Lecture 2

Perfect modules and regularity

2019-10-23

2.1 Perfect modules

Last lecture we saw the notion of (finite) f.g. proj. resolution.

Construction: Let A noetherian, $M \in \mathrm{Mod}_A^{fg}$. Then there is a fg free resolution $P_\bullet \xrightarrow{q.is} M$:

$$P_0 = A^{\oplus n_0} \twoheadrightarrow M$$

$$K = \mathrm{Ker}(P_0 \twoheadrightarrow M) \text{ is f.g.}$$

$$\Rightarrow P_1 = A^{\oplus n_1} \twoheadrightarrow K \subset P_0$$

etc...

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[d] \\ & & & & M \end{tikzcd}$$

f.g. free resolution

However, $P_\bullet$ may not be finite.

Example: $A = \mathbb{Z}/\langle 4 \rangle$, $M = \mathbb{Z}/2\mathbb{Z}$

$M \in \mathrm{Mod}_A^{fg}$ admits no finite f.g. proj. resolution.

Def: $M \in \mathrm{Mod}_A^{fg}$ is *perfect*
$\iff$ admits a finite f.g. proj. resolution

Def: $M \in \mathrm{Mod}_A^{fg}$ is *of Tor-amplitude $\leq n$*
$\iff$ $\mathrm{Tor}_i^A(M,N) := H_i(M \overset{L}{\underset{A}{\otimes}} N)$ vanishes
$\forall N \in \mathrm{Mod}_A$, $i > n$

M is *of finite Tor-amplitude*
$\iff$ of Tor-amplitude $\leq n$, for some $n \in \mathbb{N}$

Notation: $M \overset{L}{\underset{A}{\otimes}} N := P_\bullet \otimes N$ where $P_\bullet \xrightarrow{q.i.} M$
f.g. proj. resolution

Lemma:

(i) $0 \to M' \to M \twoheadrightarrow M'' \to 0$ exact
If two are of finite Tor-amplitude,
so is the third.

(ii) If M is a direct summand
of a f.T.a. module, then M is f.T.a.

Proof: (i) Tor LES. (ii) Obvious. ∎

Prop: A noetherian. $M \in \mathrm{Mod}_A^{fg}$

M perfect $\iff$ M finite Tor-amplitude

Proof:

$\Rightarrow$ If M perfect, $\exists\, P_\bullet \xrightarrow{qis} M$ fg. proj. resolution.

$\forall N \in \mathrm{Mod}_A$, $i \gg 0$,

$$\mathrm{Tor}_i^A(M,N) = H_i(P_\bullet \otimes_A N) = 0$$

since $P_\bullet$ finite.

$\Leftarrow$ Say M Tor-amplitude $\leq n$ $(n \in \mathbb{N})$.

Argue by induction on n.

$n=0$ $\Rightarrow$ M flat, f.p. $\Rightarrow$ fg. proj.
$\Rightarrow$ perfect

$n>0$ Choose an exact sequence

$$0 \to K \hookrightarrow A^{\oplus k} \twoheadrightarrow M \to 0$$

Since A noetherian, K is f.g.

K is of Tor-amplitude $\leq n-1$, by the Tor long exact sequence:

$$\underset{\substack{0 \\ \text{(assumption)}}}{\cancel{\mathrm{Tor}^A_{n+1}(M,N)}} \xrightarrow{\partial} \mathrm{Tor}^A_n(K,N) \longrightarrow \underset{\substack{0 \\ (A^{\oplus k} \text{ flat, } n>0)}}{\cancel{\mathrm{Tor}^A_n(A^{\oplus k},N)}}$$

By induction, K is perfect.

