Lecture 7
Dévissage, localization and supports

7.1. Dévissage.

Construction 1. Let $\mathcal{A}$ be an abelian category. Then $K_0(\mathcal{A})$ is the free abelian group on isomorphism classes of objects of $\mathcal{A}$, modulo the relations

$$[A] = [A'] + [A'']$$

for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$.

Example 2. Let A be a noetherian ring. Then $K_0(\mathrm{Mod}_A^{fg}) = G_0(A)$ by definition.

Definition 3. Let A be a ring and $I \subset A$ an ideal. An A-module M is called I^∞*-torsion* if it is I^k-torsion for some $k \geqslant 0$, i.e., $I^k M = 0$. If I is generated by a single element $f \in A$, we also use the term f^∞*-torsion*. Let $\mathrm{Mod}_A^{fg}(I^\infty)$ (resp. $\mathrm{Mod}_A^{fg}(f^\infty)$) denote the full subcategory of Mod_A^{fg} spanned by I^∞-torsion modules (resp. f^∞-torsion modules).

Remark 4. Let A be a noetherian ring, $I \subset A$ an ideal, and $\phi : A \twoheadrightarrow A/I$ the quotient homomorphism. Then the restriction of scalars functor

$$(-)_{[A]} : \mathrm{Mod}_{A/I}^{fg} \to \mathrm{Mod}_A^{fg}$$

lands in the full subcategory $\mathrm{Mod}_A^{fg}(I^\infty)$. Indeed, we have $IM_{[A]} = 0$ for every A/I-module M.

Theorem 5 (Dévissage)**.** *Let* A *be a noetherian ring,* $I \subset A$ *an ideal, and* $\phi : A \twoheadrightarrow A/I$ *the quotient homomorphism. Then the restriction of scalars functor induces a canonical isomorphism*

$$\phi_* : G_0(A/I) \xrightarrow{\sim} G_0(\mathrm{Mod}_A^{fg}(I^\infty)).$$

Example 6. If I is a nil ideal, then every A-module M is I^∞-torsion, so $\mathrm{Mod}_A^{fg}(I^\infty) = \mathrm{Mod}_A^{fg}$. In that case, we recover the nil-invariance property proven in §6.4: $G_0(A/I) \xrightarrow{\sim} G_0(A)$.

Proof. Note that if M is I^k-torsion, then it admits a filtration

$$0 = I^k M \subset I^{k-1} M \subset \cdots \subset IM \subset M$$

whose successive quotients are I-torsion, hence are A/I-modules. Thus we may define an inverse map

$$[M] \mapsto \sum_{i>0} [I^i M / I^{i-1} M].$$

The rest of the proof is the same as that of nil-invariance (§6.4). □

7.2. Digression: quotients of abelian categories.

Definition 7. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a non-empty full subcategory. We say that $\mathcal{B}$ is a *Serre subcategory* if, for any short exact sequence

$$0 \to \mathrm{A}' \to \mathrm{A} \to \mathrm{A}'' \to 0,$$

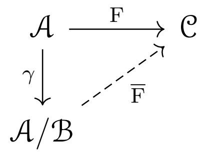
we have $\mathrm{A} \in \mathcal{B}$ iff $\mathrm{A}' \in \mathcal{B}$ and $\mathrm{A}'' \in \mathcal{B}$. In other words, $\mathcal{B}$ should be closed under subobjects, quotients, and extensions.

Remark 8. Note that if $\mathcal{B} \subseteq \mathcal{A}$ is a Serre subcategory, then it contains the zero object $0 \in \mathcal{A}$. Also, $\mathcal{B}$ is abelian and the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is an exact functor.

Theorem 9. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a Serre subcategory. Then there exists a universal exact functor*

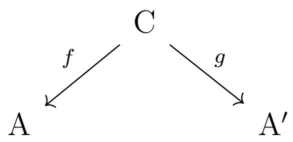
$$\gamma : \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

with kernel $\mathcal{B}$. Universality means that for any exact functor $\mathrm{F} : \mathcal{A} \to \mathcal{C}$ *with* $\mathrm{F}(b) = 0$ *for all $b \in \mathcal{B}$, there exists a unique exact functor* $\overline{\mathrm{F}} : \mathcal{A}/\mathcal{B} \to \mathcal{C}$ *making the triangle below commute.*

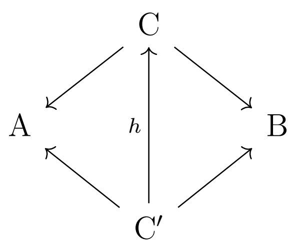


Construction 10. The quotient $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ can be described as the localization (in the sense of Gabriel–Zisman) at the class of morphisms $f : \mathrm{A} \to \mathrm{A}'$ in $\mathcal{A}$ with $\mathrm{Ker}(f) \in \mathcal{B}$ and $\mathrm{Coker}(f) \in \mathcal{B}$. We sketch a concrete construction.

