K-Theory and Intersection Theory

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Introduction

2.1

The problem of defining intersection products on the Chow groups of schemes has a long history. Perhaps the first example of a theorem in intersection theory is Bézout's theorem, which tells us that two projective plane curves C and D, of degrees c and d and which have no components in common, meet in at most cd points. Furthermore if one counts the points of $C \cap D$ with multiplicity, there are *exactly* cd points. Bezout's theorem can be extended to closed subvarieties Y and Z of projective space over a field k, $\mathbb{P}^n_k$, with $\dim(Y) + \dim(Z) = n$ and for which $Y \cap Z$ consists of a finite number of points.

When the ground field $k = \mathbb{C}$, Bezout's theorem can be proved using integral cohomology. However, prior to the development of étale cohomology for curves over fields of characteristic p, one had to use algebraic methods to prove Bezout's theorem, and there is still no cohomology theory which makes proving similar theorems over an arbitrary base, including $\mathrm{Spec}(\mathbb{Z})$, possible.

In this chapter we shall outline the two approaches to intersection theory that are currently available. One method is to reduce the problem of defining the intersection product of arbitrary cycles to intersections with divisors. The other method is to use the product in K-theory to define the product of cycles. The difference between the two perspectives on intersection theory is already apparent in the possible definitions of intersection multiplicities. The first definition, due to Weil and Samuel, first defines the multiplicities of the components of the intersection of two subvarieties Y and Z which intersect properly in a variety X, when Y is a local complete intersection in X, by reduction to the case of intersection with divisors. For general Y and Z intersecting properly in a smooth variety X, the multiplicities are defined to be the multiplicities of the components of the intersection of $Y \times Z$ with the diagonal Δ_X in $X \times X$. The key point here is that Δ_X is a local complete intersection in $X \times X$. The second construction is Serre's "tor formula", which is equivalent to taking the product, in K-theory with supports, of the classes $[\mathcal{O}_Y] \in K_0^Y(X)$ and $[\mathcal{O}_Z] \in K_0^Z(X)$. This definition works for an arbitrary regular scheme X, since it does not involve reduction to the diagonal.

The Chow ring of a smooth projective variety X over a field was first constructed using the moving lemma, which tells us that, given two arbitrary closed subvarieties Y and Z of X, Z is rationally equivalent to a cycle $\sum_i n_i[W_i]$ in which all the W_i meet Y properly. One drawback of using the moving lemma is that one expects that $[Y].[Z]$ should be able to be constructed as a cycle on $Y \cap Z$, since (for example) using cohomology with supports gives a cohomology class supported on the intersection. A perhaps less important drawback is that it does not apply to non-quasi-projective varieties.

This problem was solved, by Fulton and others, by replacing the moving lemma by reduction to the diagonal and deformation to the tangent bundle. One can then prove that intersection theory for varieties over fields is determined by intersections with Cartier divisors, see Fulton's book *Intersection Theory* [17] for details.

For a general regular scheme X, $X \times X$ will not be regular, and the diagonal map $\Delta_X \to X \times X$ will not be a regular immersion. Hence we cannot use deformation to the normal cone to construct a product on the Chow groups. In SGA6, [2], Grothendieck and his collaborators showed that, when X is regular $\mathrm{CH}^*(X)_{\mathbb{Q}} \simeq \mathrm{Gr}^*_\gamma(K_0(X))_{\mathbb{Q}}$, which has a natural ring structure, and hence one can use the product on K-theory, which is induced by the tensor product of locally free sheaves, to define the product on $\mathrm{CH}^*(X)_{\mathbb{Q}}$. Here $\mathrm{Gr}^*_\gamma(K_0(X))$ is the graded ring associated to the γ-filtration $F^\cdot_\gamma(K_0(X))$. By construction, this filtration is automatically compatible with the product structure on $K_0(X)$, and was introduced because the filtration that is more naturally related to the Chow groups, the coniveau or codimension filtration $F^\cdot_{\mathrm{cod}}(K_0(X))$, is not tautologically compatible with products. However, in SGA6 Grothendieck proved, using the moving lemma, that if X is a smooth quasi-projective variety over a field, then $F^\cdot_{\mathrm{cod}}(K_0(X))$ is compatible with products. In this chapter, using deformation to the normal cone, we give a new proof of the more general result that the coniveau filtration $F^\cdot_{\mathrm{cod}}(K_*(X))$ on the entire K-theory ring is compatible with products.