$\Longrightarrow \exists Q_\bullet \xrightarrow{\text{qis}} K$ finite f.g. proj. resolution

$$\begin{array}{ccc} P_\bullet := (\cdots \to Q_2 \to Q_1 \to Q_0 \to & A^{\oplus k} &) \\ \text{qis}\downarrow & \downarrow & \\ M[0] & M & \end{array}$$

$\rightsquigarrow$ M perfect. $\blacksquare$

<u>Remark</u>: The noetherian assumption is not necessary.

2.2 Minimal resolutions

Def: A local ring, $\underline{m} \subset A$ maximal ideal.

$k = A/\underline{m}$.

$M_\bullet$ = chain complex of A-modules

$M_\bullet$ is minimal if the differentials d_i vanish modulo $\underline{m}$.

$\iff \operatorname{Im}(d_i : M_i \to M_{i-1}) \subseteq \underline{m} M_{i-1} \quad \forall i$.

Remark: Every $M \in \mathrm{Mod}_A^{fg}$ admits a minimal free resolution.

$P_0 = A^{\oplus n_0} \twoheadrightarrow M$ corresp. to a minimal set of generators for M

$\Rightarrow P_0 \otimes_A k \xrightarrow{\sim} M \otimes_A k$ iso.

$\Rightarrow$ If $K = \ker(P_0 \twoheadrightarrow M)$, $K \subseteq \underline{m} P_0$

$P_1 := A^{\oplus n_1} \twoheadrightarrow K \subset \underline{m} P_0 \subset P_0$

continue inductively ...

Exercise: $M_\bullet \xrightarrow{\varphi} N_\bullet$ morphism of minimal complexes bounded below ($M_i = N_i = 0 \ \forall i \ll 0$) and M_i, N_i free $\forall i \in \mathbb{Z}$.

If φ is a qis $\Rightarrow$ φ iso.

Corollary: Minimal resolutions are unique up to iso.

Remark: Let $M \in \mathrm{Mod}_A^{fg}$.

Choose $P_\bullet \xrightarrow{qis} M$ a finite fg free resolution.

We can turn this into a minimal resolution:

$P_\bullet$ not minimal

$\Rightarrow$ the matrix defining some $d_i : P_i \to P_{i-1}$ has some entry which is a unit in A

$\Rightarrow$ up to a change of bases of P_i, P_{i-1}, matrix looks like $\begin{pmatrix} 1 & 0 \\ 0 & ? \end{pmatrix}$

$\Rightarrow$ $P_\bullet$ splits: $P_\bullet \simeq P'_\bullet \oplus Q_\bullet$

where $Q_\bullet = (0 \to \underset{Q_i}{A} \xrightarrow{id} \underset{Q_{i-1}}{A} \to 0)$

and $P'_\bullet = (\cdots \to P_{i+2} \to P'_{i+1} \to P'_i \to P_{i-1} \to \cdots)$

where $rk(P'_{i+1}) = rk(P_{i+1}) - 1$,

$rk(P'_i) = rk(P_i) - 1$.

and $P'_\bullet \xrightarrow{qis} M$

Keep removing such trivial summands until we get a minimal resolution.

<u>Remark</u>: $M \in \mathrm{Mod}_A^{fg}$, $P_\bullet \xrightarrow{qis} M$ minimal res.

$$\Longrightarrow \quad \mathrm{Tor}_i^A(M, k) = H_i(M \overset{L}{\underset{A}{\otimes}} k) \cong H_i(P_\bullet \otimes_A k) \cong P_i \otimes_A k$$

since $P_\bullet \otimes_A k$ has zero differentials.

Proposition: A local ring

residue field $\kappa = A/\underline{m}$ perfect

$\iff$ every $M \in \mathrm{Mod}_A^{fg}$ perfect

Proof: $M \in \mathrm{Mod}_A^{fg}$.

$\exists\, P_\bullet \xrightarrow{qis} M$ minimal f.g. free res.

