Lecture 6

Products, functoriality and nil-invariance

6.1 Products

6.2 Functoriality in K-theory

6.3 Functoriality in G-theory

6.4 Nil-invariance in G-theory

6.1 Products

Recall: There is a cup product:

$$K_0(A) \otimes K_0(A) \xrightarrow{\cup} K_0(A)$$
$$[M] \otimes [N] \longmapsto [M \otimes_A N]$$

Construction (Cup product):

$$K_0(\mathrm{Perf}_A) \otimes K_0(\mathrm{Perf}_A) \xrightarrow{\cup} K_0(\mathrm{Perf}_A)$$
$$[M_\bullet] \otimes [N_\bullet] \longmapsto [M_\bullet \overset{L}{\otimes}_A N_\bullet]$$

Well-defined: $\overset{L}{\otimes}$ preserves exact triangles (in each arg.)

$\Rightarrow K_0(\mathrm{Perf}_A)$ commutative ring with unit $[A[0]]$.

Remark:
The diagram

$$\begin{tikzcd}
K_0(A) \otimes K_0(A) \arrow[r, "\cup"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circ"] & K_0(A) \arrow[d, "\wr"] \\
K_0(\mathrm{Perf}A) \otimes K_0(\mathrm{Perf}A) \arrow[r] & K_0(\mathrm{Perf}A)
\end{tikzcd}$$

commutes, since $\overset{L}{\otimes}$ agrees with $\otimes$ on f.g. projectives.

In particular, $K_0(A) \xrightarrow{\sim} K_0(\mathrm{Perf}A)$ is a ring homomorphism.

__Lemma:__ $M_\bullet \in Ch_A$ of Tor-amplitude $[a,b]$

$N_\bullet \in Ch_A$ m-connective and __n-coconnective__ $(n \geq m \geq 0)$

$\quad$ (i.e. $H_i(N_\bullet) = 0 \quad \forall i \notin [m,n]$)

$\Rightarrow \; H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} N_\bullet) = 0 \quad \forall i \notin [a+m, b+n]$

__Proof:__

By shifting $N_\bullet$, we may assume $m = 0$.

__$n=0$:__ $\quad N_\bullet \simeq H_0(N_\bullet)[0]$

$\quad \Rightarrow$ claim follows from def. of finite Tor-amplitude

__$n>0$:__ $\quad H_n(N_\bullet)[n] \to \underset{\substack{\text{qis}\,\wr \\ N_\bullet}}{\tau_{\leq n}(N_\bullet)} \to \tau_{\leq n-1}(N_\bullet)$ exact triangle

$\Rightarrow M_\bullet \overset{L}{\underset{A}{\otimes}} H_n(N_\bullet)[n] \longrightarrow M_\bullet \overset{L}{\underset{A}{\otimes}} \tau_{\leq n}(N_\bullet) \longrightarrow M_\bullet \overset{L}{\underset{A}{\otimes}} \tau_{\leq n-1}(N_\bullet)$

exact triangle

$\Rightarrow H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} N_\bullet) \simeq H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} \tau_{\leq n}(N_\bullet))$ fits in a LES

$$\cdots \xrightarrow{\partial} \underset{\substack{\| \\ 0 \text{ for } i \notin [a+m,\, b+n]}}{H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} H_n(N_\bullet)[n])} \to H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} \tau_{\leq n} N_\bullet) \to \underset{\substack{\| \\ 0 \;\; \forall i \notin [a,\, b+n-1] \\ \text{(induction)}}}{H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} \tau_{\leq n-1} N_\bullet)} \xrightarrow{\partial} \cdots$$

$\Rightarrow$ conclude by induction. $\blacksquare$

<u>Construction</u> (Cap product):

$$\begin{tikzcd} K_0(A) \otimes G_0(A) \arrow[r, "\cap"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circ"] & G_0(A) \arrow[d, "\wr"] \\ K_0(\mathrm{Perf}_A) \otimes K_0(\mathrm{Coh}_A) \arrow[r, "\cap"] & K_0(\mathrm{Coh}\,A) \\ {[M_\bullet] \otimes [N_\bullet]} \arrow[r, mapsto] & {[M_\bullet \overset{L}{\underset{A}{\otimes}} N_\bullet]} \end{tikzcd}$$

Follows from lemma that

$$M_\bullet \in \mathrm{Perf}_A,\ N_\bullet \in \mathrm{Coh}_A \implies M_\bullet \overset{L}{\underset{A}{\otimes}} N_\bullet \in \mathrm{Coh}_A.$$

This gives $K_0(\mathrm{Coh}_A)$ the structure of $K_0(\mathrm{Perf}_A)$-module.

