Lecture 11
Divisors

11.1. The Picard group.

Definition 1. Let A be a ring. An A-module M is *invertible* iff there exists an A-module N and an A-module isomorphism $M \otimes_A N \simeq A$. Equivalently, M is finitely generated and the localization at every prime ideal $\mathfrak{p} \subset A$ is free of rank one (i.e., there exists an $A_\mathfrak{p}$-linear isomorphism $M_\mathfrak{p} \simeq A_\mathfrak{p}$).

Proposition 2. *For any invertible* A*-module* M*, the canonical evaluation homomorphism*

$$M \otimes_A \mathrm{Hom}_A(M, A) \to A$$

is invertible. In particular, $M^{\otimes -1} \simeq \mathrm{Hom}_A(M, A)$ *(since the inverse is unique up to isomorphism).*

Definition 3. The set of isomorphism classes of invertible A-modules forms an abelian group, under the operation $\otimes$. The identity element is A itself. This group is called the *Picard group* of A and is denoted Pic(A).

Construction 4. Note that there is a canonical map

$$\mathrm{Pic}(A) \to K_0(A)$$

given by $[M] \mapsto [M]$. It is a monoid homomorphism with respect to the multiplication on $K_0(A)$, and in particular induces a group homomorphism $\mathrm{Pic}(A) \to K_0(A)^\times$ valued in the group of units. It is functorial in A with respect to inverse image ϕ^* (for any ring homomorphism $\phi : A \to B$).

Definition 5. Let M be a f.g. projective A-module. The rank of M at a point $x \in |\mathrm{Spec}(A)|$ is $\mathrm{rk}_A(M, x) = \dim_\kappa(M \otimes_A \kappa(x))$. We say M is of constant rank n if $\mathrm{rk}_A(M, x) = n$ for every x.

Proposition 6. *Let* M *be a f.g. projective* A*-module. Then there exists a ring isomorphism* $A \simeq A_1 \times \cdots \times A_n$*, inducing a bijection* $|\mathrm{Spec}(A)| \simeq \coprod_i |\mathrm{Spec}(A_i)|$*, such that the function* $\mathrm{rk}_A(M, -) : |\mathrm{Spec}(A)| \to \mathbf{N}$ *is constant on each component* $\mathrm{Spec}(A_i)$*. Moreover, we then have* $M \simeq \prod_i M_i$*, where* $M_i = M \otimes_A A_i$*.*

Proof. The function $f = \mathrm{rk}_A(M, -)$ can only take finitely many values $r_1, \ldots, r_n$, and each preimage $f^{-1}(r_i)$ is necessarily a closed subset $V_A(I_i)$. Using an idempotent lifting argument (as in the proof of Sheet 6, Exercise 3), one may assume A is reduced. In that case one shows that the I_i are mutually disjoint and comaximal, so the Chinese remainder theorem yields the decomposition $A \simeq \prod_i A/I_i$. $\square$

Construction 7. For any f.g. projective A-module M, there is an invertible A-module $\det_A(M)$, called the *determinant.* If M is of constant rank n, then $\det_A(M)$ is the top exterior power $\Lambda^n_A(M)$ (which is of constant rank 1). In general, choose a decomposition $A \simeq \prod_i A_i$ such that each $M_i = M \otimes_A A_i$ is of constant rank. Then $\det_A(M) = \prod_i \det_{A_i}(M_i)$.

Proposition 8.

(i) *For any* A*-module* M *and ring homomorphism* $\phi : \mathrm{A} \to \mathrm{B}$*, there is a canonical isomorphism of invertible* B*-modules*

$$\det_{\mathrm{A}}(\mathrm{M}) \otimes_{\mathrm{A}} \mathrm{B} \simeq \det_{\mathrm{B}}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}).$$

(ii) *For any short exact sequence of* A*-modules*

$$0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}'' \to 0,$$

there is a canonical isomorphism of invertible A*-modules*

$$\det_{\mathrm{A}}(\mathrm{M}) \simeq \det_{\mathrm{A}}(\mathrm{M}') \otimes_{\mathrm{A}} \det_{\mathrm{A}}(\mathrm{M}'').$$

Proposition 9. *The assignment* $\mathrm{M} \mapsto \det_{\mathrm{A}}(\mathrm{M})$ *induces a canonical homomorphism*

$$\det_{\mathrm{A}} : \mathrm{K}_0(\mathrm{A}) \to \mathrm{Pic}(\mathrm{A})$$

which is a retraction of $\mathrm{Pic}(\mathrm{A}) \to \mathrm{K}_0(\mathrm{A})$*.*

