Lecture 13
Algebraic geometry

In this lecture, we'll explain how everything we've done so far can be interpreted in the language of algebraic geometry. For simplicity, we'll restrict our attention to k-schemes over a field k.

13.1. Affine k-schemes.

Definition 1. We define *affine n-space* over k as the affine k-scheme

$$\mathbf{A}^n_k = \mathrm{Spec}(k[\mathrm{T}_1, \dots, \mathrm{T}_n])$$

for every $n \geqslant 0$.

Definition 2. Let X be a k-scheme. A *regular function* on X is a morphism of k-schemes

$$f : \mathrm{X} \to \mathbf{A}^1_k.$$

If X is affine, say $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ for some k-algebra A, then a regular function on X is the same data as that of a k-algebra morphism $k[\mathrm{T}] \to \mathrm{A}$, or equivalently that of an element $a \in \mathrm{A}$ (the image of T).

Remark 3. Let $m : \mathbf{A}^1_k \times \mathbf{A}^1_k \to \mathbf{A}^1_k$ be the multiplication morphism, corresponding to the k-algebra homomorphism

$$k[\mathrm{T}] \to k[\mathrm{T}_1] \otimes_k k[\mathrm{T}_2] \simeq k[\mathrm{T}_1, \mathrm{T}_2]$$

which sends $\mathrm{T} \mapsto \mathrm{T}_1\mathrm{T}_2$. This induces a multiplication of regular functions $f : \mathrm{X} \to \mathbf{A}^1_k$ and $g : \mathrm{X} \to \mathbf{A}^1_k$,

$$f \cdot g : \mathrm{X} \xrightarrow{(f,g)} \mathbf{A}^1_k \times \mathbf{A}^1_k \xrightarrow{m} \mathbf{A}^1_k,$$

where the first morphism is induced by the universal property of the product. One defines addition of regular functions similarly, using the addition morphism $\mathbf{A}^1_k \times \mathbf{A}^1_k \to \mathbf{A}^1_k$. These operations turns the set of regular functions on X into a *ring* (in fact a k-algebra), which we denote $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$. We call it the *ring of regular functions* or *ring of functions* on X.

Theorem 4. *The assignments* $\mathrm{X} \mapsto \Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$ *and* $\mathrm{A} \mapsto \mathrm{Spec}(\mathrm{A})$ *induce an equivalence between the category of affine k-schemes and the opposite of the category of k-algebras.*

Proof. The main point is that if $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, the ring of functions $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$ is canonically isomorphic to A as a k-algebra. □

Remark 5. If $f : \mathrm{X} \to \mathrm{S}$ and $g : \mathrm{Y} \to \mathrm{S}$ are morphisms of k-schemes, there is a *fibred product* $\mathrm{X} \times_\mathrm{S} \mathrm{Y}$ which fits into the cartesian square

$$\begin{tikzcd} \mathrm{X} \times_\mathrm{S} \mathrm{Y} \arrow[r] \arrow[d] & \mathrm{X} \arrow[d, "f"] \\ \mathrm{Y} \arrow[r, "g"] & \mathrm{S}. \end{tikzcd}$$

The ring of functions is given by the tensor product

$$\Gamma(\mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{Y}, \mathcal{O}_{\mathrm{X} \times_{\mathrm{S}} \mathrm{Y}}) \simeq \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \otimes_{\Gamma(\mathrm{S}, \mathcal{O}_{\mathrm{S}})} \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}).$$

13.2. Zero loci.

Remark 6. Note that k-scheme morphisms $\mathrm{Spec}(k) \to \mathbf{A}^n_k$ are in bijection with k-algebra homomorphisms $k[\mathrm{T}_1, \ldots, \mathrm{T}_n] \to k$. The latter are in bijection with n-tuples $(x_1, \ldots, x_n) \in k^{\times n}$. Given such an n-tuple, we write

$$\{(x_1, \ldots, x_n)\} : \mathrm{Spec}(k) \to \mathbf{A}^n_k$$

for the corresponding morphism. In case of the tuple $(0, \ldots, 0)$, we write simply $\{0\} : \mathrm{Spec}(k) \to \mathbf{A}^n_k$ for the corresponding morphism (the "inclusion of the origin").

