Lecture 4

Algebraic K-theory of perfect complexes

4.1 Perfect complexes

A: ring. $M_\bullet$: (unbounded) chain complex of A-modules

Def: $M_\bullet$ is a perfect complex
$\iff$ $\exists P_\bullet$ bounded complex ($P_i = 0\ \forall i \gg 0, i \ll 0$)
with P_i f.g. projectives
and $P_\bullet \xrightarrow{q.i.s} M_\bullet$ a quasi-iso

Example: $M \in \mathrm{Mod}_A^{fg}$ is perfect (§2.1)
$\iff$ $M[0]$ is a perfect complex

Notation: Ch_A = category of chain complexes of A-modules.

morphisms: $M_\bullet \to N_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r] & M_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] \arrow[dr, phantom, "\circ"] & M_0 \arrow[r, "d_0"] \arrow[d, "\varphi_0"] \arrow[dr, phantom, "\circ"] & M_{-1} \arrow[r] \arrow[d, "\varphi_{-1}"] & \cdots \\
\cdots \arrow[r] & N_1 \arrow[r, "d_1"'] & N_0 \arrow[r, "d_0"'] & N_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

($\circ$ = commutative square)

Construction:

$M_\bullet \in \mathrm{Ch}(\mathrm{Mod}_A) \rightsquigarrow M_\bullet[n]$ shift by $n \in \mathbb{Z}$

$$\left(M_\bullet[n]\right)_k = M_{k-n}, \quad d_k^{M_\bullet[n]} = (-1)^n \cdot d_{k-n}^M$$

Example: $M \in \mathrm{Mod}_A$.

We denoted $M[0] = (0 \to \underset{\text{degree } 0}{M} \to 0) \in \mathrm{Ch}(\mathrm{Mod}_A)$

We have $(M[0])[n] = (0 \to \underset{\text{degree } n}{M} \to 0)$

(Set $M[n] := (M[0])[n]$.)

Remark: $H_n(M_\bullet[n]) = H_0(M_\bullet) \quad \forall n \in \mathbb{Z}$.

Constr (Mapping cone): $\varphi : M_\bullet \to N_\bullet$ morphism in $\mathrm{Ch}(\mathrm{Mod}_A)$

$$\begin{tikzcd} \arrow[r] & M_n \arrow[r, "d_n^M"] \arrow[d, "\varphi_n"'] \arrow[dr, phantom, "\circ"] & M_{n-1} \arrow[r, "d_{n-1}^M"] \arrow[d, "\varphi_{n-1}"] \arrow[dr, phantom, "\circ"] & \cdots \\ \arrow[r] & N_n \arrow[r, "d_n^N"'] & N_{n-1} \arrow[r, "d_{n-1}^N"'] & \cdots \end{tikzcd}$$

($\circ$ = commutative squares)

$\mathrm{Cone}(\varphi)_\bullet \in \mathrm{Ch}(\mathrm{Mod}_A)$ mapping cone

$\mathrm{Cone}(\varphi)_n = N_n \oplus M_{n-1} \quad \forall n \in \mathbb{Z}$

differentials:
$$\begin{tikzcd} N_{n+1} \arrow[r, "d_{n+1}^N"] & N_n \\ \oplus & \oplus \\ M_n \arrow[uur, "\varphi_n"] \arrow[r, "d_n^M"'] & M_{n-1} \end{tikzcd}$$

(in other words: $\begin{bmatrix} d^N_{n+1} & \varphi_n \\ 0 & -d^M_n \end{bmatrix} = d^{\mathrm{Cone}(\varphi)_\bullet}_{n+1}$)

<u>Exercise:</u> $\mathrm{Cone}(\varphi)_\bullet$ is in fact a chain complex, i.e., $d_n \circ d_{n+1} = 0 \quad \forall n \in \mathbb{Z}$.

