Lecture 14
Comparing K-theory and the Chow groups

14.1. From algebraic cycles to K-theory.

Construction 1. Let X be a smooth k-scheme. Recall that for a closed subscheme Y, the structure sheaf $\mathcal{O}_{\mathrm{Y}}$ defines a coherent sheaf on X. Therefore we have a canonical homomorphism

$$\gamma_{\mathrm{X}} : \mathrm{Z}^*(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})$$

given by $[\mathrm{Y}] \mapsto [\mathcal{O}_{\mathrm{Y}}]$.

Remark 2. Recall the coniveau filtration on $\mathrm{G}_0(\mathrm{X})$: for each p, $\mathrm{G}_0(\mathrm{X})^{\geqslant p}$ is the subgroup generated by classes $[\mathcal{F}]$ such that $\mathrm{codim}(\mathrm{Supp}(\mathcal{F})) \geqslant p$. Note that γ_{X} sends $\mathrm{Z}^p(\mathrm{X})$ to $\mathrm{G}_0(\mathrm{X})^{\geqslant p}$. In particular, there is an induced homomorphism

$$\gamma_{\mathrm{X}} : \mathrm{Z}^p(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})^{\geqslant p}.$$

Remark 3. We proved (§9.1) that the restriction

$$\gamma_{\mathrm{X}} : \bigoplus_{c=p}^{d} \mathrm{Z}^c(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})^{\geqslant p}$$

is surjective, where $d = \dim(\mathrm{X})$. This induces a surjection

$$\gamma_{\mathrm{X}} : \mathrm{Z}^p(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})^{\geqslant p} / \mathrm{G}_0(\mathrm{X})^{\geqslant p+1}$$

for each p.

Proposition 4. *Let* X *be a smooth k-scheme of dimension d. Let $\mathcal{F}$ be a coherent sheaf on* X *whose support is of codimension $\geqslant p$. Then we have*

$$\sum_{p \leqslant i \leqslant d} \gamma([\mathcal{F}]_{d-i}) = [\mathcal{F}]$$

in $\mathrm{G}_0(\mathrm{X})^{\geqslant p} / \mathrm{G}_0(\mathrm{X})^{\geqslant p+1}$. *In particular if* $\mathrm{Supp}(\mathcal{F})$ *is of pure codimension p, then*

$$\gamma([\mathcal{F}]_{d-p}) = [\mathcal{F}]$$

in $\mathrm{G}_0(\mathrm{X})^{\geqslant p} / \mathrm{G}_0(\mathrm{X})^{\geqslant p+1}$.

Proof. The second part of the claim was proven (in the case where X is affine) in Exercise 4 on Sheet 12. A straightforward adaption of that proof also gives our more general claim. □

14.2. Intersection products. Let X be a smooth k-scheme. Since smooth k-schemes are regular, we have a canonical isomorphism $\mathrm{G}_0(\mathrm{X}) \simeq \mathrm{K}_0(\mathrm{X})$, and in particular an intersection product coming from derived tensor product. We want to compare this with the intersection product in $\mathrm{Z}^*(\mathrm{X})$.

We first consider the case of *transverse* intersection.

Definition 5. Let $i_1 : \mathrm{Z}_1 \to \mathrm{X}$ and $i_2 : \mathrm{Z}_2 \to \mathrm{X}$ be closed immersions of k-schemes. We say that Z_1 and Z_2 *intersect transversely* in X, or more precisely that the square

$$\begin{array}{ccc}
\mathrm{Z}_1 \times_{\mathrm{X}} \mathrm{Z}_2 & \longrightarrow & \mathrm{Z}_2 \\
\downarrow & & \downarrow{\scriptstyle i_2} \\
\mathrm{Z}_1 & \xrightarrow{\ i_1\ } & \mathrm{X}
\end{array}$$

is *Tor-independent*, if there exists a covering $\mathrm{X} = \bigcup_\alpha \mathrm{U}_\alpha$ by affine Zariski opens U_α such that for every α, the induced square of commutative rings

