Lecture 1

Homological algebra crash course

1.1. Finiteness of modules

1.2. Functoriality

1.3. Structure of f.g. modules

1.4. Projective resolutions

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1.1. Finiteness of modules

A comm. ring (unital, associative)

$\mathrm{Mod}_A = \{ A\text{-modules} \}$

<u>Def</u>: $M \in \mathrm{Mod}_A$ is <u>finitely generated</u>

$\overset{\text{def}}{\iff} \exists\ A^{\oplus n} \twoheadrightarrow M$ A-linear surjection, $n \geq 0$

M is <u>finitely presented</u> $\overset{\text{def}}{\iff}$ $\exists$ exact sequence

$$A^{\oplus m} \to A^{\oplus n} \twoheadrightarrow M \to 0 \qquad m, n \geq 0$$

<u>Lemma</u>: $0 \to M' \to M \to M'' \to 0$ exact sequence

(i) M', M'' f.g. $\Rightarrow$ M f.g.

(ii) M', M'' f.p. $\Rightarrow$ M f.p.

(iii) M f.p., M' f.g. $\Rightarrow$ M'' f.p.

(iv) M f.g. $\Rightarrow$ M'' f.g.

(v) M f.g., M'' f.p. $\Rightarrow$ M' f.g.

<u>Remark</u>: If A noetherian, then

$$M \text{ f.g.} \iff M \text{ f.p.}$$

for any $M \in \mathrm{Mod}_A$.

If M f.g., choose $A^{\oplus n} \twoheadrightarrow M$.

$$\Rightarrow 0 \to K \hookrightarrow A^{\oplus n} \twoheadrightarrow M \to 0$$

Since A noetherian, K is f.g.

$$\exists\, A^{\oplus m} \twoheadrightarrow K \hookrightarrow A^{\oplus n}$$

$$\Rightarrow A^{\oplus m} \to A^{\oplus n} \twoheadrightarrow M \to 0 \quad \text{exact}$$

$\Rightarrow M$ is f.p.

<u>Example</u>: $I \subseteq A$ ideal, $A \twoheadrightarrow A/I$

$\Rightarrow A/I$ is f.g.

<u>Example</u>: A non-noetherian (e.g. $k[x_0, x_1, \dots]$)

$\Rightarrow \exists\, I \subseteq A$ ideal not f.g. (e.g. $\langle x_0, x_1, \dots \rangle$)

$$0 \to I \hookrightarrow A \twoheadrightarrow A/I \to 0$$

If A/I f.p. then I would be f.g.

$\Rightarrow$ A/I is f.g. but not f.p. A-module.

<u>Def</u>: $M \in \mathrm{Mod}_A$ is <u>projective</u>

$\overset{\text{def}}{\iff}$

$$\begin{tikzcd} & N \arrow[d, two heads, "\forall"] \\ M \arrow[r, "\forall"'] \arrow[ur, dashed] & N' \end{tikzcd}$$

$\iff$ $\mathrm{Hom}_A(M,N) \twoheadrightarrow \mathrm{Hom}_A(M,N')$ surj.

$\forall\, N \twoheadrightarrow N'$ surj.

$\iff$ $\mathrm{Hom}_A(M,-) : \mathrm{Mod}_A \to \mathrm{Mod}_A$ right-exact.

<u>Lemma</u>: M is projective (resp. f.g. proj.)

$\iff$ $M \oplus N \cong \bigoplus_{\Lambda} A$ for some set Λ and $N \in \mathrm{Mod}_A$

(resp. $M \oplus N \cong A^{\oplus n}$ for some $n \in \mathbb{N}$, $N \in \mathrm{Mod}_A$)

<u>Exercise</u>: $\varphi : M \to N$ with M, N f.g. proj.
Then neither $\mathrm{Ker}(\varphi)$ nor $\mathrm{Coker}(\varphi)$ must be projective.

1.2 Functoriality (change of rings)

$\varphi : A \to B$ ring homomorphism

$\mathrm{Mod}_B \to \mathrm{Mod}_A$ restriction of scalars

$N \mapsto N_{[\varphi]} = N_{[A]}$ exact functor

$\mathrm{Mod}_A \to \mathrm{Mod}_B$ extension of scalars

$M \mapsto M \otimes_A B$ right-exact functor

Exercise: These form an adjoint pair, i.e.

$$\mathrm{Hom}_B(M \otimes_A B, N) \simeq \mathrm{Hom}_A(M, N_{[A]})$$

for all $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$.