The objects of $\mathcal{A}/\mathcal{B}$ are the same as those of $\mathcal{A}$; we write $\gamma(\mathrm{A}) \in \mathcal{A}/\mathcal{B}$ for the object corresponding to an object $\mathrm{A} \in \mathcal{A}$. Morphisms $\gamma(\mathrm{A}) \to \gamma(\mathrm{A}')$ are equivalences classes of diagrams in $\mathcal{A}$



where f has kernel and cokernel in $\mathcal{B}$. Two such diagrams $(\mathrm{A} \leftarrow \mathrm{C} \to \mathrm{A}')$ and $(\mathrm{A} \leftarrow \mathrm{C}' \to \mathrm{A}')$ are equivalent if there exists a morphism $h : \mathrm{C}' \to \mathrm{C}$ making the diagram below commute.



The composition law is defined as follows. Given two morphisms $\gamma(\mathrm{A}) \to \gamma(\mathrm{A}')$ and $\gamma(\mathrm{A}') \to \gamma(\mathrm{A}'')$, represented by diagrams $(\mathrm{A} \leftarrow \mathrm{C} \to \mathrm{A}')$ and $(\mathrm{A}' \leftarrow \mathrm{C}' \to \mathrm{A}'')$, respectively, the composite $\gamma(\mathrm{A}) \to \gamma(\mathrm{A}'')$ is represented by

$$
\begin{array}{ccccc}
 & & \mathrm{C} \times_{\mathrm{A}'} \mathrm{C}' & & \\
 & \swarrow & & \searrow & \\
 & \mathrm{C} & & \mathrm{C}' & \\
 \swarrow & & \searrow \quad \swarrow & & \searrow \\
\mathrm{A} & & \mathrm{A}' & & \mathrm{A}''
\end{array}
$$

where the square in the middle is cartesian. The functor $\gamma : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ acts on morphisms by sending $f : \mathrm{A} \to \mathrm{A}'$ to the equivalence class of the diagram

$$
\begin{array}{ccc}
 & \mathrm{A} & \\
 /\!/ & & \searrow^{f} \\
\mathrm{A} & & \mathrm{A}'.
\end{array}
$$

7.3. Localization. Let $\mathcal{B} \subseteq \mathcal{A}$ be a Serre subcategory. Let $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ be the inclusion and $\gamma : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the quotient functor. Both functors are exact and induce canonical homomorphisms

$$\iota_* : \mathrm{K}_0(\mathcal{B}) \to \mathrm{K}_0(\mathcal{A})$$

and

$$\gamma_* : \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{A}/\mathcal{B}).$$

Theorem 11. *The sequence*

$$\mathrm{K}_0(\mathcal{B}) \xrightarrow{\iota_*} \mathrm{K}_0(\mathcal{A}) \xrightarrow{\gamma_*} \mathrm{K}_0(\mathcal{A}/\mathcal{B}) \to 0$$

is exact.

Proof. Surjectivity of γ_* is clear since γ is the identity on objects. Note that $\gamma_*\iota_* = 0$ since $\gamma \circ \iota = 0$ by construction. Therefore there is a canonical morphism

$$\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B}) := \mathrm{Coker}(\iota_*) \to \mathrm{K}_0(\mathcal{A}/\mathcal{B}).$$

We claim that an inverse is given by the assignment

$$\gamma_*[\mathrm{A}] \mapsto [\mathrm{A}],$$

where $\mathrm{A} \in \mathcal{A}$.