$$\begin{array}{ccc}
\Gamma(\mathrm{U}, \mathcal{O}_{\mathrm{U}}) & \longrightarrow & \Gamma(\mathrm{Z}_1 \cap \mathrm{U}, \mathcal{O}_{\mathrm{Z}_1 \cap \mathrm{U}}) \\
\downarrow & & \downarrow \\
\Gamma(\mathrm{Z}_2 \cap \mathrm{U}, \mathcal{O}_{\mathrm{Z}_2 \cap \mathrm{U}}) & \longrightarrow & \Gamma(\mathrm{Z}_1 \cap \mathrm{U}, \mathcal{O}_{\mathrm{Z}_1 \cap \mathrm{U}}) \otimes_{\Gamma(\mathrm{U}, \mathcal{O}_{\mathrm{U}})} \Gamma(\mathrm{Z}_2 \cap \mathrm{U}, \mathcal{O}_{\mathrm{Z}_2 \cap \mathrm{U}})
\end{array}$$

is Tor-independent.

Proposition 6. *Suppose that* Y *and* Z *intersect transversally in* X. *Then*

$$\gamma[\mathrm{Y}] \cup \gamma[\mathrm{Z}] = \gamma([\mathrm{Y}] \cup [\mathrm{Z}])$$

in $\mathrm{G}_0(\mathrm{X})$.

Proof. Exercise. Use the Tor formula to compute the right-hand side, and note that all the higher Tors vanish by the transversity assumption. □

In the more general case of proper intersection, we don't typically have this equality anymore. However, it still holds *modulo the coniveau filtration.*

Theorem 7. *Let* X *be a smooth quasi-projective* k*-scheme. Let* Y *and* Z *be integral closed subschemes. Suppose that* Y *and* Z *intersect properly, i.e., without excess component (§13.6), and are of codimension* p *and* q*, respectively. Then*

$$\gamma[\mathrm{Y}] \cup \gamma[\mathrm{Z}] = \gamma([\mathrm{Y}] \cup [\mathrm{Z}])$$

holds modulo the coniveau filtration*, i.e., in the quotient* $\mathrm{G}_0(\mathrm{X})^{\geqslant p+q}/\mathrm{G}_0(\mathrm{X})^{\geqslant p+q+1}$.

Proof. We only consider the affine case $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ for simplicity. We can write $\mathrm{Y} = \mathrm{Spec}(\mathrm{A}/\mathfrak{p})$ and $\mathrm{Z} = \mathrm{Spec}(\mathrm{A}/\mathfrak{q})$ for prime ideals $\mathfrak{p}$ and $\mathfrak{q}$.

The left-hand side is

$$\begin{aligned}\gamma[\mathrm{Y}] \cup \gamma[\mathrm{Z}] &= [\mathrm{A}/\mathfrak{p}] \cup [\mathrm{A}/\mathfrak{q}] \\ &= [\mathrm{A}/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}/\mathfrak{q}] \\ &= \sum_i (-1)^i [\mathrm{H}_i(\mathrm{A}/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}/\mathfrak{q})] \\ &= \sum_i (-1)^i [\mathrm{Tor}^{\mathrm{A}}_i(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q})].\end{aligned}$$

Let's compute the right-hand side. Since we are in the case of proper intersection, the product $[\mathrm{Y}] \cup [\mathrm{Z}] \in \mathrm{Z}^{p+q}(\mathrm{X})$ is defined by the Tor formula (§10.6):

$$[\mathrm{Y}] \cup [\mathrm{Z}] = \sum_i (-1)^i [\mathrm{Tor}^{\mathrm{A}}_i(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q})]_{d-p-q}.$$