κ perfect $\Longrightarrow$ κ finite Tor-amplitude

$\Longrightarrow \mathrm{Tor}_i^A(M, \kappa) = 0 \quad \forall\, i \gg 0$

$\cong P_i \otimes_A \kappa$

$\Longrightarrow P_i = 0 \quad \forall\, i \gg 0$ (P_i free)

$\Longrightarrow P_\bullet$ finite

$\Longrightarrow M$ perfect. $\blacksquare$

2.3 Regularity

Def: A: noetherian ring

A regular $\iff$ every $M \in \mathrm{Mod}_A^{fg}$ is perfect

Example: Suppose A is a local PID, $k = A/\underline{m}$

$\Rightarrow \underline{m} = \langle a \rangle$ for some $a \in A$

If $a = 0$ then $A = k$.

k is perfect as k-module $\Rightarrow$ A regular

If $a \neq 0$ then consider

$$\mathrm{Kosz}_A(a) = \left(0 \to A \xrightarrow{a} A \to 0\right)$$

Since A is a domain, $\mathrm{Ann}_A(a) = 0$

$\Rightarrow \mathrm{Kosz}_A(a)$ is a finite fgfree res. of k

$\Rightarrow k$ perfect A-module

$\Rightarrow A$ regular (by §2.2)

So: fields and DVR's are regular.

Theorem (Serre): A: noetherian local ring of Krull dimension d

A regular $\iff$ $\exists\, x_1, \dots, x_d \in A$ generating $\underline{m}$

Recall: Krull dimension = max. length of a chain of prime ideals

Proof of $\Leftarrow$: $d = \dim_\kappa(\underline{m} \otimes_A \kappa) = \dim_\kappa(\underline{m}/\underline{m}^2)$

Note: $(x_1, \dots, x_d)$ regular sequence

Proof: Under the conditions, A: integral domain [Matsumura, CRT, Thm. 14.3]

$x_1 \in \underline{m} \setminus \underline{m}^2 \Rightarrow x_1$ non-zero-divisor

and $A/\langle x_1 \rangle$ of Krull dim. $d-1$,

$\overline{x_2}, \dots, \overline{x_d}$ generate max. ideal

By induction, we conclude. ∎

$\Rightarrow$ $K = \mathrm{Kosz}_A(x_1, \dots, x_d) \xrightarrow{\S 3} \kappa$ finite f.g. free res.

$\Rightarrow$ κ perfect

$\Rightarrow$ A regular by §2.2. ∎

Proof of $\Rightarrow$: [Matsumura, CRT, Thm. 19.2]

Proposition: If A is regular, then $A_\mathfrak{p}$ is regular, for every $\mathfrak{p} \subset A$ prime.

More generally, $A[S^{-1}]$ is regular, for every multiplicative subset $S \subset A$.

Lemma: If $M \in \mathrm{Mod}_A^{f.g.}$ admits a finite proj. resolution, then it admits a finite f.g. proj. resolution. (A is any noetherian ring.)

Lemma: A regular ring
Then every A-module M admits a finite projective (resolution). (not necessarily f.g. proj.)

Proof of Proposition: $B := A[S^{-1}]$.

$\forall M \in \mathrm{Mod}_B$, $M \simeq M_{[A]} \otimes_A B$.

Since A regular, $M_{[A]}$ admits a finite resolution by projectives: $P_\bullet \xrightarrow{q.is} M_{[A]}$

$$\Rightarrow \quad P_\bullet \otimes_A B \xrightarrow{q.is} M_{[A]} \otimes_A B \simeq M$$

finite resolution of M (since B flat over A)

$\rightsquigarrow$ M perfect by Lemma. ∎

Proposition: A noetherian,

A regular $\iff$ $A_{\underline{m}}$ regular $\forall\, \underline{m} \subset A$ maximal

(Not obvious.)

Corollary: Any PID is regular.

Example: $\mathbb{Z}$ regular.

$k[x]$ regular (k field)

Exercise: If A is a regular ring, then $A[x_1, \dots, x_n]$ regular $\forall\, n \geq 0$.