Instead of looking at the group $K_0(X)$, we can instead filter the category of coherent sheaves on X by codimension of support. We then get a filtration on the K-theory spectrum of X and an associated spectral sequence called the Quillen, or coniveau, spectral sequence. The Chow groups appear as part of the E_2-term of this spectral sequence, while $\mathrm{Gr}^*_{\mathrm{cod}}K_*(X)$ is the E_∞ term. The natural map $\mathrm{CH}^*(X) \to \mathrm{Gr}^*_{\mathrm{cod}}(K_0(X))$ then becomes an edge homomorphism in this spectral sequence. The E_1-terms of this spectral sequence form a family of complexes $R^*_q(X)$ for $q \geq 0$, with $H^q(R^*_q(X)) \simeq \mathrm{CH}^q(X)$.

Let us write $\mathcal{R}^*_{X,q}$ for the complex of sheaves with $\mathcal{R}^*_{X,q}(U) \simeq R^*_q(U)$ for $U \subset X$ an open subset; we shall refer to these as the Gersten complexes. Gersten's conjecture (Sect. 2.5.6) implies that the natural augmentation $K_q(\mathcal{O}_X) \to \mathcal{R}^*_{X,q}$ is a quasi-isomorphism, which in turn implies Bloch's formula:

$$H^q\left(X, K_q(\mathcal{O}_X)\right) \simeq \mathrm{CH}^q(X)\ .$$

If X is a regular variety over a field, Quillen proved Gersten's conjecture, so that Bloch's formula is true in that case. (Bloch proved the $q = 2$ case by different methods.) For regular varieties over a field, this then gives another construction of a product on $CH^*(X)$, which one may prove is compatible with the product defined geometrically.

If X is a regular scheme of dimension greater than 0, the E_1-term of the associated Quillen spectral sequence does not have an obvious multiplicative structure. (Having such a product would imply that one can choose intersection *cycles* in a fashion compatible with rational equivalence.) However, there is another spectral sequence (the Brown spectral sequence) associated to the Postnikov tower of the presheaf of K-theory spectra on X which is naturally multiplicative. In general there is map from the Brown spectral sequence to the Quillen spectral sequence, which maps E_r to E_{r+1}. If this map is an isomorphism, then the Quillen spectral sequence is compatible with the product on K-theory from E_2 on, and it follows that the coniveau filtration on $K_*(X)$ is also compatible with the ring structure on K-theory. This map of spectral sequences is a quasi-isomorphism if Gersten's conjecture is true. Thus we have another proof of the multiplicativity of the coniveau filtration $F^{\cdot}_{\mathrm{cod}}(K_0(X))$, which depends on Gersten's conjecture, rather than using deformation to the normal cone.

The groups $H^p_Y(X, K_q(\mathcal{O}_X))$, for all p and q, and for all pairs $Y \subset X$ with Y a closed subset of a regular variety X, form a bigraded cohomology theory with nice properties, including homotopy invariance and long exact sequences, for pairs $Y \subset X$ with Y closed in X:

$$\ldots \to H^{p-1}\left(X - Y, K_q(\mathcal{O}_X)\right) \xrightarrow{\partial} H^p_Y\left(X, K_q(\mathcal{O}_X)\right)$$
$$\to H^p\left(X, K_q(\mathcal{O}_X)\right) \to H^p\left(X - Y, K_q(\mathcal{O}_X)\right) \to \ldots \,.$$

However, from the perspective of intersection theory, these groups contain a lot of extraneous information; if one looks at the weights of the action of the Adams operations on the Quillen spectral sequence, then it was shown in [64] that after tensoring with $\mathbb{Q}$, the spectral sequence breaks up into a sum of spectral sequences, all but one of which gives no information about the Chow groups, and that the E_1 term of the summand which computes the Chow groups can be described using Milnor K-theory tensored with the rational numbers.