To show this is well-defined, we have to show that if $\gamma(\mathrm{A}) \simeq \gamma(\mathrm{A}')$ is an isomorphism in $\mathcal{A}/\mathcal{B}$, then $[\mathrm{A}] = [\mathrm{A}']$ in $\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B})$. Such an isomorphism can be represented by a diagram

$$\mathrm{A} \xleftarrow{f} \mathrm{C} \xrightarrow{g} \mathrm{A}'$$

where f and g both have kernel and cokernel contained in $\mathcal{B}$. From the short exact sequences

$$0 \to \mathrm{Ker}(f) \hookrightarrow \mathrm{C} \twoheadrightarrow \mathrm{Im}(f) \to 0,$$
$$0 \to \mathrm{Im}(f) \hookrightarrow \mathrm{A} \twoheadrightarrow \mathrm{Coker}(f) \to 0,$$

we see that

$$[\mathrm{C}] = [\mathrm{A}] + [\mathrm{Ker}(f)] - [\mathrm{Coker}(f)] = [\mathrm{A}'] + [\mathrm{Ker}(g)] - [\mathrm{Coker}(g)].$$

In particular, we deduce $[\mathrm{A}] - [\mathrm{A}'] = 0$ in $\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B})$.

It remains to show that if

$$0 \to \gamma(\mathrm{A}') \xrightarrow{a} \gamma(\mathrm{A}) \xrightarrow{b} \gamma(\mathrm{A}'') \to 0$$

is a short exact sequence in $\mathcal{A}/\mathcal{B}$, then

$$[\mathrm{A}] = [\mathrm{A}'] + [\mathrm{A}'']$$

holds in $\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B})$. Choose a diagram

$$\begin{array}{ccc} & \mathrm{C} & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ \mathrm{A} & & \mathrm{A}'' \end{array}$$

representing the morphism b. As above, we have $[\mathrm{C}] = [\mathrm{A}] + [\mathrm{Ker}(f)] - [\mathrm{Coker}(f)]$ in $\mathrm{K}_0(\mathcal{A})$ and hence $[\mathrm{C}] = [\mathrm{A}]$ in $\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B})$ (since $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathcal{B}$). Now consider the morphism g. Since γ is exact, there is an exact sequence

$$0 \to \gamma(\mathrm{Ker}(g)) \to \gamma(\mathrm{C}) \xrightarrow{\gamma(g)} \gamma(\mathrm{A}'') \to \gamma(\mathrm{Coker}(g)) \to 0.$$

Consider the commutative diagram

$$\begin{array}{ccc} & \gamma(\mathrm{C}) & \\ {\scriptstyle \gamma(f)}\swarrow & & \searrow{\scriptstyle \gamma(g)} \\ \gamma(\mathrm{A}) & \xrightarrow{\qquad b \qquad} & \gamma(\mathrm{A}'') \end{array}$$

Since $\gamma(f)$ is an isomorphism and b is surjective, we deduce that $\gamma(g)$ is surjective. In particular, $\gamma(\mathrm{Coker}(g)) = 0$ and $\gamma(\mathrm{Ker}(g)) \simeq \gamma(\mathrm{A}')$ in $\mathcal{A}/\mathcal{B}$. Thus by above it follows that $[\mathrm{Ker}(g)] = [\mathrm{A}']$ in $\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B})$. Finally we have

$$[\mathrm{A}] = [\mathrm{C}] = [\mathrm{A}''] + [\mathrm{Ker}(g)] - [\mathrm{Coker}(g)] = [\mathrm{A}''] + [\mathrm{A}']$$

in $\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B})$, as desired. This concludes the construction of the inverse map $\mathrm{K}_0(\mathcal{A}/\mathcal{B}) \to \mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{B})$, and hence the proof. □

Theorem 12. *Let* A *be a noetherian ring and* $f \in \mathrm{A}$ *an element. Then the extension of scalars functor* $\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}} \to \mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}[f^{-1}]}$ *induces an equivalence of categories*

$$\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}/(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(f^{\infty})) \to \mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}[f^{-1}]}.$$

Proof. Exercise. □

Corollary 13 (Localization theorem)**.** *Let* A *be a noetherian ring and* $f \in \mathrm{A}$ *an element. Let* $\phi : \mathrm{A} \to \mathrm{A}[f^{-1}]$ *and* $\psi : \mathrm{A} \to \mathrm{A}/\langle f \rangle$*. Then there is an exact sequence*

$$\mathrm{G}_0(\mathrm{A}/\langle f \rangle) \xrightarrow{\psi_*} \mathrm{G}_0(\mathrm{A}) \xrightarrow{\phi^*} \mathrm{G}_0(\mathrm{A}[f^{-1}]) \to 0$$

Proof. From the two previous theorems, we have the exact sequence

$$\mathrm{K}_0(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(f^{\infty})) \to \mathrm{G}_0(\mathrm{A}) \xrightarrow{\phi^*} \mathrm{G}_0(\mathrm{A}[f^{-1}]) \to 0.$$