6.2 Functoriality in K-theory

$\varphi : A \to B$ homo. of comm. rings

$\varphi^* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ (Lecture 1)

$M \mapsto M \otimes_A B$

$\varphi_* : \mathrm{Mod}_B \to \mathrm{Mod}_A$

$N \mapsto N_{[A]}$

Recall:
- φ^* preserves f.g. projective modules (always)
- φ_* preserves f.g. projective modules
  $\Leftrightarrow$ B is a f.p. flat A-module

Construction: φ^* induces a well-defined ring homo.

$$\varphi^* : K_0(A) \to K_0(B).$$

If B is a f.g. flat A-module, then φ_* induces

$$\varphi_* : K_0(B) \to K_0(A).$$

$$N \mapsto N_{[A]}$$

Remark: φ_* is not a ring homo. (unless $\varphi : A \xrightarrow{\sim} B$ iso.)
It sends the unit $1 = [B] \in K_0(B)$ to $[B_{[A]}]$ instead of the unit $1 = [A] \in K_0(A)$.

Prop: $\varphi : A \to B$ ring homo.

(i) $M_\bullet \in \mathrm{Perf}_A \Rightarrow M_\bullet \overset{L}{\underset{A}{\otimes}} B \in \mathrm{Perf}_B$

(ii) If A noetherian,
B f.g. A-module of finite Tor-amplitude,
$N_\bullet \in \mathrm{Perf}_B \Rightarrow (N_\bullet)_{[A]} \in \mathrm{Perf}_A$
(where $(N_\bullet)_{[A]} = (\cdots \to (N_n)_{[A]} \to (N_{n-1})_{[A]} \to \cdots)$)

Proof:

(i) $P_\bullet \xrightarrow{qis} M_\bullet \qquad P_\bullet \in \mathrm{Proj}^b_A$

$\Rightarrow P_\bullet \underset{A}{\otimes} B \simeq P_\bullet \overset{L}{\underset{A}{\otimes}} B \xrightarrow{qis} M_\bullet \overset{L}{\underset{A}{\otimes}} B$

with $P_\bullet \underset{A}{\otimes} B \in \mathrm{Proj}^b_B$.

(ii) $N_\bullet \in \mathrm{Perf}_B \Leftrightarrow N_\bullet$ coherent and finite Tor-amp.

$\Rightarrow N_\bullet$ of Tor-amplitude $[a,b]$ $\quad (a \leq b \in \mathbb{Z})$

$$H_i\big((N_\bullet)_{[A]} \overset{L}{\underset{A}{\otimes}} M\big) = H_i\big(N_\bullet \overset{L}{\underset{B}{\otimes}} B \overset{L}{\underset{A}{\otimes}} M\big)$$

B of Tor-amplitude $\leq n \Rightarrow B \overset{L}{\underset{A}{\otimes}} M$ is n-coconnective

Lemma $\Rightarrow N_\bullet \overset{L}{\underset{B}{\otimes}} (B \overset{L}{\underset{A}{\otimes}} M)$ is a-connective, $(b+n)$-coconn.

$\Rightarrow H_i\big((N_\bullet)_{[A]} \overset{L}{\underset{A}{\otimes}} M\big) = 0 \quad \forall i \notin [a, b+n]$

$\Rightarrow (N_\bullet)_{[A]}$ is of finite Tor-amplitude (over A)

$\Rightarrow N_\bullet \in \mathrm{Perf}_A$. $\blacksquare$

Corollary: There are well-defined homos.

$$\varphi^*: K_0(\mathrm{Perf}_A) \longrightarrow K_0(\mathrm{Perf}_B) \qquad \varphi: A \to B \text{ arbitrary}$$
$$[M_\bullet] \longmapsto [M_\bullet \overset{L}{\underset{A}{\otimes}} B]$$

$$\varphi_*: K_0(\mathrm{Perf}_B) \longrightarrow K_0(\mathrm{Perf}_A) \qquad A \text{ noeth.}$$
$$[N_\bullet] \longmapsto [(N_\bullet)_{[A]}] \qquad B \text{ finite, f Tor}/A.$$