It is natural to ask whether one can build a "smallest" family of complexes with the same formal properties that the Gersten complexes have, and which still computes the Chow groups. As explained in Sect. 2.4.1, the "obvious" relations that must hold in a theory of "higher rational equivalence" are also the relations that define the Milnor K-theory ring as quotient of the exterior algebra of the units in a field. The remarkable fact, proved by Rost, is that for smooth varieties over a field, the obvious relations are enough, *i.e.*, the cycle complexes constructed using Milnor K-theory have all the properties that one wants. Deformation to the normal cone plays a key role in constructing the product on Rost's cycle complexes. This result is strong confirmation that to build intersection theory for smooth varieties over a field, one needs only the theory of intersections of divisors, together with deformation to the tangent bundle. Rost also proves, though this is *not* needed for his result, that the analog of Gersten's conjecture holds for the complexes built out of Milnor K-theory.

Thus for smooth varieties over a field, we have a good theory of Chow groups and higher rational equivalence, whether we use Gersten's conjecture or deformation to the normal cone. For general regular schemes, there is not an obvious analog of reduction to the diagonal and deformation to the tangent bundle. Gersten's conjecture still makes sense; however a new idea is needed in order to prove it. It is perhaps worth noting that a weaker conjecture, that for a regular local ring R, $\mathrm{CH}^p(\mathrm{Spec}(R)) = 0$ for all $p > 0$ (see [12]), is still open.

Conventions 2.1.1

- Schemes will be assumed to be separated, noetherian, finite dimensional, and excellent. See EGA IV.7.8, [36], for a discussion of excellent schemes. We shall refer to these conditions as the "standard assumptions".

Any separated scheme which is of finite type over a field or Spec($\mathbb{Z}$) automatically satisfies these hypotheses.

- By a variety we will simply mean a scheme which is of finite type over a field.
- A scheme is said to be integral if it is reduced and irreducible.
- The natural numbers are $1, 2, \ldots$.

2.2 Chow groups

In this section we shall give the basic properties of divisors and Chow groups on general schemes, and sketch the two geometric constructions of the intersection product for varieties over fields, via the moving lemma and via deformation to the normal cone.

2.2.1 Dimension and Codimension

Normally one is used to seeing the group of cycles on a scheme equipped with a grading – however it is important to remember that dimension is not always a well behaved concept. In particular, for general noetherian schemes, while one may think of cycles as *homological* objects, it is the grading by *codimension* that is well defined.

The best reference for the dimension theory of general schemes is EGA IV, §5, [36]. We shall summarize here some of the main points.

Recall that any noetherian local ring has finite Krull dimension. Therefore, if X is a noetherian scheme, any integral subscheme $Z \subset X$ with generic point $\zeta \in X$, has finite codimension, equal to the Krull dimension of the noetherian local ring $\mathcal{O}_{X,\zeta}$. We will also refer to this as the codimension of the point ζ.

1 **Definition 1** A scheme (or more generally a topological space) X is said to be:

Catenary, if given irreducible closed subsets $Y \subset Z \subset X$, all maximal chains of closed subsets between Y and Z have the same length.

Finite Dimensional, if there is a (finite) upper bound on the length of chains of irreducible closed subsets.

A scheme S, is said to be *universally catenary* if every scheme of finite type over S is catenary. Any excellent scheme is universally catenary.

2.2.2 Cycles

2 **Definition 2** Let X be a scheme (not necessarily satisfying the standard assumptions). A cycle on X is an element of the free abelian group on the set of closed integral subschemes of X. We denote the group of cycles by $Z(X)$.

Since the closed integral subschemes of X are in one to one correspondence with the points of X, with an integral subscheme $Z \subset X$ corresponding to its generic point ζ, we have:

$$Z(X) := \bigoplus_{\zeta \in X} \mathbb{Z} \,.$$

For a noetherian scheme, this group may be graded by codimension, and we write $Z^p(X)$ for the subgroup consisting of the free abelian group on the set of closed integral subschemes of codimension p in X. If every integral subscheme of X is finite dimensional, we can also grade the group $Z(X)$ by dimension, writing $Z_p(X)$ for the free abelian group on the set of closed integral subschemes of dimension p in X. If X is a noetherian, catenary and finite dimensional scheme, which is also equidimensional (*i.e.*, all the components of X have the same dimension), of dimension d then the two gradings are just renumberings of each other: $Z_{d-p}(X) = Z^p(X)$. However if X is not equidimensional, then codimension and dimension do *not* give equivalent gradings:

Example 3. Suppose that k is a field, and that $X = T \cup S$, with $T := \mathbb{A}^1_k$ and $S := \mathbb{A}^2_k$, is the union of the affine line and the affine plane, with $T \cap S = \{P\}$ a single (closed) point P. Then X is two dimensional. However any closed point in T, other than P has dimension 0, and codimension 1, while P has dimension 0 and codimension 2.