Question: What finiteness properties do these functors preserve?

Lemma: $M \mapsto M \otimes_A B$ preserves f.g., f.p., and proj.

Proof: $A^{\oplus n} \twoheadrightarrow M \to 0$

$$\Rightarrow B^{\oplus n} \cong A^{\oplus n} \otimes_A B \twoheadrightarrow M \otimes_A B \to 0$$

Similar for f.p.

M proj. $\Rightarrow$ $M \oplus N \cong \bigoplus_{\Lambda} A$

$\Rightarrow$ $(M \oplus N) \otimes_A B \cong \left(\bigoplus_{\Lambda} A\right) \otimes_A B$

$\Rightarrow$ $(M \otimes_A B) \oplus (N \otimes_A B) \cong \bigoplus_{\Lambda} B$

$\Rightarrow$ $M \otimes_A B$ proj. $\blacksquare$

<u>Lemma</u>: (i) $N \mapsto N_{[\varphi]}$ preserves f.g.

$\Longleftrightarrow$ φ exhibits B as a f.g. A-module.

(ii) $N \mapsto N_{[\varphi]}$ preserves f.g. proj.

$\Longleftrightarrow$ φ exhibits B as a f.g. proj A-module

$\Longleftrightarrow$ φ exhibits B as a f.p., flat A-module

<u>Proof</u>: The conditions are necessary: take $N = B$.

(i) $B^{\oplus n} \twoheadrightarrow N \to 0$

$\Rightarrow$ $(B^{\oplus n})_{[\varphi]} \twoheadrightarrow N_{[\varphi]} \to 0$ exact

$$(B^{\oplus n})_{[\varphi]} = (B_{[\varphi]})^{\oplus n}$$

$\Rightarrow$ If $B_{[\varphi]}$ f.g. then $N_{[\varphi]}$ f.g. $\forall N$

(ii) $N \oplus N' \cong B^{\oplus n}$

$\Rightarrow N_{[\varphi]} \oplus N'_{[\varphi]} \cong (B_{[\varphi]})^{\oplus n}$

If $B_{[\varphi]}$ is f.g. proj. then $N_{[\varphi]}$ is a direct summand of f.g. proj.

$\Rightarrow N$ is f.g. proj. ∎

<u>Prop</u> (Base change formula):

(i) $\varphi : A \to B$ $\quad N \in \mathrm{Mod}_B$, $M \in \mathrm{Mod}_A$

$$N_{[A]} \otimes_A M \xrightarrow{\sim} N \otimes_B (M \otimes_A B)$$

(ii)
$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A' & \to & B' := A' \otimes_A B \end{array}$$
ring homos.

$N \in \mathrm{Mod}_B$

Then $N_{[A]} \otimes_A A' \xrightarrow{\sim} (N \otimes_B B')_{[A']}$.

<u>Proof</u>: Obvious ∎

Proposition: $I \subseteq A$ ideal, $\varphi : A \twoheadrightarrow A/I$

Then $\mathrm{Mod}_{A/I} \longrightarrow \mathrm{Mod}_A$ is fully faithful

$N \longmapsto N_{[A]}$

with essential image

$$\{ M \in \mathrm{Mod}_A \mid IM = 0 \} \subseteq \mathrm{Mod}_A .$$

In particular,

$$N_{[A]} \otimes_A A/I \simeq N \qquad \forall N \in \mathrm{Mod}_{A/I} .$$

Proof: $N = A/I \Rightarrow (A/I)_{[A]} \otimes_A A/I$

$$\simeq A/I \otimes_A A/I$$

$$\simeq A/I$$

$N \in \mathrm{Mod}_{A/I} \Rightarrow N \simeq N \otimes_{A/I} A/I$, so

$$\left.\begin{aligned} N_{[A]} \otimes_A A/I &\simeq N \otimes_{A/I} (A/I \otimes_A A/I) \\ &\simeq N \otimes_{A/I} A/I \\ &\simeq N \end{aligned}\right] \; (\ast)$$

$$\mathrm{Hom}_{A/I}(N, N') \longrightarrow \mathrm{Hom}_A(N_{[A]}, N'_{[A]})$$

$$\text{(adjunction)} \quad \simeq \mathrm{Hom}_{A/I}(N_{[A]} \otimes_A A/I,\ N')$$

$$\circledast \quad \simeq \mathrm{Hom}_{A/I}(N, N')$$

$\Longrightarrow$ fully faithful. $\quad (\forall N, N' \in \mathrm{Mod}_{A/I})$

Remains to describe ess. image.