If each $\mathrm{Tor}^{\mathrm{A}}_i(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q})$ had support of pure codimension $p+q$, then we would be done by the proposition in §14.1. But we only know this for $i = 0$: $\mathrm{Tor}^{\mathrm{A}}_0(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q}) = \mathrm{A}/(\mathfrak{p}+\mathfrak{q})$ has support $\mathrm{V}(\mathfrak{p}) \cap \mathrm{V}(\mathfrak{q})$ of pure codimension $p+q$ by the assumption that the intersection is proper. We also know that $\mathrm{A}/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}/\mathfrak{q}$ has support contained inside $\mathrm{V}(\mathfrak{p}) \cap \mathrm{V}(\mathfrak{q})$ by §8.5, so in particular we at least have

$$\mathrm{codim}(\mathrm{Supp}(\mathrm{Tor}^{\mathrm{A}}_i(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q}))) \geqslant p+q$$

for all i. It follows from §14.1 that the difference between the right- and left-hand sides is

$$\sum_{p+q<c\leqslant d} \sum_{i>0} (-1)^i \gamma[\mathrm{Tor}^{\mathrm{A}}_i(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q})]_{d-c}.$$

But each $[\mathrm{Tor}^{\mathrm{A}}_i(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q})]_{d-c}$ lives in $\mathrm{Z}^c(\mathrm{X})$ and is sent by γ to $\mathrm{G}_0(\mathrm{X})^{\geqslant c} \subseteq \mathrm{G}_0(\mathrm{X})^{\geqslant p+q+1}$ (since $c > p+q$). So, the difference between the two sides of the formula lives in $\mathrm{G}_0(\mathrm{X})^{\geqslant p+q+1}$. □

14.3. Flat inverse image.

Proposition 8. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a flat morphism of k-schemes. Then the square*

$$\begin{array}{ccc} \mathrm{Z}^p(\mathrm{Y}) & \xrightarrow{\gamma_{\mathrm{Y}}} & \mathrm{G}_0(\mathrm{Y})^{\geqslant p} \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{Z}^p(\mathrm{X}) & \xrightarrow{\gamma_{\mathrm{X}}} & \mathrm{G}_0(\mathrm{X})^{\geqslant p} \end{array}$$

commutes. That is, $f^*(\gamma_{\mathrm{Y}}[\mathrm{Z}]) = \gamma_{\mathrm{X}}(f^*[\mathrm{Z}])$ for every integral closed subscheme $\mathrm{Z} \subset \mathrm{Y}$.

Proof. Exercise. Similar to the analogue for intersection products (§14.2). Since the morphism is flat, we don't need to pass to the quotient by $\mathrm{G}_0(\mathrm{Y})^{\geqslant p+1}$. □

14.4. Rational equivalence. We would now like to understand whether the homomorphism

$$\gamma_{\mathrm{X}} : \mathrm{Z}^*(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})$$

respects rational equivalence, i.e., whether it descends to a homomorphism from the Chow group $\mathrm{CH}^*(\mathrm{X})$.

Recall the description of rational equivalence from last lecture (§13.6).

Construction 9. Let Z be an integral closed subscheme of $\mathrm{X} \times \mathbf{A}^1_k$ of codimension $p-1$. We set

$$\partial^0[\mathrm{Z}] := [\mathrm{Z}] \cup [\mathrm{X} \times \{0\}] \in \mathrm{Z}^p(\mathrm{X}).$$

and similarly $\partial^1[\mathrm{Z}] = [\mathrm{Z}] \cup [\mathrm{X} \times \{1\}]$. We extend ∂^0 and ∂^1 to cycles by linearity. Then $\mathrm{CH}^p(\mathrm{X})$ is the cokernel

$$\mathrm{Z}^{p-1}(\mathrm{X} \times \mathbf{A}^1_k) \xrightarrow{\partial^0 - \partial^1} \mathrm{Z}^p(\mathrm{X}) \twoheadrightarrow \mathrm{CH}^p(\mathrm{X}) \to 0.$$

Remark 10. Let $[\mathrm{Z}] \in \mathrm{Z}^{p-1}(\mathrm{X} \times \mathbf{A}^1_k)$. Our question amounts to whether the equality

$$\gamma(\partial^0[\mathrm{Z}]) = \gamma(\partial^1[\mathrm{Z}])$$

holds in $\mathrm{G}_0(\mathrm{X})$.

