Lecture 9

The coniveau filtration and algebraic cycles

From now on, all rings will be implicitly assumed noetherian.

9.1. The coniveau filtration.

Construction 1. Let A be a (noetherian) ring. For each $n \in \mathbf{N}$, let $\mathrm{G}_0(\mathrm{A})^{\geqslant n}$ denote the subgroup of $\mathrm{G}_0(\mathrm{A})$ generated by classes $[\mathrm{M}]$ where $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$ has $\mathrm{codim}(\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})) \geqslant n$.

Proposition 2. *The subgroup* $\mathrm{G}_0(\mathrm{A})^{\geqslant n}$ *is generated by classes* $[\mathrm{A}/\mathfrak{p}]$*, where* $\mathfrak{p}$ *is a prime ideal such that* $\mathrm{V}(\mathfrak{p})$ *is of codimension* $\geqslant n$*.*

Proof. Let $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$ such that $\mathrm{codim}(\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})) \geqslant n$. Let $0 = \mathrm{M}_0 \subset \mathrm{M}_1 \subset \cdots \subset \mathrm{M}_n = \mathrm{M}$ be a filtration whose successive quotients are of the form $\mathrm{A}/\mathfrak{p}_i$, with $\mathfrak{p}_i \subset \mathrm{A}$ prime ideals. Then we have seen that $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) = \bigcup_i \mathrm{V}(\mathfrak{p}_i)$. Thus

$$\mathrm{codim}(\mathrm{V}(\mathfrak{p}_i)) \geqslant \mathrm{codim}(\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})) \geqslant n.$$

As we have

$$[\mathrm{M}] = \sum_i [\mathrm{M}_i/\mathrm{M}_{i-1}] = \sum_i [\mathrm{A}/\mathfrak{p}_i]$$

in $\mathrm{G}_0(\mathrm{A})$, the claim follows. ☐

Proposition 3. *Let* A *be an irreducible ring. Then we have*

$$\mathrm{G}_0(\mathrm{A})^{\geqslant 0}/\,\mathrm{G}_0(\mathrm{A})^{\geqslant 1} \simeq \mathbf{Z}.$$

Proof. Note that $\mathrm{G}_0(\mathrm{A})^{\geqslant 0} = \mathrm{G}_0(\mathrm{A})$. To define a map from left to right we proceed as usual: given $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$, choose a filtration where the successive quotients are of the form $\mathrm{A}/\mathfrak{p}$ with $\mathfrak{p}$ prime. On classes $[\mathrm{A}/\mathfrak{p}]$, the map is defined as follows. If $\mathfrak{p}$ is the (unique) minimal prime ideal, then we send it to 1. Otherwise it is of codimension $\geqslant 1$ so we are forced to send it to 0. Arguing with the butterfly lemma again we see that this construction is independent of the choice of filtration and gives a well-defined map from left to right. An inverse map is given by sending $1 \in \mathbf{Z}$ to $[\mathrm{A}/\mathfrak{p}]$, where $\mathfrak{p}$ is the minimal prime ideal. ☐

Remark 4. If A is not irreducible, then a straightforward adaptation of this argument shows that the quotient $\mathrm{G}_0(\mathrm{A})^{\geqslant 0}/\,\mathrm{G}_0(\mathrm{A})^{\geqslant 1}$ is isomorphic to a direct sum of copies of $\mathbf{Z}$ indexed by the irreducible components.

9.2. Multiplicities of modules.

Notation 5. Let A be a ring and $x \in \left|\mathrm{Spec}(\mathrm{A})\right|$ a point. We let $\mathfrak{p}(x)$ denote the corresponding prime ideal, i.e.,

$$\mathfrak{p}(x) := \mathrm{Ker}(\mathrm{A} \to \kappa(x)).$$

Lemma 6. *Let* A *be a ring and* M *a f.g.* A*-module. Let* η *be a generic point of* $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})$ *and let* $\mathfrak{p}(\eta)$*. Then the* $\mathrm{A}_{\mathfrak{p}(\eta)}$*-module* $\mathrm{M}_{\mathfrak{p}(\eta)}$ *is of finite length.*