Proof: $(-)\overset{L}{\underset{A}{\otimes}} B$ preserves exact triangles, so
$\Rightarrow \varphi^*$ well-defined.
$(-)_{[A]}$ also clearly preserves exact triangles
(it is an exact functor $\mathrm{Ch}_B \to \mathrm{Ch}_A$)
$\Rightarrow \varphi_*$ well-defined. ∎

Remark: $\varphi^*: K_0(\mathrm{Perf}_A) \to K_0(\mathrm{Perf}_B)$ is a ring homo.

$$[A[0]] \longmapsto [B[0]]$$
$$[M_\bullet] \cup [N_\bullet] \mapsto [M_\bullet \overset{L}{\underset{A}{\otimes}} B] \cup [N_\bullet \overset{L}{\underset{A}{\otimes}} B]$$
$$= [M_\bullet \overset{L}{\underset{A}{\otimes}} N_\bullet] \qquad = [(M_\bullet \overset{L}{\underset{A}{\otimes}} N_\bullet) \underset{A}{\otimes} B]$$

The diagram

$$\begin{tikzcd} K_0(A) \arrow[r, "\varphi^*"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circ"] & K_0(B) \arrow[d, "\wr"] \\ K_0(\mathrm{Perf}_A) \arrow[r, "\varphi^*"] & K_0(\mathrm{Perf}_B) \end{tikzcd}$$

commutes.

If B finite flat $/A$ (A noetherian), then

$$\begin{tikzcd} K_0(B) \arrow[r, "\varphi_*"] \arrow[d, "\wr"'] & K_0(A) \arrow[d, "\wr"] \\ K_0(\mathrm{Perf}_B) \arrow[r, "\varphi_*"] & K_0(\mathrm{Perf}_A) \end{tikzcd}$$

commutes.

<u>Notation</u>: From now on, we identify $K_0(A) = K_0(\mathrm{Perf}_A)$.

<u>Prop</u> (Base change formula):

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \arrow[d, "\psi'"] \\ A' \arrow[r, "\varphi'"'] & B' = A' \otimes_A B \end{tikzcd}$$

φ arbitrary

A noetherian, A' f.g., f.T.a./A

Suppose the square is <u>Tor-independent</u>,

i.e. $A' \overset{L}{\underset{A}{\otimes}} B$ acyclic in positive degrees.

$$\begin{tikzcd} K_0(A') \arrow[r, "(\varphi')^*"] \arrow[d, "\psi_*"'] \arrow[dr, phantom, "\circ"] & K_0(B') \arrow[d, "(\psi')_*"] \\ K_0(A) \arrow[r, "\varphi^*"'] & K_0(B) \end{tikzcd}$$

Then $\psi'_*(\varphi')^* = \varphi^*\psi_*$.

Proof: $M'_\bullet \in \mathrm{Perf}_{A'}$

$$\varphi^* \psi_* [M'_\bullet] = \varphi^* [(M'_\bullet)_{[A]}] = [(M'_\bullet)_{[A]} \overset{L}{\underset{A}{\otimes}} B]$$

$$(\psi')_* (\varphi')^* [M'_\bullet] = (\psi')_* [M'_\bullet \overset{L}{\underset{A'}{\otimes}} B'] = [(M'_\bullet \overset{L}{\underset{A'}{\otimes}} B')_{[B]}]$$

$$\Rightarrow (M'_\bullet)_{[A]} \overset{L}{\underset{A}{\otimes}} B \underset{\text{qis}}{\simeq} M'_\bullet \overset{L}{\underset{A'}{\otimes}} A' \overset{L}{\underset{A}{\otimes}} B$$
$$\underset{\text{qis}}{\simeq} M'_\bullet \overset{L}{\underset{A'}{\otimes}} (A' \underset{A}{\otimes} B) \qquad \text{(tor-independence)}$$
$$\simeq M'_\bullet \overset{L}{\underset{A'}{\otimes}} B'. \quad \blacksquare$$

Prop (Projection formula):

$\varphi : A \to B$ $\qquad$ A noeth, B f.g., f.T.a. $/A$.