Let's note the following consequence of $\mathbf{A}^1$-invariance for G-theory:

Proposition 11. *Let* X *be a noetherian k-scheme. Let i_0 and i_1 be the inclusions of the closed subschemes* $\mathrm{X} \times \{0\}$ *and* $\mathrm{X} \times \{1\}$ *in* $\mathrm{X} \times \mathbf{A}^1_k$. *Then we have the equality of homomorphisms*

$$i_0^* = i_1^* : \mathrm{G}_0(\mathrm{X} \times \mathbf{A}^1) \to \mathrm{G}_0(\mathrm{X}).$$

Proof. Let $p : \mathrm{X} \times \mathbf{A}^1_k \to \mathrm{X}$ be the projection and consider the diagram

$$\begin{array}{ccc} \mathrm{G}_0(\mathrm{X} \times \mathbf{A}^1) & \rightrightarrows & \mathrm{G}_0(\mathrm{X}) \\ {\scriptstyle p^*}\uparrow & \nearrow\!\!\nearrow & \\ \mathrm{G}_0(\mathrm{X}) & & \end{array}$$

where the two horizontal arrows are i_0^* and i_1^*. By $\mathbf{A}^1$-homotopy invariance, the vertical arrow p^* is an isomorphism. Therefore, it will suffice to show $i_0^* p^* = i_1^* p^*$ (i.e., that the diagonal composites are the same). Since $p \circ i_0 = \mathrm{id} = p \circ i_1$, both of these maps are the identity. □

Remark 12. Another formulation is

$$\alpha \cup [\mathcal{O}_{\mathrm{X} \times \{0\}}] = \alpha \cup [\mathcal{O}_{\mathrm{X} \times \{1\}}] \in \mathrm{G}_0(\mathrm{X} \times \mathbf{A}^1_k)$$

for every $\alpha \in \mathrm{G}_0(\mathrm{X} \times \mathbf{A}^1_k)$. After all, $i_0^*[\mathcal{F}] = [\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathrm{X}}} \mathcal{O}_{\mathrm{X} \times \{0\}}] = [\mathcal{F}] \cup [\mathcal{O}_{\mathrm{X} \times \{0\}}]$ by definition, and similarly for i_1.

Theorem 13. *The homomorphism*

$$\gamma_{\mathrm{X}} : \mathrm{Z}^p(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})^{\geqslant p}$$

sends $\mathrm{R}^p(\mathrm{X})$ *to* $\mathrm{G}_0(\mathrm{X})^{\geqslant p+1}$*, and induces a homomorphism*

$$\gamma_{\mathrm{X}} : \mathrm{CH}^p(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})^{\geqslant p} / \mathrm{G}_0(\mathrm{X})^{\geqslant p+1}$$

for every p*.*

Proof. As discussed, we need to show

$$\gamma(\partial^0[\mathrm{Z}]) = \gamma(\partial^1[\mathrm{Z}])$$

whenever $\mathrm{Z} \subset \mathrm{X} \times \mathbf{A}^1$ is an integral closed subscheme of codimension $p-1$. In other words, we want

$$\gamma([\mathrm{Z}] \cup [\mathrm{X} \times \{0\}]) = \gamma([\mathrm{Z}] \cup [\mathrm{X} \times \{1\}])$$

modulo the coniveau filtration. But by §14.2 we have

$$\gamma([\mathrm{Z}] \cup [\mathrm{X} \times \{0\}]) = \gamma[\mathrm{Z}] \cup \gamma[\mathrm{X} \times \{0\}] = [\mathcal{O}_{\mathrm{Z}}] \cup [\mathcal{O}_{\mathrm{X}\times\{0\}}]$$

and similarly for the right-hand side. Hence the claim follows from the equality

$$[\mathcal{O}_{\mathrm{Z}}] \cup [\mathcal{O}_{\mathrm{X}\times\{0\}}] = [\mathcal{O}_{\mathrm{Z}}] \cup [\mathcal{O}_{\mathrm{X}\times\{1\}}]$$

which is the previous remark with $\alpha = [\mathcal{O}_{\mathrm{Z}}]$. □

14.5. Multiplicity of the coniveau filtration. The following theorem was proven by Grothendieck using Chow's moving lemma:

Theorem 14. *Let* X *be a smooth quasi-projective* k*-scheme. Then the coniveau filtration on* $\mathrm{G}_0(\mathrm{X})$ *is multiplicative, i.e.,*

$$x \in \mathrm{G}_0(\mathrm{X})^{\geqslant p}, y \in \mathrm{G}_0(\mathrm{X})^{\geqslant q} \implies x \cup y \in \mathrm{G}_0(\mathrm{X})^{\geqslant p+q}.$$

Remark 15. Recall from §8.5 that

$$\mathrm{Supp}(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) \subseteq \mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(\mathcal{G})$$

for perfect complexes $\mathcal{F}$ and $\mathcal{G}$. However, since $\mathrm{Supp}(\mathcal{F})$ and $\mathrm{Supp}(\mathcal{G})$ need not intersect properly, their intersection may have excess components. So $[\mathcal{F}] \in \mathrm{G}_0(\mathrm{X})^{\geqslant p}$ and $[\mathcal{G}] \in \mathrm{G}_0(\mathrm{X})^{\geqslant q}$ does not obviously imply that $[\mathcal{F}] \cup [\mathcal{G}] \in \mathrm{G}_0(\mathrm{X})^{\geqslant p+q}$. One has to use Chow's moving lemma to be able to reduce to the case of proper intersection.

14.6. The comparison.

Definition 16. Denote the graded pieces of the coniveau filtration by

$$\mathrm{Gr}^p \mathrm{G}_0(\mathrm{X}) = \mathrm{G}_0(\mathrm{X})^{\geqslant p} / \mathrm{G}_0(\mathrm{X})^{\geqslant p+1}$$

for every p.

Remark 17. As we saw in §14.1, we have surjections

$$\gamma_{\mathrm{X}} : \mathrm{Z}^p(\mathrm{X}) \to \mathrm{Gr}^p\, \mathrm{G}_0(\mathrm{X}).$$

for all p. In §14.4, we saw that these induce surjections

$$\gamma_{\mathrm{X}} : \mathrm{CH}^p(\mathrm{X}) \to \mathrm{Gr}^p\, \mathrm{G}_0(\mathrm{X}).$$

These are compatible with flat inverse images (§14.3). By §14.5, the graded abelian group

$$\mathrm{Gr}^*\, \mathrm{G}_0(\mathrm{X}) := \bigoplus_p \mathrm{Gr}^p\, \mathrm{G}_0(\mathrm{X})$$

inherits a ring structure from $\mathrm{G}_0(\mathrm{X})$. The induced map

$$\gamma_{\mathrm{X}} : \mathrm{CH}^*(\mathrm{X}) \to \mathrm{Gr}^*\, \mathrm{G}_0(\mathrm{X})$$

is a homomorphism of graded rings (essentially follows from §14.2).

In particular we see that the graded ring $\mathrm{Gr}^*\, \mathrm{G}_0(\mathrm{X})$ is a quotient of $\mathrm{CH}^*(\mathrm{X})$ by some subgroup. It turns out that this subgroup is always torsion (but can be nonzero):

Theorem 18. *Let* X *be a smooth quasi-projective* k*-scheme. The homomorphism* γ_{X} *induces an isomorphism*

$$\mathrm{CH}^*(\mathrm{X}) \otimes \mathbf{Q} \to \mathrm{Gr}^*\, \mathrm{G}_0(\mathrm{X}) \otimes \mathbf{Q}.$$

There is a "Chern character" map from K-theory to Chow. The Grothendieck–Riemann–Roch theorem implies that it becomes an inverse after tensoring with $\mathbf{Q}$.