Construction 7. Let X be a k-scheme. Let $f_1, \ldots, f_n \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ be regular functions on X. These are morphisms $f_i : \mathrm{X} \to \mathbf{A}^1_k$ which together define a morphism $f : \mathrm{X} \to (\mathbf{A}^1_k)^{\times n} = \mathbf{A}^n_k$. The *zero locus* $\mathrm{Z} = \mathrm{Z}(f_1, \ldots, f_n)$ is defined by the cartesian square

$$\begin{tikzcd}
\mathrm{Z} \arrow[r] \arrow[d] & \mathrm{X} \arrow[d, "f"] \\
\mathrm{Spec}(k) \arrow[r, "\{0\}"] & \mathbf{A}^n_k
\end{tikzcd}$$

The ring of functions on Z is thus given by the tensor product

$$\begin{aligned}
\Gamma(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}) &= \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \otimes_{\Gamma(\mathbf{A}^n_k, \mathcal{O}_{\mathbf{A}^n_k})} \Gamma(\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)}) \\
&\simeq \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \otimes_{k[\mathrm{T}_1, \ldots, \mathrm{T}_n]} k \\
&\simeq \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) / \langle f_1, \ldots, f_n \rangle.
\end{aligned}$$

Definition 8. A morphism of affine k-schemes $i : \mathrm{Z} \to \mathrm{X}$ is a *closed immersion* if the induced map of rings of functions

$$\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \Gamma(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}})$$

is surjective. In other words, if $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{Z} = \mathrm{Spec}(\mathrm{B})$, then the condition is that $\mathrm{A} \to \mathrm{B}$ is surjective.

Example 9. Let X be an affine k-scheme. Then for any collection of regular functions $f_1, \ldots, f_n$, the inclusion of the zero locus

$$\mathrm{Z}(f_1, \ldots, f_n) \to \mathrm{X}$$

is a closed immersion. Moreover, every closed immersion $i : \mathrm{Z} \to \mathrm{X}$ is of this form (choose generators f_i for the kernel of $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \Gamma(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}})$).

13.3. Algebraic k-varieties. The following definitions can be generalized to non-affine schemes, but we'll restrict to the affine case in this lecture for simplicity.

Definition 10. An *algebraic k-scheme* is a k-scheme of finite type. An affine k-scheme X is algebraic iff there exists a cartesian square of k-schemes

$$\begin{tikzcd}
\mathrm{X} \arrow[r] \arrow[d] & \mathbf{A}^n_k \arrow[d, "f"] \\
\mathrm{Spec}(k) \arrow[r, "\{0\}"] & \mathbf{A}^m_k.
\end{tikzcd}$$

In other words, X is the zero locus of some regular functions $f_1, \ldots, f_m$ on some ambient affine space. Equivalently, the ring of regular functions $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$ is of the form

$$\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) \simeq k[\mathrm{T}_1, \ldots, \mathrm{T}_n]/\langle f_1, \ldots, f_m \rangle,$$

i.e., it is *of finite type* as a k-algebra.

Definition 11. An affine k-scheme X is *reduced* if the ring of regular functions $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$ is reduced (i.e., its only nilpotent element is zero). It is *irreducible* if the reduction $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})_{\mathrm{red}}$ is an integral domain. It is *integral* if it is reduced and irreducible, or equivalently $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$ is an integral domain.

Definition 12. An *affine algebraic k-variety* is an integral algebraic affine k-scheme. Equivalently, it is an affine k-scheme X whose ring of functions $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$ is an integral domain that is of finite type as a k-algebra.

13.4. Coherent sheaves.

Remark 13. Let X be a k-scheme. The ring of regular functions $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$ can be identified with the ring of global sections of the structure sheaf $\mathcal{O}_\mathrm{X}$ (which is the reason for the notation). More generally, given any *quasi-coherent $\mathcal{O}_\mathrm{X}$-module* $\mathcal{F}$ on X, its global sections form a module

$$\Gamma(\mathrm{X}, \mathcal{F})$$

over the ring $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$. We will usually say "quasi-coherent sheaf" instead of "quasi-coherent $\mathcal{O}_\mathrm{X}$-module".