Proof. Let $\mathfrak{p} = \mathfrak{p}(\eta)$. For every non-maximal prime ideal $\mathfrak{q} \subset \mathrm{A}_{\mathfrak{p}}$, we have $(\mathrm{M}_{\mathfrak{p}})_{\mathfrak{q}} = 0$. By Sheet 2, Exercise 4 it follows then that $\mathrm{M}_{\mathfrak{p}}$ is of finite length. □

Definition 7. Let A be a ring and M a f.g. A-module. Let η be a generic point of $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})$. The *multiplicity* of M at η is the integer

$$\mathrm{mult}_{\mathrm{A},\eta}(\mathrm{M}) := \ell_{\mathrm{A}_{\mathfrak{p}(\eta)}}(\mathrm{M}_{\mathfrak{p}(\eta)}).$$

Remark 8. Choose any filtration of M where the successive quotients are of the form $\mathrm{A}/\mathfrak{p}$ with $\mathfrak{p}$ prime. The number of times the prime ideal $\mathfrak{p}(\eta)$ appears in this way is exactly the multiplicity $\mathrm{mult}_{\mathrm{A},\eta}(\mathrm{M})$.

9.3. Algebraic cycles. Algebraic cycles are a convenient way to record multiplicities.

Definition 9. Let A be a ring. The *dimension* of A is the maximal length n of a chain

$$\varnothing \subsetneq \mathrm{Y}_0 \subsetneq \mathrm{Y}_1 \subsetneq \cdots \subsetneq \mathrm{Y}_n \subseteq \big|\mathrm{Spec}(\mathrm{A})\big|$$

of irreducible closed subsets of $\big|\mathrm{Spec}(\mathrm{A})\big|$. (The zero ring is of dimension -1 by convention.) For a closed subset $\mathrm{Y} = \mathrm{V}(\mathrm{I}) \subseteq \big|\mathrm{Spec}(\mathrm{A})\big|$, the dimension of Y is the dimension of A/I. (This is well-defined since if $\mathrm{V}(\mathrm{I}) = \mathrm{V}(\mathrm{J})$ then $\big|\mathrm{Spec}(\mathrm{A}/\mathrm{I})\big| \simeq \big|\mathrm{Spec}(\mathrm{A}/\mathrm{J})\big|$.) We say Y is of *pure dimension* d if all its irreducible components are of dimension d.

Example 10. Any field k is of dimension 0. More generally, any nonzero artinian ring is of dimension 0.

Example 11. For a field k, the ring $k[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ is of dimension n.

Definition 12. Let A be a ring. An *algebraic cycle of dimension* k on A (or k*-cycle for short*) is a formal linear combination

$$\alpha = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \cdot [\mathrm{V}(\mathfrak{p})]$$

where $\mathfrak{p}$ ranges over a set of prime ideals of A, such that each $\mathrm{V}(\mathfrak{p})$ is a k-dimensional subset of $\big|\mathrm{Spec}(\mathrm{A})\big|$, and $n_{\mathfrak{p}}$ are integers. For each $\mathfrak{p}$, the integer $n_{\mathfrak{p}}$ is called the *multiplicity* of the cycle α at $\mathfrak{p}$. We let $\mathrm{Z}_k(\mathrm{A})$ denote the set of algebraic cycles of dimension k, which is thus a free abelian group. By $\mathrm{Z}_*(\mathrm{A})$ we denote the graded abelian group $\bigoplus_{k\geqslant 0} \mathrm{Z}_k(\mathrm{A})$.

Construction 13 (Cycle associated to a module)**.** Let M be an A-module with support of dimension d. For any integer $k \leqslant d$, we define a k-cycle $[\mathrm{M}]_k \in \mathrm{Z}_k(\mathrm{A})$ as follows. Let η_α be the generic points of $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})$ for which the integral subset $\mathrm{V}(\mathfrak{p}(\eta_\alpha))$ is of dimension k. Set

$$[\mathrm{M}]_k := \sum_{\alpha} \mathrm{mult}_{\mathrm{A},\eta_\alpha}(\mathrm{M}) \cdot [\mathrm{V}(\mathfrak{p}(\eta_\alpha))].$$

If $k > d$, we set $[\mathrm{M}]_k = 0$.