$\varphi_* : K_0(B) \longrightarrow K_0(A)$ is a $K_0(A)$-module homo.

That is:

$$\varphi_*(x) \cup y = \varphi_*(x \cup \varphi^*(y))$$

$\forall x \in K_0(B)$, $y \in K_0(A)$.

Proof: Follows from

$$(N_\bullet)_{[A]} \overset{L}{\underset{A}{\otimes}} M_\bullet \underset{\text{qis}}{\simeq} N_\bullet \overset{L}{\underset{B}{\otimes}} (M_\bullet \underset{A}{\otimes} B) \qquad M_\bullet \in \mathrm{Perf}_A$$

$N_\bullet \in \mathrm{Perf}_B$. $\blacksquare$

Prop: $\varphi: A \to B$ $\quad \psi: B \to C$ ring homos.

(i) $\psi^* \circ \varphi^* = (\psi \circ \varphi)^* : K_0(A) \to K_0(C)$

(ii) If A noeth, B f.g., f.T.a./A, C f.g., f.T.a./B

$\Longrightarrow \quad \varphi_* \circ \psi_* = (\psi \circ \varphi)_* : K_0(C) \to K_0(A)$.

Proof: From the definitions.

6.3 Functoriality in G-theory

(all rings noetherian)

$\varphi : A \to B \quad B$ f.g. $/A$

$\Rightarrow \mathrm{Mod}_B^{fg} \to \mathrm{Mod}_A^{fg}, \quad N \mapsto N_{[A]} \quad$ exact

$\Rightarrow \varphi_* : G_0(B) \to G_0(A)$

$\varphi : A \to B \quad B$ flat $/A$

$\Rightarrow \mathrm{Mod}_A^{fg} \to \mathrm{Mod}_B^{fg}, \quad M \mapsto M \otimes_A B \quad$ exact

$\Rightarrow \varphi^* : G_0(A) \to G_0(B)$

<u>Exercise:</u> (using Lemma in §6.1)

If $\varphi : A \to B$, B f.T.a. $/A$, $M_\bullet \in \mathrm{Coh}_A \Rightarrow M_\bullet \otimes_A B \in \mathrm{Coh}_B$.

<u>Prop:</u> There are well-defined homos.

$\varphi_* : K_0(\mathrm{Coh}_B) \to K_0(\mathrm{Coh}_A) \qquad \varphi : A \to B, \ B$ f.g. $/A$

$\varphi^* : K_0(\mathrm{Coh}_A) \to K_0(\mathrm{Coh}_B) \qquad \varphi : A \to B, \ B$ f.T.a. $/A$

They are compatible with the above maps via the isos. $G_0(A) \xrightarrow{\sim} K_0(\mathrm{Coh}_A)$, $G_0(B) \xrightarrow{\sim} K_0(\mathrm{Coh}_B)$.

They are functorial $\quad (\psi^* \circ \varphi^* = (\psi \circ \varphi)^*,$

$\varphi_* \circ \psi_* = (\psi \circ \varphi)_*$

when these make sense).

They satisfy the base change formula for Tor-independent squares.

They satisfy the projection formula:

$\varphi : A \to B, \quad B$ f.g. $/A \quad \Rightarrow \varphi_* : K_0(\mathrm{Coh}_B) \to K_0(\mathrm{Coh}_A)$

is a $K_0(A)$-module homo

Equivalently, $\quad \varphi_*(x) \cap y = \varphi_*(x \cap \varphi^*(y)) \in K_0(\mathrm{Coh}_A)$

$\forall x \in K_0(\mathrm{Coh}_B), \ y \in K_0(A)$

<u>Proof</u>: Almost the same as K-theory version ■

<u>Notation</u>: From now on, identify $G_0(A) = K_0(\mathrm{Coh}_A)$.

6.4 Nil-invariance in G-theory (all rings noetherian)

Def: A: noetherian, $I \subseteq A$ ideal

I is a <u>nil ideal</u> $\Leftrightarrow \forall x \in I,\ x^n = 0\ \forall n \gg 0$

$\underset{\text{(noeth.)}}{\Leftrightarrow}\quad I^n = 0 \quad \forall n \gg 0$

Theorem: $I \subseteq A$ nil ideal. $\varphi : A \twoheadrightarrow A/I$.