Definition 4 If X is a general noetherian scheme, we write $X^{(p)}$ for the set of points $x \in X$, which are of "codimension p", *i.e.*, such that the integral closed subscheme $\overline{\{x\}} \subset X$ has codimension p, or equivalently, the local ring $\mathcal{O}_{X,x}$ has Krull dimension p. 4

We also write $X_{(p)}$ for the set of points $x \in X$ such that the closed subset $\overline{\{x\}} \subset X$ is finite dimensional of dimension p.

Observe that

$$Z^p(X) \simeq \bigoplus_{x \in X^{(p)}} \mathbb{Z} \,,$$

while, if X is finite dimensional,

$$Z_q(X) \simeq \bigoplus_{x \in X_{(q)}} \mathbb{Z} \,.$$

Cycles of codimension 1, *i.e.* elements of $Z^1(X)$, are also referred to as Weil divisors.

If $Z \subset X$ is an closed integral subscheme, we will write $[Z]$ for the associated cycle, and will refer to it as a "prime cycle". An element $\zeta \in Z(X)$ will then be written $\zeta = \sum_i n_i [Z_i]$.

If $\mathcal{M}$ is a coherent sheaf of $\mathcal{O}_X$-modules, let us write $\mathrm{supp}(\mathcal{M}) \subset X$ for its support. For each irreducible component Z of the closed subset $\mathrm{supp}(\mathcal{M}) \subset X$, the stalk $\mathcal{M}_\zeta$ of $\mathcal{M}$ at the generic point ζ of Z is an $\mathcal{O}_{X,\zeta}$-module of finite length.

5 **Definition 5** The cycle associated to $\mathcal{M}$ is:

$$[\mathcal{M}] := \sum_\zeta \ell(\mathcal{M}_\zeta)[Z] \, ,$$

where the sum runs over the generic points ζ of the irreducible components Z of $\mathrm{supp}(\mathcal{M})$, and $\ell(\mathcal{M}_\zeta)$ denotes the length of the Artinian $\mathcal{O}_\zeta$ module $\mathcal{M}_\zeta$.

If $Y \subset X$ is a closed subscheme, we set $[Y] := [\mathcal{O}_Y]$; notice that if Y is an integral closed subscheme then this is just the prime cycle $[Y]$.

It will also be convenient to have:

6 **Definition 6** Let $W \subset X$ be a closed subset. Then $Z^p_W(X) \subset Z^p(X)$ is the subgroup generated by those cycles supported in W, i.e., of the form $\sum_i n_i[Z_i]$ with $Z_i \subset W$.

If $U \subset V \subset X$ are Zariski open subsets, and $\zeta = \sum_i n_i[Z_i] \in Z^p(V)$ is a codimension p cycle, then $\zeta|_U := \sum_i n_i[Z_i \cap U]$ is a codimension p cycle on U. The maps $Z^p(V) \to Z^p(U)$, for all pairs $U \subset V$ define a sheaf on the Zariski topology of X, $\mathcal{Z}^p_X$ which is clearly flasque. Note that $Z^p_W(X) = H^0_W(X, \mathcal{Z}^p_X)$, and also that if U is empty then $Z^p(U)$ is the free abelian group on the empty set, *i.e.*, $Z^p(U) \simeq 0$.

2.2.3 Dimension Relative to a Base

Dimension can behave in ways that seem counter-intuitive. For example, if $U \subset X$ is a dense open subset of a scheme, U may have strictly smaller dimension than X. The simplest example of this phenomenon is if X is the spectrum of a discrete valuation ring, so $\dim(X) = 1$, and U is the Zariski open set consisting of the generic point, so $\dim(U) = 0$. One consequence of this phenomenon is that the long exact sequence of Chow groups associated to the inclusion of an open subset into a scheme will not preserve the grading by dimension.