Remark 14. The noetherian hypothesis implies that there are only finitely many generic points η_α. In particular, the sum appearing in the previous construction is finite.

Example 15. Let $\mathfrak{p}$ be a prime ideal and take $\mathrm{M} = \mathrm{A}/\mathfrak{p}$. Let d be the dimension of the support $\mathrm{Supp}_\mathrm{A}(\mathrm{M}) = \mathrm{V}(\mathfrak{p})$. Then the associated d-cycle $[\mathrm{A}/\mathfrak{p}]_d$ is the same as the cycle $[\mathrm{V}(\mathfrak{p})] \in \mathrm{Z}_d(\mathrm{A})$. Indeed we have

$$\mathrm{mult}_{\mathrm{A},\eta}(\mathrm{A}/\mathfrak{p}) = \ell_{\mathrm{A}_\mathfrak{p}}((\mathrm{A}/\mathfrak{p})_\mathfrak{p}) = \ell_{\mathrm{A}_\mathfrak{p}}(\kappa(\mathfrak{p})) = 1,$$

where $\eta = [\mathrm{A} \to \kappa(\mathfrak{p})]$ is the generic point of $\mathrm{V}(\mathfrak{p})$.

9.4. Rational equivalence.

Exercise 16. Let A be an integral domain of dimension d. For any nonzero element $f \in \mathrm{A}$, the quotient ring $\mathrm{A}/\langle f\rangle$ is of dimension $d-1$.

Construction 17. Let A be a ring and $\mathrm{V}(\mathfrak{p})$ a $(k+1)$-dimensional integral subset of $|\mathrm{Spec}(\mathrm{A})|$ (where $k \geqslant 0$). If $f \in \mathrm{A}$ is an element such that $f \notin \mathfrak{p}$, then $(\mathrm{A}/\mathfrak{p})/f(\mathrm{A}/\mathfrak{p}) \simeq \mathrm{A}/(\mathfrak{p} + \langle f\rangle)$ is of dimension k. Regarding $(\mathrm{A}/\mathfrak{p})/f(\mathrm{A}/\mathfrak{p})$ as an A-module, there is an associated k-cycle. This is the *principal divisor* defined by f in $\mathrm{V}(\mathfrak{p})$:

$$\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f) := [(\mathrm{A}/\mathfrak{p})/f(\mathrm{A}/\mathfrak{p})]_k \in \mathrm{Z}_k(\mathrm{A}).$$

Definition 18. Let $\mathrm{R}_k(\mathrm{A})$ denote the subgroup of $\mathrm{Z}_k(\mathrm{A})$ generated by elements of the form $\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f)$ for all $(k+1)$-dimensional integral subsets $\mathrm{V}(\mathfrak{p})$ and elements $f \notin \mathfrak{p}$. We say that two k-cycles $\alpha, \beta \in \mathrm{Z}_k(\mathrm{A})$ are *rationally equivalent* if their difference belongs to $\mathrm{R}_k(\mathrm{A})$.

Construction 19. The *Chow group* of k-cycles on A is the quotient

$$\mathrm{CH}_k(\mathrm{A}) := \mathrm{Z}_k(\mathrm{A})/\mathrm{R}_k(\mathrm{A}),$$

for every integer $k \geqslant 0$.

9.5. Direct images.

Construction 20. Let $\phi : \mathrm{A} \twoheadrightarrow \mathrm{A}/\mathrm{I}$ be a surjective ring homomorphism. Any closed integral subset $\mathrm{V}_{\mathrm{A}/\mathrm{I}}(\mathfrak{p}) \subseteq |\mathrm{Spec}(\mathrm{A}/\mathrm{I})|$ can be regarded, via the inclusion $|\mathrm{Spec}(\mathrm{A}/\mathrm{I})| \simeq \mathrm{V}_\mathrm{A}(\mathrm{I}) \subseteq |\mathrm{Spec}(\mathrm{A})|$, as a subset of $|\mathrm{Spec}(\mathrm{A})|$. As a subset of $|\mathrm{Spec}(\mathrm{A})|$, it is the closed integral subset defined by the prime ideal $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Thus there is for each $k \geqslant 0$ a group homomorphism

$$\phi_* : \mathrm{Z}_k(\mathrm{A}/\mathrm{I}) \to \mathrm{Z}_k(\mathrm{A})$$

given by $[\mathrm{V}_{\mathrm{A}/\mathrm{I}}(\mathfrak{p})] \mapsto [\mathrm{V}_\mathrm{A}(\phi^{-1}(\mathfrak{p}))]$.