Theorem 14. *Let* X *be an* affine *k-scheme.*

(i) *The assignment $\mathcal{F} \mapsto \Gamma(\mathrm{X}, \mathcal{F})$ defines an equivalence from the category of quasi-coherent sheaves on X to the category of modules over $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$.*

(ii) *If X is moreover* noetherian*, then the equivalence in (i) restricts to an equivalence from the category of coherent sheaves on X to the category of f.g. modules over $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$.*

(iii) *The equivalence in (i) restricts to an equivalence from the category of finite locally free sheaves on X to the category of f.g. projective modules over $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$.*

Remark 15. Let X be a k-scheme. If $\mathcal{F}$ is a perfect complex on X (roughly, a chain complex of quasi-coherent sheaves which restricts to perfect complexes over some open covering), then its derived global sections $\mathbf{R}\Gamma(\mathrm{X}, \mathcal{F})$ form a perfect complex over $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$. If X is affine, there is an equivalence between the category of perfect complexes on X and $\mathrm{Perf}_{\Gamma(\mathrm{X},\mathcal{O}_{\mathrm{X}})}$.

Remark 16. Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of finite type k-schemes. Then there is a pair of adjoint functors

$$f^* : \mathrm{Qcoh}(\mathrm{Y}) \to \mathrm{Qcoh}(\mathrm{X}), \quad f_* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\mathrm{Y}).$$

If X and Y are affine, these are identified with extension and restriction of scalars along $\Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) \to \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$, respectively. The inverse image functor f^* preserves the subcategories of coherent and finite locally free sheaves. The direct image functor f_* preserves coherent sheaves if it is proper, and finite locally free sheaves if it is finite and flat. If X and Y are affine, $f : \mathrm{X} \to \mathrm{Y}$ is proper iff it is finite iff $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ is f.g. as a $\Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}})$-module.

Example 17. Any closed immersion $i : \mathrm{Z} \to \mathrm{X}$ is finite (hence *a fortiori* proper). In particular, the direct image of the structure sheaf is a coherent sheaf $i_*(\mathcal{O}_{\mathrm{Z}})$ on X. It corresponds to the $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$-module

$$\Gamma(\mathrm{X}, i_*(\mathcal{O}_{\mathrm{Z}})) \simeq \Gamma(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}).$$

For example, for $i : \mathrm{Spec}(\mathrm{A}/\mathrm{I}) \to \mathrm{Spec}(\mathrm{A})$, $i_*(\mathcal{O}_{\mathrm{Z}})$ is the coherent sheaf corresponding to A/I viewed as an A-module. Sometimes we may abuse notation and write simply $\mathcal{O}_{\mathrm{Z}}$ instead of $i_*(\mathcal{O}_{\mathrm{Z}})$.

13.5. K-theory.

Construction 18. Let X be a k-scheme. Its algebraic K-theory $\mathrm{K}_0(\mathrm{X})$ is defined as the abelian group freely generated by quasi-isomorphism classes of perfect complexes, modulo relations coming from exact triangles. By construction, $\mathrm{K}_0(\mathrm{X}) = \mathrm{K}_0(\mathrm{Perf}_{\Gamma(\mathrm{X},\mathcal{O}_{\mathrm{X}})}) \simeq \mathrm{K}_0(\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}))$. If X is noetherian, then the algebraic G-theory $\mathrm{G}_0(\mathrm{X})$ is constructed as the abelian group freely generated by iso. classes of coherent sheaves on X, modulo relations coming from exact sequences of coherent sheaves.

Recall that $\mathrm{K}_0(\mathrm{Perf}_{\mathrm{A}}) \simeq \mathrm{K}_0(\mathrm{A})$ for any ring A. This generalizes as follows.

Theorem 19. *Let* X *be a quasi-projective* k*-scheme (e.g. affine schemes are quasi-projective). Then* $\mathrm{K}_0(\mathrm{X})$ *is canonically isomorphic to the group completion of the monoid of iso. classes of locally free sheaves on* X.