<u>Example:</u> $M \xrightarrow{\varphi} N$ morphism in Mod_Λ

$\Rightarrow M[0] \xrightarrow{\varphi} N[0]$ morphism in $\mathrm{Ch}(\mathrm{Mod}_\Lambda)$

$$\Rightarrow \mathrm{Cone}(\varphi)_\bullet = (0 \to \underset{\deg 1}{M} \xrightarrow{\varphi} \underset{\deg 0}{N} \to 0)$$

<u>Example:</u> $M_\bullet \xrightarrow{0} N_\bullet$ zero morphism

$\Rightarrow \mathrm{Cone}(0)_\bullet = M_\bullet[1] \oplus N_\bullet$

<u>Remark:</u> For every $\varphi: M_\bullet \to N_\bullet$, we have a diagram:

$$M_\bullet \xrightarrow{\varphi} N_\bullet \xrightarrow{\mathrm{can}} \mathrm{Cone}(\varphi)_\bullet \xrightarrow{\mathrm{can}} M_\bullet[1]$$

(can = canonical morphism)

<u>Def:</u> An <u>exact triangle</u> of chain complexes of A-modules is a diagram

$$M'_\bullet \longrightarrow M_\bullet \longrightarrow M''_\bullet \longrightarrow M'_\bullet[1]$$

in Ch_A, whose image in $D(A)$ is isomorphic to a diagram of the form

$$A_\bullet \xrightarrow{\varphi} B_\bullet \xrightarrow{can} \mathrm{Cone}(\varphi)_\bullet \xrightarrow{can} A_\bullet[1].$$

That is:

$$\begin{tikzcd}
M'_\bullet \arrow[r] \arrow[d, "\wr"] & M_\bullet \arrow[r] \arrow[d, "\wr"] & M''_\bullet \arrow[r] \arrow[d, "\wr"] & M'_\bullet[1] \arrow[d, "\wr"] \\
A_\bullet \arrow[r, "\varphi"] & B_\bullet \arrow[r] & \mathrm{Cone}(\varphi)_\bullet \arrow[r] & A_\bullet[1]
\end{tikzcd}$$

in the derived category $D(A)$.

⚠ The vertical arrows are isomorphisms in $D(A)$, not necessarily represented by morphisms in Ch_A (only zig-zags).

__Notation:__ We often write an exact triangle

$$M'_\bullet \to M_\bullet \to M''_\bullet \to M'_\bullet[1]$$

simply as

$$M'_\bullet \to M_\bullet \to M'_\bullet .$$

__Prop:__ $M_\bullet \to N_\bullet \to K_\bullet \to M_\bullet[1]$ exact triangle
$\Longrightarrow$ long exact sequence of homology

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(M_\bullet) \arrow[r] & H_n(N_\bullet) \arrow[r] & H_n(K_\bullet) \arrow[dll, "\partial"'] & \\
& H_{n-1}(M_\bullet) \arrow[r] & H_{n-1}(N_\bullet) \arrow[r] & H_{n-1}(K_\bullet) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

__Corollary:__ $M_\bullet \xrightarrow{\varphi} N_\bullet$ is a quasi-iso
$\Longleftrightarrow$ $\mathrm{Cone}(\varphi)_\bullet$ is acyclic $\left(H_i(\mathrm{Cone}(\varphi)_\bullet) = 0 \ \forall i\right)$ $\blacksquare$
$\Longleftrightarrow$ $M_\bullet \xrightarrow{\varphi} N_\bullet \to 0$ exact triangle

__Lemma:__ $M_\bullet \xrightarrow{\varphi} N_\bullet \xrightarrow{\psi} K_\bullet \xrightarrow{\partial} M_\bullet[1]$ exact
$\Longleftrightarrow$ $N_\bullet \xrightarrow{\psi} K_\bullet \xrightarrow{\partial} M_\bullet[1] \xrightarrow{\varphi[1]} N_\bullet[1]$ exact $\blacksquare$

__Remark:__ In the proofs, can assume $K_\bullet = \mathrm{Cone}(\varphi)_\bullet$ since $H_*(\mathrm{Cone}(\varphi)_\bullet) \xrightarrow{\sim} H_*(K_\bullet)$ are isos.

4.2 K-theory of perfect complexes

<u>Construction</u>: A: ring.