$\Rightarrow \quad \varphi_* : G_0(A/I) \xrightarrow{\sim} G_0(A)$ iso.

Observation: $M \in \mathrm{Mod}_A^{fg}$,

$0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ filtration.

$$\Rightarrow \quad \begin{array}{l} 0 \to M_{n-1} \hookrightarrow M_n \twoheadrightarrow M_n/M_{n-1} \to 0 \\ 0 \to M_{n-2} \hookrightarrow M_{n-1} \twoheadrightarrow M_{n-1}/M_{n-2} \to 0 \\ \qquad\vdots \\ 0 \to \underset{\overset{\|}{0}}{M_0} \hookrightarrow M_1 \twoheadrightarrow \underset{\overset{\wr}{M_1}}{M_1/M_0} \to 0 \end{array}$$

short exact

$$\Rightarrow \quad [M] = [M_n] = [M_n/M_{n-1}] + [M_{n-1}] = \cdots = \sum_{i=1}^{n} [M_i/M_{i-1}] \quad \text{in } G_0(A).$$

Proof of Theorem:

I nilpotent $\Longrightarrow \exists n \in \mathbb{N}$ s.th. $I^n = 0$.

$M \in \mathrm{Mod}_A^{fg}$

$M_i := I^{n-i} \cdot M \quad \Longrightarrow$ finite filtration $(M_i)_{0 \le i \le n}$

$0 = M_0 \subset \dots \subset M_n = M$.

M_i / M_{i-1} are all annihilated by I

$I \cdot (M_1 / M_0) = I \cdot I^{n-1} \cdot M = 0$

$I \cdot (M_2 / M_1) = I \cdot I^{n-2} M / I^{n-1} M = I^{n-1} M / I^{n-1} M = 0$

...

$\Longrightarrow M_i / M_{i-1}$ is in ess. image of $\mathrm{Mod}_{A/I} \hookrightarrow \mathrm{Mod}_A$ (lecture 1)

$\Longrightarrow [M] = \sum_{i=1}^n \varphi_* [N_i] = \varphi_* \left(\sum_{i=1}^n [N_i] \right)$

where $N_i = M_i / M_{i-1}$ viewed as A/I-modules

$\rightarrow \varphi$ surjective

Inverse map $G_0(A) \xrightarrow{f} G_0(A/I)$:

$f[M] := \sum_{i=1}^n [N_i]$ (choose a filtration and set N_i as above)

- $f[M]$ doesn't depend on the filtration:
Choose another filtration. By butterfly lemma (a.k.a. Zassenhaus lemma, c.f. [Bourbaki, Algebra I,

Chap. I, §7, Theorem 5]) the two filtrations have "equivalent refinements". Thus we can assume the second filtration is a refinement of the first.

Suffices: $0 = M_0 \subset \cdots \subset M_{i-1} \subset M_i \subset \cdots \subset M_n = M$
(refinement) $0 = M_0 \subset \cdots \subset M_{i-1} \subset N \subset M_i \subset \cdots \subset M_n = M$

$\Rightarrow$ $0 \to N/M_{i-1} \hookrightarrow M_i/M_{i-1} \twoheadrightarrow M_i/N \to 0$ exact
$\Rightarrow$ $[M_i/M_{i-1}] = [N/M_{i-1}] + [M_i/N]$
$\Rightarrow$ claim follows.

- Well-definedness of $f: G_0(A) \to G_0(A/I)$
  $0 \to M' \to M \to M'' \to 0$ exact in Mod_A^{fg}
  want: $f[M] = f[M'] + f[M'']$
  Choose filtrations for M', M'' with quotients $\in \mathrm{Mod}_{A/I}$.
  Filtration on $M'' = M/M'$ $\Leftrightarrow$ filtration of M containing M'
  combine with filtration of M' $\Rightarrow$ filtration of M with quotients $\in \mathrm{Mod}_{A/I}$.
  Using this to compute $f[M]$, we get
  $f[M] = f[M'] + f[M'']$.

▶ $f \circ \varphi_* = \mathrm{id}$.

$N \in \mathrm{Mod}_{A[I]}$ $\implies$ $N_{[A]}$ has trivial filtration

$\implies$ $f(\varphi_*[N]) = f[N_{[A]}] = [N]$.

$\implies$ φ_* injective. ■