Remark 20. Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of affine algebraic k-schemes. The functorialities on locally free sheaves induce an inverse image homomorphism $f^* : \mathrm{K}_0(\mathrm{Y}) \to \mathrm{K}_0(\mathrm{X})$ and, if f is finite and of finite Tor-amplitude (e.g. finite and flat), a direct image homomorphism $f_* : \mathrm{K}_0(\mathrm{X}) \to \mathrm{K}_0(\mathrm{Y})$. Similarly on G-theory we get an inverse image $f^* : \mathrm{G}_0(\mathrm{Y}) \to \mathrm{G}_0(\mathrm{X})$ if f is of finite Tor-amplitude (e.g. flat), and a direct image $f_* : \mathrm{G}_0(\mathrm{X}) \to \mathrm{G}_0(\mathrm{Y})$ if f is finite. These satisfy the base change and projection formulas we proved in Lecture 6. For example, if there is a cartesian square of affine algebraic k-schemes

$$\begin{tikzcd} \mathrm{X}' \arrow[r, "g"] \arrow[d, "q"] & \mathrm{Y}' \arrow[d, "p"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \end{tikzcd}$$

with p finite and of finite Tor-amplitude, then there is a base change formula

$$f^* p_* = q_* g^* : \mathrm{K}_0(\mathrm{Y}') \to \mathrm{K}_0(\mathrm{X}).$$

Definition 21. A k-scheme X is called *regular* if every coherent sheaf on X is quasi-isomorphic to a perfect complex. It is *smooth* over k iff $\mathrm{X} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is regular.

Theorem 22. *If* X *is a regular k-scheme, there is a canonical isomorphism*

$$\mathrm{K}_0(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X}).$$

In particular if X is smooth over k, then this holds. This is a K-theoretic analogue of *Poincaré duality* for singular co/homology of smooth manifolds.

Theorem 23. *Consider the flat morphism $f : \mathbf{A}^n_k \to \mathrm{Spec}(k)$. For every $n \geqslant 0$, inverse image induces an isomorphism*

$$f^* : \mathrm{G}_0(\mathrm{Spec}(k)) \to \mathrm{G}_0(\mathbf{A}^n_k).$$

More generally, for any noetherian k-scheme X*, inverse image along $f_{\mathrm{X}} : \mathrm{X} \times \mathbf{A}^n_k \to \mathrm{X}$ induces an isomorphism*

$$f_{\mathrm{X}}^* : \mathrm{G}_0(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X} \times \mathbf{A}^n_k).$$

In the affine case, this is a restatement of Sheet 10, Exercise 1. If X is smooth over k (or just regular), then the same holds for K-theory. This property is called $\mathbf{A}^1$-*homotopy invariance* and is analogous to homotopy invariance for singular cohomology of smooth manifolds (where the affine line $\mathbf{A}^1_k$ is replaced by the real line).

13.6. Chow groups.

Construction 24. Let X be a k-scheme.

(i) We let $\mathrm{Z}_n(\mathrm{X})$ be the free abelian group on integral closed subschemes of dimension n.

(ii) Given an integral closed subscheme $\mathrm{Z} \subset \mathrm{X}$ of dimension $n+1$ and a nonzero regular function $f \in \Gamma(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}})$, consider the zero locus $\mathrm{W} := \mathrm{Z}(f)$, which is equipped with a closed immersion $i : \mathrm{W} \to \mathrm{X}$. We define the principal divisor of f as the n-cycle associated to the coherent sheaf $i_*(\mathcal{O}_{\mathrm{W}})$:

$$\mathrm{div}_{\mathrm{Z}}(f) = [i_*(\mathcal{O}_{\mathrm{W}})]_n \in \mathrm{Z}_n(\mathrm{X}).$$

Here the construction $[-]_n$ is the n-cycle associated to a coherent sheaf, defined just like the n-cycle associated to a module (§9.3).

(iii) As Z and f vary, the subgroup of $\mathrm{Z}_n(\mathrm{X})$ generated by the classes $\mathrm{div}_{\mathrm{Z}}(f)$ is denoted $\mathrm{R}_n(\mathrm{X})$. Elements of $\mathrm{R}_n(\mathrm{X})$ are the n-cycles that are *rationally equivalent to zero.*

(iv) The *Chow groups* of X, denoted $\mathrm{CH}_n(\mathrm{X})$, are the quotients $\mathrm{Z}_n(\mathrm{X})/\mathrm{R}_n(\mathrm{X})$, for each $n \geqslant 0$. We set $\mathrm{CH}_n(\mathrm{X}) = 0$ for $n < 0$.

Remark 25. Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of algebraic k-schemes. If f is flat of relative dimension d, there are inverse image homomorphisms $f^* : \mathrm{CH}_n(\mathrm{Y}) \to \mathrm{CH}_{n+d}(\mathrm{X})$. If f is proper, there are direct image homomorphisms $f_* : \mathrm{CH}_n(\mathrm{X}) \to \mathrm{CH}_n(\mathrm{Y})$.