Proof. The fact that it descends to $\mathrm{K}_0(\mathrm{A})$ follows from point (ii) of the previous proposition. The fact that it is a retraction follows from the canonical isomorphism $\det_{\mathrm{A}}(\mathrm{M}) \simeq \Lambda^1_{\mathrm{A}}(\mathrm{M}) \simeq \mathrm{M}$ when M is invertible (hence of constant rank one). □

Remark 10. Using the isomorphism $\mathrm{K}_0(\mathrm{Perf}_{\mathrm{A}}) \simeq \mathrm{K}_0(\mathrm{A})$, we obtain a notion of determinant of a perfect complex. Explicitly,

$$\det_{\mathrm{A}}(\mathrm{M}_\bullet) \simeq \otimes_{i \in \mathbf{Z}} \det_{\mathrm{A}}(\mathrm{M}_\bullet)^{\otimes(-1)^i}.$$

11.2. Effective Cartier divisors.

Definition 11. Let A be a ring. An *effective Cartier divisor* on A is a surjective ring homomorphism $\phi : \mathrm{A} \twoheadrightarrow \mathrm{A}/\mathrm{I}$ that is quasi-smooth of relative dimension -1. Recall this means that for every point $x \in \left|\mathrm{Spec}(\mathrm{A}/\mathrm{I})\right| \simeq \mathrm{V}(\mathrm{I})$, corresponding to a prime ideal $\mathfrak{p} \subset \mathrm{A}$, the localized ideal $\mathrm{I}_{\mathfrak{p}}$ is generated by a single element which is a non-zero-divisor.

Example 12. For any non-zero-divisor $f \in \mathrm{A}$, $\phi : \mathrm{A} \to \mathrm{A}/\langle f \rangle$ is an effective Cartier divisor. Warning: not every effective Cartier divisor is of this form.

Proposition 13. *Let* I *be an ideal of* A*. Then* $\phi : \mathrm{A} \to \mathrm{A}/\mathrm{I}$ *is an effective Cartier divisor iff* I *is invertible as an* A*-module.*

Proof. Suppose that ϕ is an effective Cartier divisor. It will suffice to show that, for every prime $\mathfrak{p} \subset \mathrm{A}$, $\mathrm{I}_{\mathfrak{p}}$ is free of rank one as an $\mathrm{A}_{\mathfrak{p}}$-module. If $[\mathrm{A} \to \kappa(\mathfrak{p})] \in \mathrm{V}(\mathrm{I})$, then by assumption there exists an element $f_{\mathfrak{p}} \in \mathrm{A}_{\mathfrak{p}}$ and an exact sequence of $\mathrm{A}_{\mathfrak{p}}$-modules

$$0 \to \mathrm{A}_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} \mathrm{A}_{\mathfrak{p}} \twoheadrightarrow \mathrm{A}_{\mathfrak{p}}/\mathrm{I}_{\mathfrak{p}} \to 0.$$

In particular, multiplication by $f_{\mathfrak{p}}$ gives an isomorphism $\mathrm{I}_{\mathfrak{p}} = f_{\mathfrak{p}}\mathrm{A}_{\mathfrak{p}} \simeq \mathrm{A}_{\mathfrak{p}}$. Now suppose $\mathfrak{p}$ is a prime such that $[\mathrm{A} \to \kappa(\mathfrak{p})] \notin \mathrm{V}(\mathrm{I})$, i.e., $\mathrm{I} \not\subset \mathfrak{p}$. In this case the inclusion $\mathrm{I}_{\mathfrak{p}} \subseteq \mathrm{A}_{\mathfrak{p}}$ is easily seen to be an equality. Thus $\mathrm{I}_{\mathfrak{p}}$ is free of rank one for every prime ideal $\mathfrak{p} \subset \mathrm{A}$.

Conversely, assume I is invertible. Then for every prime ideal $\mathfrak{p}$, $I_{\mathfrak{p}}$ is free of rank one and comes with a canonical $A_{\mathfrak{p}}$-module injection $I_{\mathfrak{p}} \to A_{\mathfrak{p}}$. Choose a basis, i.e., an element $f_{\mathfrak{p}} \in I_{\mathfrak{p}}$ such that multiplication by $f_{\mathfrak{p}}$ induces an isomorphism $f_{\mathfrak{p}} : A_{\mathfrak{p}} \to I_{\mathfrak{p}}$. Through the inclusion $I_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$, we can view $f_{\mathfrak{p}}$ as an element of $A_{\mathfrak{p}}$ which fits in a short exact sequence

$$0 \to A_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} A_{\mathfrak{p}} \twoheadrightarrow A_{\mathfrak{p}}/I_{\mathfrak{p}} \to 0.$$

Thus, $A \twoheadrightarrow A/I$ is an effective Cartier divisor. □

Remark 14. It follows that an effective Cartier divisor on A is the same data as that of an invertible ideal I, or equivalently an invertible A-module M together with an A-linear injection $M \hookrightarrow A$. Note however that two A-linear injections $M \hookrightarrow A$ and $N \hookrightarrow A$ can have the same image in A. This happens if there exists an A-linear isomorphism $M \to N$ such that the diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

commutes. In that case, we regard them as the same effective Cartier divisor.

