we can reduce the question to studying the K-theory of affine singular schemes.

Let $\mathrm{K}^{\mathrm{cn}}(\mathrm{X}) := \mathrm{K}(\mathrm{X})_{\geqslant 0}$ denote the connective cover of nonconnective K-theory. It turns out that the connective part can actually be built out of colimits from the K-theory spectra of regular schemes.

Theorem 3.2 (Bhatt–Lurie). *Consider the commutative diagram*

$$\begin{tikzcd} (\mathrm{Sm}^{\mathrm{aff}}_{/\mathbf{Z}})^{\mathrm{op}} \arrow[r, "\mathrm{K}^{\mathrm{cn}}"] \arrow[d, hook] & \mathrm{Spt} \\ (\mathrm{Sch}^{\mathrm{aff}}_{/\mathbf{Z}})^{\mathrm{op}} \arrow[ur, "\mathrm{K}^{\mathrm{cn}}"'] & \end{tikzcd}$$

where $\mathrm{Sch}^{\mathrm{aff}}_{/\mathbf{Z}}$ *denotes the category of affine schemes over* $\mathrm{Spec}(\mathbf{Z})$ *and* $\mathrm{Sm}^{\mathrm{aff}}_{/\mathbf{Z}}$ *denotes the full subcategory of affine schemes which are* smooth *over* $\mathrm{Spec}(\mathbf{Z})$*. This diagram is a left Kan extension, i.e. the diagonal arrow is the left Kan extension of the horizontal arrow.*

3.3. The nonconnective part $\mathrm{K}(\mathrm{X})_{<0}$ is difficult to get a handle on. The following result, which resolves a conjecture made by Weibel in 1980, represents a significant breakthrough.

Theorem 3.4 (Kerz–Strunk–Tamme). *Let* X *be a noetherian scheme of finite dimension* d*. Then we have* $\mathrm{K}_i(\mathrm{X}) := \pi_i\mathrm{K}(\mathrm{X}) = 0$ *for* $i < -d$*. Further, the canonical map* $\mathrm{K}_{-d}(\mathrm{X}) \to \mathrm{K}_{-d}(\mathrm{X} \times \mathbf{A}^1)$ *is invertible.*

This is the ultimate goal of these lectures.

4. . Much of the proof of Theorem 3.4 resolves around understanding the extent to which K-theory satisfies descent by blow-ups.

4.1. One of the main tools is the following, originally proved by Thomason for classical schemes:

Theorem 4.2 (KST). *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a regular immersion of derived schemes. Let* $\tilde{\mathrm{X}}$ *be the* derived blow-up *of* X *in* Z *and let* $\mathrm{E} = \mathbf{P}(\mathrm{N}_{\mathrm{Z}/\mathrm{X}})$ *be the virtual exceptional divisor. Then K-theory sends the square*

$$\begin{tikzcd} \mathrm{E} \arrow[r, hook] \arrow[d] & \tilde{\mathrm{X}} \arrow[d] \\ \mathrm{Z} \arrow[r, hook] & \mathrm{X} \end{tikzcd}$$

to a homotopy cartesian square of spectra

$$\begin{tikzcd} \mathrm{K}(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{K}(\mathrm{Z}) \arrow[d] \\ \mathrm{K}(\tilde{\mathrm{X}}) \arrow[r] & \mathrm{K}(\mathrm{E}). \end{tikzcd}$$

The derived blow-up is a construction that agrees with the classical blow-up when X and Z are classical, but has the feature of being stable under arbitrary derived base change.

4.3. Define an *abstract blow-up square* to be a cartesian square

$$\begin{tikzcd} \mathrm{E} \arrow[r, hook] \arrow[d] & \tilde{\mathrm{X}} \arrow[d, "p"] \\ \mathrm{Z} \arrow[r, hook, "i"] & \mathrm{X} \end{tikzcd}$$

where i is a closed immersion, p is proper, and the induced morphism $(\tilde{\mathrm{X}} - \mathrm{E})_{\mathrm{red}} \to (\mathrm{X} - \mathrm{Z})_{\mathrm{red}}$ induces an isomorphism on underlying topological spaces.

K-theory does not satisfy descent with respect to abstract blow-up squares (or even with respect to blow-ups of arbitrary closed immersions). Forcing homotopy invariance somehow "corrects" this:

Theorem 4.4 (Cisinski)**.** *The presheaf* KH *satisfies descent with respect to arbitrary abstract blow-up squares.*

In fact, we even have the following striking statement:

Theorem 4.5 (KST)**.** *The map* $\mathrm{K}^{\mathrm{cn}} \to \mathrm{KH}$ *is a* cdh-sheafification *when restricted to noetherian (derived) schemes of finite dimension.*

The cdh topology is generated by Nisnevich squares and abstract blow-up squares, so this statement means roughly that KH is obtained from connective K-theory by universally imposing descent with respect to abstract blow-ups.

Remark 4.6. When restricted to schemes over a field of characteristic zero, this result was proven by Haesemeyer using resolution of singularities. One of the main innovations of Kerz–Strunk–Tamme is the use of the *Zariski–Riemann space* to avoid resolution of singularities.

References.

[1] R.W. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories.*
[2] M. Kerz, F. Strunk, G. Tamme *Algebraic K-theory and descent for blow-ups.*