However, dimension is well behaved with respect to proper morphisms:

7 **Theorem 7** Let $f : W \to X$ be a proper surjective morphism between integral schemes which satisfy our standing hypotheses. Then $\dim(W) = \dim(X) + \mathrm{tr.\,deg.}_{k(X)} k(W)$. In particular, if f is birational and proper, then $\dim(W) = \dim(X)$.

Proof [36], proposition 5.6.5.

This leads to the notion of *relative dimension*. By a theorem of Nagata, any morphism $f : X \to S$ of finite type is compactifiable. *I.e.*, it may be factored as $\overline{f} \cdot i$:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

with i an open immersion with dense image, and $\overline{f}$ proper.

Lemma 8 If S and X are as above, the dimension of $\overline{X}$ minus the dimension of S is independent of the choice of compactification $\overline{X}$. 8

Proof This is a straightforward consequence of Theorem 7.

Therefore we may make the following definition:

Definition 9 Let S be a fixed base, satisfying our standing hypotheses. If X is a scheme of finite type over S, we set $\dim_S(X) := \dim(\overline{X}) - \dim(S)$, where $\overline{X}$ is any compactification of X over S. 9

The key feature of relative dimension is that if X is a scheme of finite type over S and $U \subset X$ is a dense open, then $\dim_S(X) = \dim_S(U)$. It follows that the grading of the Chow groups of schemes of finite type over S by dimension relative to S is compatible with proper push-forward. In this respect, Chow homology behaves like homology with locally compact supports; see [15], where this is referred to as LC-homology.

Note that if the base scheme S is the spectrum of a field, or of the ring of integers in a number field, relative dimension and dimension give equivalent gradings, differing by the dimension of the base, on the cycle groups.

An equivalent approach to the definition of $\dim_S(X)$ may be found in Fulton's book on intersection theory ([17]) using transcendence degree.

Cartier Divisors 2.2.4

The starting point for intersection theory from the geometric point of view is the definition of intersection with a Cartier divisor.

Let X be a scheme. If $U \subset X$ is an open set, a section $f \in \mathcal{O}_X(U)$ is said to be *regular* if, for every $x \in U$, its image in the stalk $\mathcal{O}_{X,x}$ is a non-zero divisor. The regular sections clearly form a subsheaf $\mathcal{O}_{X,\mathrm{reg}}$ of the sheaf of monoids (with respect to multiplication) $\mathcal{O}_X$. The sheaf of total quotient rings $\mathcal{K}_X$ is the localization of $\mathcal{O}_X$ with respect to $\mathcal{O}_{X,\mathrm{reg}}$. Note that the sheaf of units $\mathcal{K}_X^*$ is the sheaf of groups associated to the sheaf of monoids $\mathcal{O}_{X,\mathrm{reg}}$, and that the natural map $\mathcal{O}_X \to \mathcal{K}_X$ is injective.

10 **Definition 10** Write $\mathcal{D}\mathrm{iv}_X$ for the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. The group $\mathrm{Div}(X)$ of Cartier divisors on X is defined to be $H^0(X, \mathcal{D}\mathrm{iv}_X)$. Note that we will view this as an *additive* group.

If $D \in \mathrm{Div}(X)$, we write $|D|$ for the *support* of D, which is of codimension 1 in X if D is non-zero. A Cartier divisor D is said to be *effective* if it lies in the image of

$$H^0\big(X, \mathcal{O}_{X,\mathrm{reg}}\big) \to H^0\big(X, \mathcal{D}\mathrm{iv}_X \simeq \mathcal{K}_X^*/\mathcal{O}_X^*\big) .$$

For details, see [37] IV part 4, §21. See also the article of Kleiman [43] for pathologies related to the sheaf of total quotient rings on a non-reduced scheme.