$N \in \mathrm{Mod}_{A/I} \Longrightarrow I \cdot (N_{[A]}) = 0$ obviously

Conversely take $M \in \mathrm{Mod}_A$ with $IM = 0$.

Then $M \simeq (M/IM)_{[A]}$, so M is in the essential image. $\blacksquare$

1.3. Structure of f.g. modules

$M \in \mathrm{Mod}_A$

<u>Def</u>: $x \in M$ $\quad \mathrm{Ann}_A(x) = \{ a \in A \mid ax = 0 \}$

$\mathrm{Ann}_A(M) = \{ a \in A \mid ax = 0 \ \forall x \in M \}$

$x \in M \rightsquigarrow Ax \subseteq M$ submodule

$$\begin{array}{ccc} A & \xrightarrow{\cdot x} & M \\ & & \cup\!\!\mid \\ & \twoheadrightarrow & Ax \end{array}$$

multiplication by x

Ax – image

$0 \to \mathrm{Ann}_A(x) \hookrightarrow A \xrightarrow{\cdot x} M$ kernel

$0 \to \mathrm{Ann}_A(x) \hookrightarrow A \twoheadrightarrow Ax \to 0$ exact

$\Longrightarrow \quad Ax \simeq A / \mathrm{Ann}_A(x). \quad \forall x \in M$

<u>Proposition</u>: $M \in \mathrm{Mod}_A$ f.g.
Then there exists an A-module filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_i/M_{i-1} \simeq A/I_i$ where $I_i \subseteq A$ ideals.
(If A noetherian, can take <u>prime</u> ideals.)

<u>Proof</u>: $A^{\oplus m} \twoheadrightarrow M$ $\Longleftrightarrow$ $x_1, \dots, x_m \in M$ sys. of generators

$M_1 := A \cdot x_1 \subseteq M$

$M_1/M_0 \simeq A \cdot x_1 \simeq A/\mathrm{Ann}(x_1)$.

Consider M/M_1, which has $m-1$ generators.
By induction on m, can assume $\exists$ filtration for M/M_1 (with quotients A/I's).

Recall $\{$submodules of $M/M_1\}$
$\simeq \{$submodules of M containing $M_1\}$
$\Longrightarrow$ get a filtration

$$M_2 \subset M_3 \subset \cdots \subset M_n = M$$

containing M_1, as desired. ∎

1.4. Projective resolutions

$$\mathrm{Mod}_A^{\mathrm{fgproj}} \subseteq \mathrm{Mod}_A^{\mathrm{fg}} \subseteq \mathrm{Mod}_A$$

very few, very good objs. — lots of objects

Study of $M \in \mathrm{Mod}_A^{\mathrm{fg}}$ can be "reduced" to A/I's by §1.3.

Want to understand A/I in terms of good objects $\in \mathrm{Mod}_A^{\mathrm{fgproj}}$.

Example: $a \in A$ $(a \neq 0)$

$I = \langle a \rangle \subseteq A$ principal ideal

$A \xrightarrow{\cdot a} A \twoheadrightarrow A/I \to 0$ finite pres.

Assume a non-zero-divisor

$\iff$ $A \xrightarrow{\cdot a} A$ injective.

Then

$\implies 0 \to A \xrightarrow{\cdot a} A \twoheadrightarrow A/I \to 0$

exact.

Def: (fgproj/fgfree) **resolution** of $M \in \mathrm{Mod}_A^{fg}$ is

$$P_\bullet = (\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0)$$

chain complex (with P_i fgproj/fgfree)

plus a quasi-isomorphism of chain complexes

$$P_\bullet \xrightarrow{qis} M[0].$$

$P_\bullet$ is **finite** if $P_i = 0 \quad \forall i \gg 0$

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r, "d_n"] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
 & & & \cdots & 0 \arrow[r, equal] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

quasi-iso. means $H_i(P_\bullet) \xrightarrow{\sim} H_i(M[0]) \ \forall i$

$$\iff \begin{array}{ccccc}
H_n(P_\bullet) & \cdots & H_1(P_\bullet) & & H_0(P_\bullet) = P_0/\mathrm{Im}(d_1) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & & H_1(M[0]) = 0 & & H_0(M[0]) = M
\end{array}$$