Lemma 21. *With $\phi : \mathrm{A} \twoheadrightarrow \mathrm{A}/\mathrm{I}$ as above, we have*

$$\phi_*(\mathrm{R}_k(\mathrm{A}/\mathrm{I})) \subseteq \mathrm{R}_k(\mathrm{A})$$

for every k.

Proof. It suffices to show that for every closed integral subset $\mathrm{V}(\mathfrak{p})$ of dimension $k{+}1$ and every element $f \in \mathrm{A}/\mathrm{I}$ not contained in $\mathfrak{p}$, the cycle $\phi_*(\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f)) \in \mathrm{Z}_k(\mathrm{A})$ is rationally equivalent to 0. The contraction $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$ is a prime ideal of A. If $\tilde{f} \in \mathrm{A}$ is an element lifting f, then $\tilde{f} \notin \mathfrak{q}$ since $f \notin \mathfrak{p}$. Then since $(\mathrm{A}/\mathfrak{q})/\tilde{f}(\mathrm{A}/\mathfrak{q}) \simeq ((\mathrm{A}/\mathrm{I})/\mathfrak{p})/f((\mathrm{A}/\mathrm{I})/\mathfrak{p})$, it follows from the definitions that

$$\phi_*(\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f)) = \mathrm{div}_{\mathrm{V}(\mathfrak{q})}(\tilde{f}),$$

whence the claim. □

Construction 22. Let $\phi : \mathrm{A} \twoheadrightarrow \mathrm{A}/\mathrm{I}$ be a surjective ring homomorphism. By the lemma, the homomorphism $\phi_* : \mathrm{Z}_k(\mathrm{A}/\mathrm{I}) \to \mathrm{Z}_k(\mathrm{A})$ descends to a canonical homomorphism

$$\phi_* : \mathrm{CH}_k(\mathrm{A}/\mathrm{I}) \to \mathrm{CH}_k(\mathrm{A})$$

for every k. We call this the homomorphism of *direct image* along ϕ.

9.6. Inverse images.

Definition 23. Let $\phi : \mathrm{A} \to \mathrm{B}$ be a flat ring homomorphism. We say that ϕ is *of relative dimension* $d \geqslant 0$ if, for every closed integral subset $\mathrm{V}(\mathfrak{p}) \subseteq |\mathrm{Spec}(\mathrm{A})|$ of dimension n, the closed subset $\mathrm{V}(\mathfrak{p}\mathrm{B}) \subseteq |\mathrm{Spec}(\mathrm{B})|$ is of pure dimension $n + d$.

Remark 24. Note that $\phi : \mathrm{A} \to \mathrm{B}$ induces a canonical map

$$f : |\mathrm{Spec}(\mathrm{B})| \to |\mathrm{Spec}(\mathrm{A})|$$

sending a point $x = [\mathrm{B} \to \kappa]$ to $f(x) = [\mathrm{A} \to \mathrm{B} \to \kappa]$. For a closed integral subset $\mathrm{V}(\mathfrak{p}) \subseteq |\mathrm{Spec}(\mathrm{A})|$ we have $f^{-1}(\mathrm{V}(\mathfrak{p})) = \mathrm{V}(\mathfrak{p}\mathrm{B}) \subseteq |\mathrm{Spec}(\mathrm{B})|$. Thus we can interpret the previous definition in terms of the fibres of the morphism f.

Example 25. For any element $f \in \mathrm{A}$, the localization homomorphism $\phi : \mathrm{A} \to \mathrm{A}[f^{-1}]$ is flat of relative dimension 0.