There is a long exact sequence:

$$0 \to H^0\big(X, \mathcal{O}_X^*\big) \to H^0\big(X, \mathcal{K}_X^*\big) \to H^0\big(X, \mathcal{K}_X^*/\mathcal{O}_X^*\big) \to$$
$$H^1\big(X, \mathcal{O}_X^*\big) \to H^1\big(X, \mathcal{K}_X^*\big) \to \dots$$

Recall that a Cartier divisor is said to be *principal* if it is in the image of

$$H^0\big(X, \mathcal{K}_X^*\big) \to H^0\big(X, \mathcal{D}\mathrm{iv}_X = \mathcal{K}_X^*/\mathcal{O}_X^*\big) .$$

Two Cartier divisors are said to be linearly equivalent if their difference is principal. From the long exact sequence above we see that there is always an *injection* of the group of linear equivalence classes of Cartier divisors into the Picard group $H^1(X, \mathcal{O}_X^*)$ of isomorphism classes of rank one locally free sheaves. If $H^1(X, \mathcal{K}_X^*) \simeq 0$ (for example if X is reduced), this injection becomes an isomorphism. Note that there are examples of schemes for which the map from the group of Cartier divisors to $\mathrm{Pic}(X)$ is *not* surjective. See the paper [44] of Kleiman for an example.

More generally, if $W \subset X$ is a closed subset, we can consider $H^1_W(X, \mathcal{O}_X^*)$, *i.e.*, the group of isomorphism classes of pairs $(\mathcal{L}, s)$ consisting of an invertible sheaf $\mathcal{L}$ and a non-vanishing section $s \in H^0(X - W, \mathcal{L})$. Then one has:

11 **Lemma 11** If X is reduced and irreducible, so that $\mathcal{K}_X^*$ is the constant sheaf, and W if has codimension at least one, then

$$H^0_W\big(X, \mathcal{K}_X^*/\mathcal{O}_X^*\big) \simeq H^1_W\big(X, \mathcal{O}_X^*\big) .$$

If D is a Cartier divisor on X, the subsheaf of $\mathcal{K}_X^*$ which is the inverse image of D is an $\mathcal{O}_X^*$ torsor – the sheaf of equations of D. The $\mathcal{O}_X$ submodule of $\mathcal{K}_X$ generated by this subsheaf is invertible; *i.e.*, it is a fractional ideal. The *inverse* of this sheaf is denoted $\mathcal{O}_X(D)$ and its class is the image of D in $H^1(X, \mathcal{O}_X^*)$ under the boundary map, see [37].

Cap Products with Cartier Divisors and the Divisor Homomorphism 2.2.5

There is a natural map from the group of Cartier divisors to the group of Weil divisors:

Lemma 12 Let X be a scheme. Then there is a unique homomorphism of sheaves: 12

$$\mathrm{div} : \mathcal{D}\mathrm{iv}_X \to \mathcal{Z}^1_X \, ,$$

such that, if $U \subset X$ is an open set and $f \in \mathcal{O}_{X,\mathrm{reg}}(U)$ is a regular element, then $\mathrm{div}(f) = [\mathcal{O}_U/(f)]$ – the cycle associated to the codimension one subscheme with equation f.

Proof See [37] §21.6.

If X is regular, or more generally locally factorial, one can show that this map is an isomorphism.

If X is an integral scheme, then since $\mathcal{K}_X$ is the constant sheaf associated to the function field $\mathbf{k}(X)$ of X, we get a homomorphism, also denoted div:

$$\mathrm{div} : \mathbf{k}(X)^* \to Z^1(X)$$

Remark 13 Observe that if X is a scheme and $D \subset X$ is a codimension 1 subscheme, 13
the ideal sheaf $\mathcal{I}_D$ of which is locally principal, then the Cartier divisor given by the local generators of $\mathcal{I}_D$ has divisor equal to the cycle $[D] = [\mathcal{O}_D]$.

Definition 14 Suppose that $D \in \mathrm{Div}(X)$ is a Cartier divisor, and that $Z \subset X$ is 14
an irreducible subvariety, such that $|D| \cap Z$ is a proper subset of Z. The Cartier divisor D determines an invertible sheaf $\mathcal{O}_X(D)$, equipped with a trivialization outside of $|D|$. Restricting $\mathcal{O}_X(D)$ to Z, we get an invertible sheaf $\mathcal{L}$ equipped with a trivialization s on $Z-(Z\cap|D|)$. Since Z is irreducible, and $Z\cap|D|$ has codimension at least 1 in Z, by Lemma 11 $H^1_{Z\cap|D|}(Z, \mathcal{O}^*_Z) \simeq H^0_{Z\cap|D|}(Z, \mathcal{D}\mathrm{iv}_Z)$, and hence the pair $(\mathcal{L}, s)$ determines a Cartier divisor on Z, which we write $D|_Z$, and which we call the *restriction* of D to Z.