$K_0(\mathrm{Perf}_A) :=$ free abelian group
on generators: quasi-iso. classes $[M_\bullet]$
of perfect complexes $M_\bullet$
modulo relations: $[M_\bullet] = [M'_\bullet] + [M''_\bullet]$
for all exact triangles $M'_\bullet \to M_\bullet \to M''_\bullet$

<u>Example</u>: $M_\bullet \in \mathrm{Perf}_A$ perfect complex, $n \in \mathbb{Z}$
$\Rightarrow [M_\bullet[n]] = (-1)^n \cdot [M_\bullet] \in K_0(\mathrm{Perf}_A)$

<u>Proof</u>: $M_\bullet \longrightarrow 0 \longrightarrow M_\bullet[1]$ is an exact triangle

$$\begin{tikzcd} M_\bullet \arrow[r, "0"] & 0 \arrow[r] \arrow[dr] & M_\bullet[1] \arrow[r, "id"] & M_\bullet[1] \\ & & \mathrm{Cone}(0) \arrow[u, "iso"'] \arrow[ur] & \end{tikzcd}$$

by an example we already saw

$\Rightarrow [M_\bullet[1]] = -[M_\bullet]$. Taking $M_\bullet = M_\bullet[-1]$, also get
$[M_\bullet[-1]] = -[M_\bullet]$, since $M_\bullet = M_\bullet[-1][1]$.
$\Rightarrow$ Claim follows by induction. ∎

Theorem: There is a canonical iso. of ab. groups:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\sim} & K_0(\mathrm{Perf}_A) \\ [M] & \longmapsto & [M[0]]. \end{array}$$

Well-definedness: Let $M, N \in \mathrm{Mod}_A^{\mathrm{fgproj}}$.

$\Rightarrow M[-1] \xrightarrow{0} N[0] \longrightarrow M \oplus N[0] \longrightarrow M[0]$

is an exact triangle.

$\Rightarrow [M \oplus N[0]] = [N[0]] - [M[-1]] = [N[0]] + [M[0]]$

in $K_0(\mathrm{Perf}_A)$. $\blacksquare$

Remark: We can factor the map:

$$\begin{array}{ccccc} K_0(A) & \xrightarrow[(1)]{} & K_0(\mathrm{Proj}_A^b) & \xrightarrow[(2)]{} & K_0(\mathrm{Perf}_A) \\ [M] & \longmapsto & [M[0]] & & \\ & & [M_\bullet] & \longmapsto & [M_\bullet] \end{array}$$

where $K_0(\mathrm{Proj}_A^b)$ is a variant of $K_0(\mathrm{Perf}_A)$, where the generators are quasi-iso. classes of objects of

$$\mathrm{Proj}_A^b = \{\text{bounded complexes } P_\bullet \text{ with } P_i \text{ f.g. projective } \forall i \in \mathbb{Z}\}.$$

(questionable notation)

The relations are the same.

(**Note:** $\mathrm{Proj}_A^b \subsetneq \mathrm{Perf}_A$ strict inclusion.)

<u>Proof</u> that (1) is an iso:

Consider $\chi : K_0(\mathrm{Proj}^b_A) \longrightarrow K_0(A)$

$$[P_\bullet] \longmapsto \sum_{i\in\mathbb{Z}} (-1)^i \cdot [P_i]$$

<u>Exercise</u>: $\chi((\mathrm{Cone}(\varphi)_\bullet) = \chi(Q_\bullet) - \chi(P_\bullet)$ for any $P_\bullet \xrightarrow{\varphi} Q_\bullet$

$P_\bullet \xrightarrow{\varphi} Q_\bullet$ qis $\implies$ $K_\bullet = \mathrm{Cone}(\varphi)_\bullet$ acyclic

$\Rightarrow 0 \to K_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} K_1 \overset{d_1}{\twoheadrightarrow} K_0 \longrightarrow 0$ exact

$$\left.\begin{array}{l}
0 \to \underset{\substack{\| \\ \mathrm{Im}(d_2)}}{\mathrm{Ker}(d_1)} \hookrightarrow K_1 \twoheadrightarrow K_0 \to 0 \\[2em]
0 \to \underset{\substack{\| \\ \mathrm{Im}(d_3)}}{\mathrm{Ker}(d_2)} \to K_2 \twoheadrightarrow \mathrm{Im}(d_2) \to 0 \\[2em]
\quad\vdots \\
0 \to \underset{\substack{\| \\ 0}}{\mathrm{Ker}(d_n)} \to K_n \overset{\sim}{\twoheadrightarrow} \mathrm{Im}(d_n) \to 0
\end{array}\right\}\text{short exact seqs.}$$

$$\begin{aligned}
\implies \chi(K_\bullet) &= [K_0] - [K_1] + \cdots + (-1)^n \cdot [K_n] \\
&= [K_0] - ([K_0] + [\mathrm{Im}(d_2)]) + \cdots + (-1)^n \cdot [K_n] \\
&= \cdots = 0
\end{aligned}$$

$\implies \chi(P_\bullet) = \chi(Q_\bullet)$

$\implies \chi$ only depends on qis. class.