Remark 26. If X is a smooth algebraic k-scheme, then we similarly define $\mathrm{Z}^n(\mathrm{X})$, $\mathrm{R}^n(\mathrm{X})$, and $\mathrm{CH}^n(\mathrm{X})$, graded by codimension, so that $\mathrm{Z}^n(\mathrm{X}) = \mathrm{Z}_{d-n}(\mathrm{X})$ and $\mathrm{CH}^n(\mathrm{X}) \simeq \mathrm{CH}_{d-n}(\mathrm{X})$ if X is of pure dimension d. This way we impose “Poincaré duality” for Chow co/homology.

Theorem 27. *Let* X *be an algebraic k-scheme. Then for every integer $n \geqslant 0$, inverse image along the flat morphism* $f : \mathrm{X} \times \mathbf{A}^n_k \to \mathrm{X}$ *induces isomorphisms*

$$f^* : \mathrm{CH}_m(\mathrm{X}) \to \mathrm{CH}_{m+n}(\mathrm{X} \times \mathbf{A}^n_k)$$

for all $m \in \mathbf{Z}$, and in particular $\mathrm{CH}_*(\mathrm{X}) \simeq \mathrm{CH}_*(\mathrm{X} \times \mathbf{A}^n_k)$.

Remark 28. Let X be a smooth affine algebraic k-scheme. Two integral closed subschemes Y and Z intersect *properly* or *without excess* if its irreducible components are all of dimension $\leqslant \dim(\mathrm{Y}) + \dim(\mathrm{Z}) - \dim(\mathrm{X})$. We have seen that there is an intersection product turning $\mathrm{CH}^*(\mathrm{X})$ into a graded ring, which is computed by Serre’s Tor formula in case of proper intersections. This is true also for non-affine schemes, at least under the very mild hypothesis of *quasi-projectivity.*

Remark 29. An alternative description of rational equivalence is as follows. Let X be an algebraic k-variety. Let $\mathrm{Z} \subset \mathrm{X} \times \mathbf{A}^1_k$ be an integral closed subscheme of dimension $n+1$. Form the diagram of cartesian squares

$$\begin{tikzcd}
\mathrm{Z}^0 \arrow[r] \arrow[d] & \mathrm{Z} \arrow[d] & \mathrm{Z}^1 \arrow[l] \arrow[d] \\
\mathrm{X} \times \{0\} \arrow[r] & \mathrm{X} \times \mathbf{A}^1_k & \mathrm{X} \times \{1\} \arrow[l]
\end{tikzcd}$$

Assume that Z intersects both $\mathrm{X} \times \{0\}$ and $\mathrm{X} \times \{1\}$ properly (inside $\mathrm{X} \times \mathbf{A}^1_k$), so that Z^0 and Z^1 are of dimension $\leqslant n$. Identifying $\mathrm{X} \times \{0\} \simeq \mathrm{X} \simeq \mathrm{X} \times \{1\}$, one may think of Z^0 and Z^1 as closed subschemes of X. One can show then that the two n-cycles $[\mathcal{O}_{\mathrm{Z}^0}]_n$ and $[\mathcal{O}_{\mathrm{Z}^1}]_n$ in $\mathrm{Z}_n(\mathrm{X})$ are rationally equivalent. Moreover, the whole subgroup $\mathrm{R}_n(\mathrm{X}) \subset \mathrm{Z}_n(\mathrm{X})$ consists of cycles of the following more general form: for any $(n+1)$-cycle $\sum_i n_i[\mathrm{Z}_i] \in \mathrm{Z}_{n+1}(\mathrm{X} \times \mathbf{A}^1_k)$ for which each Z_i intersects both $\mathrm{X} \times \{0\}$ and $\mathrm{X} \times \{1\}$ properly,

$$\sum_i n_i \left([\mathcal{O}_{\mathrm{Z}_i^0}]_n - [\mathcal{O}_{\mathrm{Z}_i^1}]_n \right) \in \mathrm{R}_n(\mathrm{X}).$$

Roughly speaking, we may think of rational equivalence as a relation which identifies two algebraic cycles if there is an $\mathbf{A}^1$-family of algebraic cycles, i.e. a family of cycles parametrized by the affine line, which deforms one to the other.