Definition 15 We define the *cap product* $D \cap [Z]$ to be $\mathrm{div}(D|_Z) \in Z^1(Z)$. 15

2.2.6 Rational Equivalence

16 **Definition 16** If X is a scheme, the direct sum

$$R(X) := \bigoplus_{\zeta \in X} \mathbf{k}(\zeta)^*$$

where $\mathbf{k}(\zeta)^*$ is the group of units in the residue field of the point ζ, will be called the group of "K_1-chains" on X. For a noetherian scheme, this group has a natural grading, in which

$$R^q(X) := \bigoplus_{x \in X^{(q)}} \mathbf{k}(x)^* \ .$$

We call this the group of codimension q *K_1-chains*.

If X is finite dimensional, then we can also grade $R(X)$ by dimension:

$$R_p(X) := \bigoplus_{x \in X_{(p)}} \mathbf{k}(x)^* \ .$$

If X is catenary and equidimensional, these gradings are equivalent.

The sum of the homomorphisms $\mathrm{div} : \mathbf{k}(Z)^* \to Z^1(Z)$, as Z runs through all integral subschemes $Z \subset X$, induces a homomorphism, for which we use the same notation:

$$\mathrm{div} : R(X) \to Z(X) \ .$$

We say that a cycle in the image of div is *rationally equivalent to zero*.

17 **Definition 17** If X is a general scheme, then we set the (ungraded) Chow group of X equal to:

$$\mathrm{CH}(X) := \mathrm{coker}(\mathrm{div}) \ .$$

Now suppose that X satisfies our standing assumptions. The homomorphism div is of pure degree -1 with respect to the grading by dimension (or by relative dimension for schemes over a fixed base), and we set $\mathrm{CH}_q(X)$ equal to the cokernel of

$$\mathrm{div} : R_{q+1}(X) \to Z_q(X) \ .$$

The homomorphism div is not in general of pure degree $+1$ with respect to the grading by codimension, unless X is equidimensional, but it does increase codimension by at least one, and so we define $\mathrm{CH}^p(X)$ to be the cokernel of the induced map:

$$\mathrm{div} : \bigoplus_{x\in X^{(p-1)}} \mathbf{k}(x)^* \to \bigoplus_{x\in X^{(p)}} \mathbb{Z}$$

$$f = \sum_x \{f_x\} \mapsto \sum_x \mathrm{div}(\{f_x\}) \, .$$

We refer to these as the Chow groups of *dimension q* and *codimension p* cycles on X, respectively.

If X is equidimensional of dimension d, then dimension and codimension are compatible, *i.e.*, $\mathrm{CH}^p(X) \simeq \mathrm{CH}_{d-p}(X)$.

The classical definition of rational equivalence of cycles on a variety over a field was that two cycles α and β were rationally equivalent if there was a family of cycles ζ_t, parameterized by $t \in \mathbb{P}^1$, with $\zeta_0 = \alpha$ and $\zeta_\infty = \beta$. More precisely, suppose that W is an irreducible closed subvariety of $X \times \mathbb{P}^1$, which is flat over $\mathbb{P}^1$ (*i.e.*, not contained in a fiber of $X \times \{t\}$, for $t \in \mathbb{P}^1$). For each $t \in \mathbb{P}^1$, $(X \times \{t\})$ is a Cartier Divisor, which is the pull back, via the projection to $\mathbb{P}^1$, of the Cartier divisor $[t]$ corresponding to the point $t \in \mathbb{P}^1$. Then one sets, for $t \in \mathbb{P}^1$, $W_t := [W].(X \times \{t\})$. (Note that $[W]$ and the Cartier divisor $(X \times \{t\})$ intersect properly). The cycle $W_\infty - W_0$ is then said to be rationally equivalent to zero. (Note that $[0]$ and $[\infty]$ are linearly equivalent Cartier divisors.) More generally a cycle is rationally equivalent to zero if it is the sum of such cycles.