$P_\bullet \xrightarrow{\varphi} Q_\bullet \to K_\bullet$ exact triangle in Proj^b_A

$$\begin{tikzcd} \Rightarrow P_\bullet \arrow[r, "\varphi"] & Q_\bullet \arrow[r] \arrow[dr, bend right] & K_\bullet \arrow[r] & P_\bullet[1] \\ & & \mathrm{Cone}(\varphi)_\bullet \arrow[u, "\text{q.i.}"] \arrow[ur] & \end{tikzcd}$$

$\Rightarrow \chi(K_\bullet) = \chi(\mathrm{Cone}(\varphi)_\bullet) = \chi(Q_\bullet) - \chi(P_\bullet)$

$\Rightarrow \chi$ well-defined.

▶ Check: $K_0(A) \longrightarrow K_0(\mathrm{Proj}^b_A) \xrightarrow{\chi} K_0(A)$ identity.

$[M] \longmapsto [M[0]] \longmapsto \chi(M[0])$

$\chi(M[0]) = [M]$ by def.

▶ Check: $K_0(\mathrm{Proj}^b_A) \xrightarrow{\chi} K_0(A) \longrightarrow K_0(\mathrm{Proj}^b_A)$ identity.

$\Leftrightarrow [P_\bullet] = \sum_{i\in\mathbb{Z}} (-1)^i [P_i]$ in $K_0(\mathrm{Proj}^b_A)$ for every $P_\bullet \in \mathrm{Proj}^b_A$.

Induction on the length of $P_\bullet$.

If $P_\bullet$ concentrated in degree n

$\Rightarrow P_\bullet = P_n[n] \Rightarrow [P_\bullet] = [P_n[n]] = (-1)^n \cdot [P_n]$.

Induction step:

$$\begin{tikzcd} P_\bullet = (0 \arrow[r] & P_b \arrow[r] & \cdots \arrow[r] & P_a \arrow[r] & 0) & (a<b) \\ Q_\bullet = \arrow[u] & & (0 \arrow[r] & P_a \arrow[r] \arrow[u, "\mathrm{id}"] & 0) & \end{tikzcd}$$

Exercise: There is an exact triangle

$$Q_\bullet \longrightarrow P_\bullet \longrightarrow K_\bullet$$

where

$$\begin{tikzcd} K_\bullet = (0 \arrow[r] & P_b \arrow[r] & \cdots \arrow[r] & P_{a+1} \arrow[r] & 0 \arrow[r] & 0) \\ P_\bullet = (0 \arrow[r] & P_b \arrow[r] \arrow[u] & \cdots \arrow[r] & P_{a+1} \arrow[r] \arrow[u] & P_a \arrow[r] \arrow[u] & 0) \end{tikzcd}$$

$K_\bullet$ has length less than that of $P_\bullet$

$$\Rightarrow \chi(K_\bullet) = \sum_{a+1 \le i \le b} (-1)^i \cdot [P_i] \quad \text{by induction}$$

$$\begin{aligned} \Rightarrow \chi(P_\bullet) &= \chi(Q_\bullet) + \chi(K_\bullet) \quad \text{by exact triangle} \\ &= \chi(P_a[a]) + \chi(K_\bullet) \\ &= (-1)^a \cdot [P_a] + \sum_{a+1 \le i \le b} (-1)^i \cdot [P_i] \\ &= \sum_{a \le i \le b} (-1)^i \cdot [P_i]. \quad \blacksquare \end{aligned}$$

This concludes the proof that (1) is an iso.