Example 26. For every ring A and every $n \geqslant 0$, the homomorphism $\mathrm{A} \to \mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ is flat of relative dimension n.

Construction 27. Let $\phi : \mathrm{A} \to \mathrm{B}$ be a flat ring homomorphism of relative dimension d. Then there are canonical homomorphisms

$$\phi^* : \mathrm{Z}_k(\mathrm{A}) \to \mathrm{Z}_{k+d}(\mathrm{B})$$

sending $[\mathrm{V}(\mathfrak{p})] \mapsto [\mathrm{B}/\mathfrak{p}\mathrm{B}]_{k+d}$.

Theorem 28. *Let* $\phi : \mathrm{A} \to \mathrm{B}$ *be as above. Then we have*

$$\phi^*(\mathrm{R}_k(\mathrm{A})) \subseteq \mathrm{R}_{k+d}(\mathrm{B})$$

for every k.

Lemma 29. *Let* A *be a ring and* M *a f.g.* A*-module whose support is of dimension* $\leqslant k + 1$. *Denote by* Λ *the set of irreducible components* $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{Supp}_{\mathrm{A}}(\mathrm{M})$ *of* *dimension $k+1$, and let $f \in \mathrm{A}$ be an element with $f \notin \mathfrak{p}$ for every $\mathrm{V}(\mathfrak{p}) \in \Lambda$. Then we have*

$$[\mathrm{M}/f\mathrm{M}]_k - [{}_f\mathrm{M}]_k = \sum_{\mathrm{V}(\mathfrak{p})\in\Lambda} \ell_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p})\,\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f)$$

in $\mathrm{Z}_k(\mathrm{A})$, where ${}_f\mathrm{M} \subseteq \mathrm{M}$ is the submodule of f-torsion elements. In particular if f is a non-zero-divisor on M, then $[\mathrm{M}/f\mathrm{M}]_k \in \mathrm{Z}_k(\mathrm{A})$ is rationally equivalent to zero.

Proof. Note that $f \notin \mathfrak{p}$ implies that $\mathrm{M}/f\mathrm{M}$ and ${}_f\mathrm{M}$ both have support of dimension $\leqslant k$.

Assume first that $\mathrm{M} = \mathrm{A}/\mathfrak{q}$ where $\mathfrak{q}$ is a prime ideal. If its support $\mathrm{V}(\mathfrak{q})$ is $(k+1)$-dimensional, then it has only one irreducible component of dimension $k+1$ (namely, $\mathrm{V}(\mathfrak{q})$ itself). The assumption is then that $f \notin \mathfrak{q}$, so in particular the image of f in the integral domain $\mathrm{A}/\mathfrak{q}$ is a non-zero-divisor and ${}_f\mathrm{M} = 0$. Thus the left-hand side is $[(\mathrm{A}/\mathfrak{q})/f(\mathrm{A}/\mathfrak{q})]_k$. Since $\mathrm{M}_\mathfrak{q} = \mathrm{A}_\mathfrak{q}/\mathfrak{q}\mathrm{A}_\mathfrak{q} = \kappa(\mathfrak{q})$ is of length 1, the right-hand side is $\mathrm{div}_{\mathrm{V}(\mathfrak{q})}(f)$. Hence the desired equality holds by definition.

Otherwise, $\mathrm{V}(\mathfrak{q})$ is of dimension $\leqslant k$. In that case Λ is empty and the right-hand side vanishes trivially. If $f \in \mathfrak{q}$, then $\mathrm{M}/f\mathrm{M} = (\mathrm{A}/\mathfrak{q})/f(\mathrm{A}/\mathfrak{q}) = \mathrm{A}/\mathfrak{q}$ and similarly ${}_f\mathrm{M} = {}_f(\mathrm{A}/\mathfrak{q}) = \mathrm{A}/\mathfrak{q}$, so the left-hand side also vanishes. If $f \notin \mathfrak{q}$, then since $\mathrm{V}(\mathfrak{q})$ is of dimension $\leqslant k$, both $\mathrm{M}/f\mathrm{M}$ and ${}_f\mathrm{M}$ have supports of dimension $\leqslant k-1$, hence again the left-hand side vanishes.