Remark: $P_\bullet$ is acyclic in positive degrees, i.e.,

$$H_i(P_\bullet) = 0 \quad \forall i > 0.$$

<u>Ex</u>: $a \in A$ non-zero-divisor

$$\begin{array}{rcl} P_\bullet & = & (0 \longrightarrow A \xrightarrow{\cdot a} A \longrightarrow 0) \\ \text{q-is} \downarrow & & \qquad\qquad \downarrow \qquad \downarrow \\ (A/\langle a\rangle)[0] & = & (\quad 0 \longrightarrow A/\langle a\rangle \longrightarrow 0) \end{array}$$

is a projective resolution (free in fact).

$$H_1(P_\bullet) = \ker(A \xrightarrow{\cdot a} A) = 0$$

$$H_0(P_\bullet) = A/\langle a\rangle$$

<u>Construction</u> (Koszul complex):

$a_1, \dots, a_n \in A$ $\qquad I = \langle a_1, \dots, a_n\rangle \subseteq A$

$$\mathrm{Kosz}_A(a_1, \dots, a_n) := \bigotimes_{i=1}^{n} (A \xrightarrow{\cdot a_i} A)$$

(tensor product of chain complexes)

$$n=2: \quad \mathrm{Kosz}_A(a,b) = \left(A \xrightarrow{\binom{b}{a}} A \oplus A \xrightarrow{(-a,\, b)} A\right)$$

More generally $M \in \mathrm{Mod}_A$ f.g. proj, rank n,

$u: M \to A$ A-linear map.

$\rightsquigarrow \mathrm{Kosz}_A(M)$ Koszul complex

$$0 \to \Lambda^n_A(M) \xrightarrow{d_n} \Lambda^{n-1}_A(M) \xrightarrow{d_{n-1}} \cdots \to \Lambda^1_A(M) \xrightarrow{d_1} A \to 0$$

$$\Lambda^1_A(M) = M \xrightarrow{u} A$$

$\Lambda^i_A(M) =$ exterior powers

$$d_k(x_1 \wedge \dots \wedge x_k) = \sum_{i=1}^{k} (-1)^{i+1} u(x_i) \cdot x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_k$$

$$H_0(\mathrm{Kosz}_A(u)) = A/\mathrm{Im}(u).$$

<u>Def</u>: $(a_1, \dots, a_n)$ is <u>Koszul-regular</u>

$\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Kosz}_A(a_1, \dots, a_n)$ is acyclic in pos. degrees.

More generally, $u: M \to A$ as above is Koszul-regular if $\mathrm{Kosz}_A(u)$ is acyclic in pos. degrees.

$\Rightarrow$ $\mathrm{Kosz}_A(a_1, \dots, a_n) \xrightarrow{\text{qis}} A/\langle a_1, \dots, a_n \rangle$ <u>free resolutions</u>

$\mathrm{Kosz}_A(u) \xrightarrow{\text{qis}} A/\mathrm{Im}(u)$

Def: A sequence $(a_1, \dots, a_n)$ is <u>regular</u>

$\overset{\text{def}}{\iff}$ a_1 is a non-zero-divisor in A
a_2 is a non-zero-divisor in $A/\langle a_1 \rangle$
$\vdots$
a_n is a non-zero-divisor in $A/\langle a_1, \dots, a_{n-1} \rangle$
and $\langle a_1, \dots, a_n \rangle \neq A$.

Prop: $(a_1, \dots, a_n)$ regular sequence

$\Longrightarrow$ Koszul-regular sequence.

Example: $A = k[x,y,z] / \langle (x-1) \cdot z \rangle$

$a_1 = x$ non-zero-div. in A
$a_2 = (x-1) \cdot y$ non-zero-div. in $A/\langle x \rangle \cong k[y]$
$\langle a_1, a_2 \rangle = \langle x, y \rangle \neq A$
$\Rightarrow$ (a_1, a_2) regular sequence.

But: a_2 zero-divisor : $(x-1) \cdot y \cdot z = 0$ in A.
$\Longrightarrow$ (a_2, a_1) <u>not</u> regular
(but still Koszul-regular)