The claim follows by combining this with the dévissage theorem (§7.1). □

Corollary 14. *Let* A *be a regular ring and* $f \in \mathrm{A}$ *an element. Let* $\phi : \mathrm{A} \to \mathrm{A}[f^{-1}]$. *Then there is an exact sequence*

$$\mathrm{G}_0(\mathrm{A}/\langle f\rangle) \to \mathrm{K}_0(\mathrm{A}) \xrightarrow{\phi^*} \mathrm{K}_0(\mathrm{A}[f^{-1}]) \to 0,$$

where the first map is $\mathrm{G}_0(\mathrm{A}/\langle f\rangle) \to \mathrm{G}_0(\mathrm{A}) \simeq \mathrm{K}_0(\mathrm{A})$.

Proof. Recall that $\mathrm{A}[f^{-1}]$ is also regular (§2.3). Thus the claim follows by combining the previous Corollary with the comparison of K-theory and G-theory for regular rings (§5.3), and observing that $\phi^* : \mathrm{K}_0(\mathrm{A}) \to \mathrm{K}_0(\mathrm{A}[f^{-1}])$ and $\phi^* : \mathrm{G}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A}[f^{-1}])$ are compatible with this comparison. □

7.4. Spectrum of a ring.

Definition 15. Let A be a commutative ring. The *underlying set of the Zariski spectrum* of A, denoted

$$\bigl|\mathrm{Spec}(\mathrm{A})\bigr|,$$

is the set of equivalence classes of morphisms $\mathrm{A} \to \kappa$, where κ is a field. Two morphisms $\mathrm{A} \to \kappa_1$ and $\mathrm{A} \to \kappa_2$ are equivalent if there exists a field κ_3 and a commutative square

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \kappa_2 \\ \downarrow & & \downarrow \\ \kappa_1 & \longrightarrow & \kappa_3. \end{array}$$

Example 16 (Fields)**.** Let k be a field. Then $\bigl|\mathrm{Spec}(k)\bigr|$ is the set of equivalence classes $[k \to \kappa]$, where κ is a field. But every $k \to \kappa$ is equivalent to the identity $k \to k$, so $\bigl|\mathrm{Spec}(k)\bigr| \simeq \{*\}$.

Example 17 (The dual numbers)**.** Let k be a field and $\mathrm{A} = k[\varepsilon]/\langle \varepsilon^2\rangle$. The data of a field κ and a ring homomorphism $\mathrm{A} \to \kappa$ is the same as that of a field extension κ/k and an element $x \in \kappa$ satisfying $x^2 = 0$. But then $x = 0$ necessarily. Thus every equivalence class $[\mathrm{A} \to \kappa]$ is equal to $[\mathrm{A} \to k]$, and we get $\bigl|\mathrm{Spec}(\mathrm{A})\bigr| \simeq \{*\}$.

Example 18 (The integers)**.** Since $\mathbf{Z}$ is the initial commutative ring, specifying the data of a field κ and a ring homomorphism $\mathbf{Z} \to \kappa$ is the same as specifying the field κ. Moreover, $\mathbf{Z} \to \kappa$ factors through either $\mathbf{Q}$ or $\mathbf{F}_p$, where p is a prime, depending on the characteristic of κ. Therefore we find

$$\bigl|\mathrm{Spec}(\mathbf{Z})\bigr| = \bigl\{[\mathbf{Z} \to \mathbf{Q}]\bigr\} \cup \bigl\{[\mathbf{Z} \to \mathbf{F}_p] \mid p \text{ prime}\bigr\}.$$

Example 19 (Polynomial rings)**.** Let k be a field and $\mathrm{A} = k[\mathrm{T}]$ the polynomial ring. The data of a field κ and a ring homomorphism $k[\mathrm{T}] \to \kappa$ is the same as that of a field extension κ/k and an element $\alpha \in \kappa$. Moreover, the morphism $k[\mathrm{T}] \to \kappa$

will factor through the subfield $k(\alpha) \subseteq \kappa$ generated by α, so $[\mathrm{A} \to \kappa] = [\mathrm{A} \to k(\alpha)]$. Therefore we have

$$\left|\mathrm{Spec}(k[\mathrm{T}])\right| = \left\{[k[\mathrm{T}] \to k(\alpha)] \mid \kappa/k \text{ a field extension, } \alpha \in \kappa\right\}.$$