This shows the case where $\mathrm{M} = \mathrm{A}/\mathfrak{q}$. In general, one reduces to this case as follows. Fix an element $f \in \mathrm{A}$ and say a f.g. A-module M is *f-good* if its support is of dimension $\leqslant k+1$ and $f \notin \mathfrak{p}$ for every prime $\mathfrak{p}$ corresponding to an $(k+1)$-dimensional irreducible component of $\mathrm{Supp}_\mathrm{A}(\mathrm{M})$. One shows that both sides of the formula are additive in short exact sequences of f-good modules (details omitted). Then the claim follows for any f-good M by choosing a filtration of M whose successive quotients are of the form $\mathrm{A}/\mathfrak{q}$ with $\mathfrak{q}$ prime. □

Proof of Theorem. It suffices to show that, for every $(k+1)$-dimensional $\mathrm{V}(\mathfrak{p}) \subseteq |\mathrm{Spec}(\mathrm{A})|$, we have

$$\phi^*(\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f)) \in \mathrm{R}_{k+d}(\mathrm{B}).$$

By definition, $\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f) = [\mathrm{M}]_k$ where $\mathrm{M} = (\mathrm{A}/\mathfrak{p})/f(\mathrm{A}/\mathfrak{p})$. We first show that $\phi^*(\mathrm{div}_{\mathrm{V}(\mathfrak{p})}(f)) = [\mathrm{N}]_{k+d}$, where $\mathrm{N} = (\mathrm{B}/\mathfrak{p}\mathrm{B})/f(\mathrm{B}/\mathfrak{p}\mathrm{B})$. Choose a filtration of M where the successive quotients are $\mathrm{A}/\mathfrak{q}_i$ for prime ideals $\mathfrak{q}_i \subset \mathrm{A}$. Then $[\mathrm{M}]_n = \sum[\mathrm{V}(\mathfrak{q}_i)]$ where the sum is taken over i such that $\mathrm{V}(\mathfrak{q}_i)$ is n-dimensional. Tensoring with the flat A-module B produces a filtration of N where the successive quotients are $\mathrm{B}/\mathfrak{q}_i\mathrm{B}$. By definition, $\phi^*[\mathrm{V}(\mathfrak{q}_i)] = [\mathrm{B}/\mathfrak{q}_i\mathrm{B}]_{k+d}$ for each i. When $\mathrm{V}(\mathfrak{q}_i)$ is of dimension $< k$ then the irreducible components of $\mathrm{V}(\mathfrak{q}_i\mathrm{B})$ have dimension $< k+d$ (since ϕ is flat of relative dimension d). Therefore we get

$$[\mathrm{N}]_{k+d} = \sum[\mathrm{B}/\mathfrak{q}_i\mathrm{B}]_{k+d} = \sum \phi^*[\mathrm{V}(\mathfrak{q}_i)] = \phi^*[\mathrm{M}]_k$$

where the sums are taken over i such that $\mathrm{V}(\mathfrak{q}_i)$ is k-dimensional.

Since $V(\mathfrak{p})$ is of dimension $k+1$ and ϕ is flat of relative dimension d, $V(\mathfrak{p}B)$ is of pure dimension $k+d+1$. Since $A/\mathfrak{p}$ is an integral domain, f is a non-zero-divisor (as it is nonzero). Since $\phi : A \to B$ is flat, its image in B is still a non-zero-divisor (e.g., $\mathrm{Kosz}_B(f) \simeq \mathrm{Kosz}_A(f) \otimes_A B$ is still acyclic in positive degrees). Thus the previous lemma shows that $[N]_{k+d} = [(B/\mathfrak{p}B)/f(B/\mathfrak{p}B)]$ is rationally equivalent to zero. □

Construction 30. Let $\phi : A \to B$ be a flat ring homomorphism of relative dimension d. Then for every k, the homomorphism $\phi^* : Z_k(A) \to Z_{k+d}(B)$ descends to a canonical homomorphism

$$\phi^* : \mathrm{CH}_k(A) \to \mathrm{CH}_{k+d}(B)$$

which we call the homomorphism of *inverse image* along f.