<u>Proof</u> that (2) is an iso:

$$K_0(\mathrm{Proj}^b_A) \longrightarrow K_0(\mathrm{Perf}_A)$$
$$[P_\bullet] \longmapsto [P_\bullet]$$

Note: if $M_\bullet$ perfect $\Longrightarrow \exists$ qis $\underset{\mathrm{Proj}^b_A}{\overset{\in}{P_\bullet}} \xrightarrow{\varphi} M_\bullet$

$\Longrightarrow$ $P_\bullet \xrightarrow{\varphi} M_\bullet \longrightarrow 0$ exact triangle

$\Longrightarrow$ $[M_\bullet] = [P_\bullet]$

So the sets of generators are the same.

Relations:

$$\begin{tikzcd}
M_\bullet \arrow[r, "\varphi"] & N_\bullet \arrow[r] \arrow[dr] & K_\bullet & \text{exact} \\
& & \mathrm{Cone}(\varphi)_\bullet \arrow[u, "\text{qis}"'] & \\
P_\bullet \arrow[uu, "\text{qis}"] \arrow[r, dotted, "\psi"] \arrow[uur, phantom, "h"] & Q_\bullet \arrow[uu, "\text{qis}"'] \arrow[r, dotted] & \mathrm{Cone}(\psi)_\bullet \arrow[u, dotted, "\text{qis}"'] & \in \mathrm{Ch}^b_A
\end{tikzcd}$$

(h = commutes up to homotopy)

Given exact triangle of perfect complexes (above row), choose $P_\bullet, Q_\bullet \in \mathrm{Proj}^b_A$ resolutions of $M_\bullet, N_\bullet$.
Since P_i projective, can lift to a map $P_\bullet \xrightarrow{\psi} Q_\bullet$ making the square commute up to homotopy ⊛.

$\Longrightarrow$ induced qis $\mathrm{Cone}(\psi)_\bullet \to \mathrm{Cone}(\varphi)_\bullet$

$\Longrightarrow$ $[P_\bullet] + [\mathrm{Cone}(\psi)_\bullet] = [Q_\bullet] \in K_0(\mathrm{Proj}^b_A)$. ∎

⊛ See [Ro], Thm. 3.1.7 for a proof.

4.3 G-theory of coherent complexes

It is also useful to have a version of $G_0(A)$ built out of chain complexes.

Definition: A: noetherian ring,
$M_\bullet$: complex of A-modules.

$M_\bullet$ is coherent $\iff$ $M_\bullet$ homologically bounded
$(H_i(M_\bullet) = 0 \ \forall i \gg 0, i \ll 0)$
and $H_i(M_\bullet) \in \mathrm{Mod}_A^{fg} \ \forall i \in \mathbb{Z}$

Construction:
$K_0(\mathrm{Coh}_A) :=$ free abelian group
on generators $[M_\bullet]$ = quasi-iso. classes of coherent complexes $M_\bullet$
modulo relations $[M'_\bullet] + [M''_\bullet] = [M_\bullet]$
for all exact triangles $M'_\bullet \to M_\bullet \to M''_\bullet$

Example: Every perfect complex is coherent.
$\Rightarrow$ canonical map $K_0(\mathrm{Perf}_A) \longrightarrow K_0(\mathrm{Coh}_A)$.

Example: If $M \in \mathrm{Mod}_A^{fg}$, then $M[0]$ is coherent.

$0 \to M' \to M \to M'' \to 0$ s.e.s. in Mod_A^{fg}

$\Rightarrow M'[0] \xrightarrow{\varphi} M[0] \to M''[0]$ exact triangle

$$\begin{tikzcd}
(0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0) = \mathrm{Cone}(\varphi)_\bullet \arrow[d, "\text{qis}"] \\
(0 \arrow[r] & 0 \arrow[r] & M'' \arrow[r] & 0) = M''[0]
\end{tikzcd}$$

$\Rightarrow [M[0]] = [M'[0]] + [M''[0]]$ in $K_0(\mathrm{Coh}_A)$

$\Rightarrow$ canonical map $G_0(A) \longrightarrow K_0(\mathrm{Coh}_A)$.

Theorem: A noetherian. $G_0(A) \xrightarrow{\sim} K_0(\mathrm{Coh}_A)$ iso.