11.3. Non-effective Cartier divisors.

We generalize the notion of effective Cartier divisor to allow rational functions (i.e., to allow poles).

Remark 15. Let A be an integral domain. Recall that we view elements $f \in A$ as "regular functions" on the scheme Spec(A). Similarly, elements $f/g \in \mathrm{Frac}(A)$ are "rational functions" on Spec(A). For example, 3/4 is a rational function on $\mathrm{Spec}(\mathbf{Z})$ with a pole of order 2 at the point $[\mathbf{Z} \to \mathbf{F}_2]$.

Definition 16. Let A be an integral domain. A *Cartier divisor* on A is an invertible A-module M together with an A-linear injection $M \hookrightarrow \mathrm{Frac}(A)$.

Remark 17. We regard two pairs $(M, M \hookrightarrow \mathrm{Frac}(A))$ and $(N, N \hookrightarrow \mathrm{Frac}(A))$ as the same Cartier divisor if there is an A-module isomorphism $M \simeq N$ which commutes with the injections into Frac(A). This way, a Cartier divisor on A is the same datum as an sub-A-module of Frac(A).

Remark 18. Cartier divisors are also called *invertible fractional ideals* (but note that they are not necessarily ideals).

Remark 19. Note that if M is invertible, injectivity of $M \to \mathrm{Frac}(A)$ is equivalent to being nonzero. Indeed, injectivity can be checked on localizations, and every nonzero $A_{\mathfrak{p}}$-module homomorphism $A_{\mathfrak{p}} \simeq M_{\mathfrak{p}} \to \mathrm{Frac}(A)_{\mathfrak{p}} \simeq \mathrm{Frac}(A_{\mathfrak{p}})$ is injective (as $A_{\mathfrak{p}}$ is an integral domain).

Example 20. If I is an invertible ideal of A, then $I \hookrightarrow A \hookrightarrow \mathrm{Frac}(A)$ is injective. Thus effective Cartier divisors are Cartier divisors.

Example 21. Any nonzero element $f/g \in \mathrm{Frac}(\mathrm{A})^\times$, induces an A-linear injection $\mathrm{A} \hookrightarrow \mathrm{Frac}(\mathrm{A})$, $a \mapsto af/g$. The pair $(\mathrm{A}, \mathrm{A} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$ is called the *principal Cartier divisor* defined by f/g, and is denoted $\mathrm{div}_\mathrm{A}(f/g)$.

Construction 22. The tensor product of two Cartier divisors $(\mathrm{M}, u : \mathrm{M} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$ and $(\mathrm{N}, v : \mathrm{N} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$ is defined as $(\mathrm{M} \otimes_\mathrm{A} \mathrm{N}, u \otimes v)$, where $u \otimes v$ is the composite

$$u \otimes v : \mathrm{M} \otimes_\mathrm{A} \mathrm{N} \hookrightarrow \mathrm{Frac}(\mathrm{A}) \otimes_\mathrm{A} \mathrm{Frac}(\mathrm{A}) \xrightarrow{\mathrm{mult}} \mathrm{Frac}(\mathrm{A}).$$

Since $u \otimes v$ is nonzero, it is indeed injective. The unit with respect to this product is $(\mathrm{A}, \mathrm{A} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$, where $\mathrm{A} \hookrightarrow \mathrm{Frac}(\mathrm{A})$ is the canonical injection.

Proposition 23. *The set of Cartier divisors on* A *forms an abelian group under tensor product.*

Proof. Let $\mathrm{D} = (\mathrm{M}, u : \mathrm{M} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$ be a Cartier divisor. Let $\mathrm{D}^{-1} = (\mathrm{M}^{\otimes -1}, v : \mathrm{M}^{\otimes -1} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$, where v is defined as follows. Fix a nonzero element $m \in \mathrm{M}$ (an invertible module is nonzero). Under the isomorphism $\mathrm{M}^{\otimes -1} \simeq \mathrm{Hom}_\mathrm{A}(\mathrm{M}, \mathrm{A})$, v sends

$$(\phi \in \mathrm{Hom}_\mathrm{A}(\mathrm{M}, \mathrm{A})) \quad \mapsto \quad \phi(m)/m.$$

One checks that D^{-1} is inverse to D. ☐

Notation 24. We let Cart(A) denote the abelian group of Cartier divisors on A.