We can say more. Either $k[\mathrm{T}] \to k(\alpha)$ is injective or not injective, depending on whether α is transcendental or algebraic. In the first case, it factors through the field of fractions and induces an isomorphism $k(\mathrm{T}) \simeq k(\alpha)$. In the second, it induces an isomorphism $k[\mathrm{T}]/\langle f\rangle \simeq k(\alpha)$, where f is an (irreducible) minimal polynomial of α. Thus we can write

$$\left|\mathrm{Spec}(k[\mathrm{T}])\right| = \left\{[k[\mathrm{T}] \to k(\mathrm{T})]\right\} \cup \left\{[k[\mathrm{T}] \to k[\mathrm{T}]/\langle f\rangle] \mid f \text{ irred. polynomial}\right\}.$$

If k is algebraically closed, then

$$\left|\mathrm{Spec}(k[\mathrm{T}])\right| = \left\{[k[\mathrm{T}] \to k(\mathrm{T})]\right\} \cup \left\{[k[\mathrm{T}] \xrightarrow{\mathrm{T}\mapsto\alpha} k] \mid \alpha \in k\right\}.$$

Definition 20. A *point* p of a commutative ring A is an equivalence class $[\mathrm{A} \to \kappa]$, i.e., an element $p \in \left|\mathrm{Spec}(\mathrm{A})\right|$.

Definition 21. The *residue field* of a point p is a field $\kappa(p)$, together with a ring homomorphism $\mathrm{A} \to \kappa(p)$, such that every homomorphism $\mathrm{A} \to \kappa$ equivalent to p factors through $\kappa(p)$.

Remark 22. Let A be a ring. Let $\phi : \mathrm{A} \to \kappa$ be a ring homomorphism with κ a field. Since κ is local, ϕ factors through a local homomorphism $\mathrm{A}_{\mathfrak{p}} \to \kappa$, where $\mathfrak{p} = \mathrm{Ker}(\phi)$. Since it kills the maximal ideal, it factors further through $\kappa(\mathfrak{p}) = \mathrm{A}_{\mathfrak{p}}/\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$. Thus we have $[\mathrm{A} \to \kappa] = [\mathrm{A} \to \kappa(\mathfrak{p})]$ in $\left|\mathrm{Spec}(\mathrm{A})\right|$. This shows that every point $p = [\mathrm{A} \to \kappa]$ has a (unique) residue field $\kappa(p) = \kappa(\mathfrak{p})$ where $\mathfrak{p} = \mathrm{Ker}(\mathrm{A} \to \kappa)$.

Remark 23. Using the above remark, one can show that $\left|\mathrm{Spec}(\mathrm{A})\right|$ is in bijection with the set of prime ideals $\mathfrak{p} \subset \mathrm{A}$.

Definition 24. A point $p \in \left|\mathrm{Spec}(\mathrm{A})\right|$ is *closed* if the homomorphism $\mathrm{A} \to \kappa(p)$ is surjective.

Definition 25. If A is an integral domain, a *generic point* is a point $\eta \in \left|\mathrm{Spec}(\mathrm{A})\right|$ such that $\mathrm{A} \to \kappa(\eta)$ is injective. From the universal property of the field of fractions $\mathrm{A} \to \mathrm{Frac}(\mathrm{A})$, it follows that $\kappa(\eta) = \mathrm{Frac}(\mathrm{A})$. In particular A admits a unique generic point.

Example 26. Above we showed that the residue fields of the points of $\mathbf{Z}$ are $\mathbf{Q}$ and $\mathbf{F}_p$. All points are closed except the generic point $[\mathbf{Z} \to \mathbf{Q}]$. Similarly we saw that the residue fields of the points of $k[\mathrm{T}]$ are $k(\mathrm{T})$ and k (when k is algebraically closed). All points are closed except the generic point $[k[\mathrm{T}] \to k(\mathrm{T})]$.

Example 27. The definition of generic point could be made more generally, but it is not very useful when A is not an integral domain. For example, note that for $\mathrm{A} = k[\varepsilon]/\langle\varepsilon^2\rangle$ there is no injection $\phi : \mathrm{A} \to \kappa$ where κ is a field. Indeed ϕ factors through $\mathrm{A} \twoheadrightarrow k \to \kappa$ and the first map is not injective. Thus A admits no generic point (it has only a closed point $[\mathrm{A} \twoheadrightarrow k]$).