Key point: For any $M_\bullet$ coherent,
$$[M_\bullet] = \sum_{i \in \mathbb{Z}} (-1)^i \cdot [H_i(M)[0]] \quad \text{in } K_0(\mathrm{Coh}_A).$$

And $[H_i(M)] \in G_0(A)$ since $H_i(M) \in \mathrm{Mod}_A^{fg}$.

4.A. Appendix: the derived category

Remark: A quasi-isomorphism $M_\bullet \longrightarrow N_\bullet$ does not necessarily admit a section $N_\bullet \longrightarrow M_\bullet$, even up to homotopy.

Example: Let $f \in A \setminus \{0\}$ be a non-zero-divisor. Then $M_\bullet = \mathrm{Kosz}_A(f)_\bullet$ is a free resolution of $N_\bullet = A/\langle f \rangle$, so we have a quasi-iso.

$$M_\bullet \xrightarrow[\text{qis}]{} N_\bullet .$$

Let's try to construct a section $N_\bullet \longrightarrow M_\bullet$.

$$\begin{tikzcd}
N_\bullet \arrow[d, dotted] & = & (0 \arrow[r] \arrow[d] & A/\langle f \rangle \arrow[r] \arrow[d, dotted] & 0) \\
M_\bullet & = (0 \arrow[r] & A \arrow[r, "f"] & A \arrow[r] & 0)
\end{tikzcd}$$

A-linear map $A/\langle f \rangle \longrightarrow A$

$\Longleftrightarrow$ A-linear map $A \longrightarrow A$ killing $\langle f \rangle$

$\Longleftrightarrow$ an element $a \in A$ s.th. $af = 0$.

If A is a domain, then $a = 0$ necessarily.

So the only morphism $N_\bullet \to M_\bullet$ is the zero map.

<u>Definition:</u> $M_\bullet, N_\bullet \in Ch_A$ are <u>quasi-isomorphic</u>
$\Leftrightarrow$ there exists a zigzag

$$M_\bullet \xleftarrow{qis} L^1_\bullet \xrightarrow{qis} L^2_\bullet \xleftarrow{qis} \cdots \xleftarrow{qis} L^k_\bullet \xrightarrow{qis} N_\bullet .$$

<u>Notation:</u> $M_\bullet \underset{qis}{\sim} N_\bullet$

<u>Construction:</u> There exists a universal functor

$$Ch_A \longrightarrow D(A)$$

inverting quasi-isomorphisms.

Universality:

$\forall F : Ch_A \to \mathcal{C}$ s.th.
$F(s)$ iso. $\forall$ qis s,
$\exists! \ \tilde{F} : D(A) \to \mathcal{C}$
extending F.

$$\begin{tikzcd} Ch_A \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathcal{C} \\ D(A) \arrow[ur, dashed, "\exists! \tilde{F}"'] & \end{tikzcd}$$

commutes up to natural iso.

<u>Sketch</u>: Define a directed graph:

- For every object $M_\bullet \in Ch_A$, add a vertex $[M_\bullet]$ to the graph.
- For every morphism $M_\bullet \xrightarrow{\varphi} N_\bullet$ in Ch_A, add a corresponding edge $[\varphi]$ from $[M_\bullet]$ to $[N_\bullet]$
- For every quasi-isomorphism $M_\bullet \xrightarrow{\varphi} N_\bullet$, add an edge $[\varphi]^{-1}$ from $[N_\bullet]$ to $[M_\bullet]$

This graph freely generates a category, whose objects are the vertices, and morphisms are finite sequences of edges. The identity morphisms are given by empty sequences of edges, and composition is concatenation.

Finally, we impose these relations on the morphisms:

- $[id_{M_\bullet}] = id_{[M_\bullet]}$ $\quad \forall M_\bullet$
- $[\psi] \circ [\varphi] = [\psi \circ \varphi]$ $\quad \forall \varphi, \psi$ morphisms
- $[\varphi]^{-1} \circ [\varphi] = id_{[M_\bullet]}$ $\quad \forall M_\bullet \xrightarrow{\varphi} N_\bullet$ qis
- $[\varphi] \circ [\varphi]^{-1} = id_{[N_\bullet]}$.

This satisfies the universal property of $D(A)$.

<u>Reference</u>: [Gelfand-Manin, III, §2]