11.4. Cartier divisors and the Picard group.

Proposition 25. *Let* A *be an integral domain. There is an exact sequence of abelian groups*

$$0 \to \mathrm{A}^\times \to \mathrm{Frac}(\mathrm{A})^\times \xrightarrow{\mathrm{div}_\mathrm{A}} \mathrm{Cart}(\mathrm{A}) \twoheadrightarrow \mathrm{Pic}(\mathrm{A}) \to 0.$$

Proof. Consider the map $\mathrm{Cart}(\mathrm{A}) \to \mathrm{Pic}(\mathrm{A})$ sending $(\mathrm{M}, \mathrm{M} \hookrightarrow \mathrm{Frac}(\mathrm{A})) \mapsto [\mathrm{M}]$, which is clearly a group homomorphism. To show it is surjective, we have to embed any invertible A-module M into Frac(A). Consider the homomorphism

$$\mathrm{M} \to \mathrm{M} \otimes_\mathrm{A} \mathrm{Frac}(\mathrm{A}) \simeq \mathrm{M}_{\langle 0 \rangle}$$

Since it is nonzero, it is injective. Since M is invertible, $\mathrm{M}_{\langle 0 \rangle} \simeq \mathrm{A}_{\langle 0 \rangle} = \mathrm{Frac}(\mathrm{A})$. Thus we have constructed a Cartier divisor $(\mathrm{M}, \mathrm{M} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$.

The kernel is the subgroup of Cartier divisors $(\mathrm{M}, \mathrm{M} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$, such that M is isomorphic to A. Let $f \in \mathrm{M}$ be the image of $1 \in \mathrm{A}$ under such an isomorphism; then M is the principal Cartier divisor $\mathrm{div}_\mathrm{A}(f)$.

Finally let $f \in \mathrm{Frac}(\mathrm{A})^\times$ such that $\mathrm{div}_\mathrm{A}(f)$ is equal to $(\mathrm{A}, \mathrm{A} \hookrightarrow \mathrm{Frac}(\mathrm{A}))$ (with the canonical injection). This means that $f\mathrm{A} = \mathrm{A}$ in Frac(A). But $f\mathrm{A} \subseteq \mathrm{A}$ implies $f \in \mathrm{A}$, and then $\mathrm{A} \subseteq f\mathrm{A}$ implies that f is a unit. ☐

11.5. From Weil divisors to Cartier divisors.

Lemma 26. *Let* A *be a locally factorial ring (i.e.,* $A_{\mathfrak{p}}$ *is factorial for every prime ideal* $\mathfrak{p}$*). For every integral subset* $V(\mathfrak{p}) \subset |\mathrm{Spec}(A)|$ *of codimension* 1*,* $\mathfrak{p}$ *is invertible as an* A*-module.*

Proof. It will suffice to show $\mathfrak{p}_{\mathfrak{q}} \simeq A_{\mathfrak{q}}$ for every prime ideal $\mathfrak{q}$. Note that $V(\mathfrak{p}_{\mathfrak{q}})$ is still of codimension 1 in $|\mathrm{Spec}(A_{\mathfrak{p}})|$. Therefore, as $A_{\mathfrak{q}}$ is factorial, a lemma used in the proof of Exercise 3 on Sheet 8 yields that $\mathfrak{p}_{\mathfrak{q}}$ is a principal ideal. It is then generated by some non-zero-divisor f, multiplication with which induces an isomorphism $f : A_{\mathfrak{q}} \to \mathfrak{p}_{\mathfrak{q}}$. □

Definition 27. Let A be a noetherian ring. A *Weil divisor* on A is an element of the free abelian group $Z^1(A)$ generated by integral subsets $V(\mathfrak{p}) \subset |\mathrm{Spec}(A)|$ of codimension 1. If A is an integral domain of finite type over a field k, then $Z^1(A) = Z_{d-1}(A)$, where $d = \dim(A)$.

Construction 28. Let A be regular. Then A is in particular locally factorial, so the lemma applies. Let $Z^1(A)$ be the free abelian group on integral subsets $V(\mathfrak{p})$ of codimension 1. Then there is a canonical homomorphism

$$Z^1(A) \to \mathrm{Cart}(A)$$

sending $[V(\mathfrak{p})] \mapsto (\mathfrak{p}, \mathfrak{p} \hookrightarrow A \hookrightarrow \mathrm{Frac}(A))$.