It can be shown that this definition agrees with the one given previously, though we shall not use this fact here.

Just as we sheafified the cycle functors to get flasque sheaves $\mathcal{Z}_X^p$, we have flasque sheaves $\mathcal{R}_X^q$, with $\mathcal{R}_X^q(U) = R^q(U)$. The divisor homomorphism then gives a homomorphism:

$$\mathrm{div} : \mathcal{R}_X^{q-1} \to \mathcal{Z}_X^q$$

and an isomorphism:

$$\mathrm{CH}^q(X) \simeq \mathbb{H}^1(X, \mathcal{R}_X^{q-1} \to \mathcal{Z}_X^q) \, .$$

Lemma 18 The map $\mathrm{Div}(X) \to \mathrm{CH}^1(X)$ induced by div factors through $\mathrm{Pic}(X)$, and vanishes on principal divisors. **18**

Proof If $x \in X^{(0)}$ is a generic point of X, then the local ring $\mathcal{O}_{X,x}$ is Artinian, and $\mathcal{O}_{X,\mathrm{reg},x} = \mathcal{O}_{X,x}^*$. Hence $\mathcal{K}_{X,x} = \mathcal{O}_{X,x}$, and so there is a natural homomorphism

$\mathcal{K}^*_{X,x} \to \mathbf{k}(x)^*$. Therefore there is a commutative diagram of maps of sheaves of abelian groups:

$$\begin{tikzcd}
\mathcal{O}^*_X \arrow[r] \arrow[d] & \mathcal{K}^*_X \arrow[r] \arrow[d] & \mathcal{R}^0_X \arrow[d] \\
0 \arrow[r] & \mathcal{D}\mathrm{iv}_X \arrow[r, "\mathrm{div}"] & \mathcal{Z}^1_X
\end{tikzcd}$$

Hence we get maps:

$$\mathrm{Div}(X) \to H^1(X, \mathcal{O}^*_X) \simeq \mathbb{H}^1(X, \mathcal{K}^*_X \to \mathcal{D}\mathrm{iv}_X)$$
$$\to \mathbb{H}^1(X, \mathcal{R}^0_X \to \mathcal{Z}^1_X) \simeq \mathrm{CH}^1(X) \,.$$

It follows by a diagram chase that the map from Cartier divisors on X to Weil divisors X induces a map from linear equivalence classes of Cartier divisors to rational equivalence classes of Weil divisors which factors through $\mathrm{Pic}(X)$.

From this lemma and the intrinsic contravariance of $H^1_W(X, \mathcal{O}^*_X)$, we get the following proposition.

19 **Proposition 19** Let $f : Y \to X$ be a morphism of varieties. Let $W \subset X$ be a closed subset, suppose that $\phi \in H^0_W(X, \mathcal{D}\mathrm{iv}_X)$ is a Cartier divisor with supports in W, and that $\zeta \in Z^p_T(Y)$ is a cycle supported in a closed subset $T \subset Y$. Then there is a natural "cap" product cycle class $\phi \cap \zeta \in \mathrm{CH}^{p+1}_{(T \cap f^{-1}(W))}(Y)$.

Note that if ϕ above is a principal effective divisor, given by a regular element $g \in \Gamma(X, \mathcal{O}_X)$, which is invertible on $X - W$, and if $\zeta = [Z]$ is the cycle associated to a reduced irreducible subvariety $Z \subset X$, with $f(Z) \nsubseteq W$, then $f^*(g)|_Z$ is again a regular element, and $\phi \cap \zeta$ is the divisor associated to $f^*(g)|_Z$ discussed in Definition 15.

Finally, the other situation in which we can pull back Cartier divisors is if $f : X \to Y$ is a flat morphism of schemes. Since f is flat, if $x \in X$, and $t \in \mathcal{O}_{Y,f(x)}$ is a regular element, then $f^*(t)$ is a regular element in $\mathcal{O}_{X,x}$. It follows that there is a pull-back

$$f^* : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X \,,$$

and hence an induced map on Cartier divisors.

Basic Properties of Chow Groups 2.2.7

Functoriality

Let $f : X \to Y$ be a morphism of schemes.

If f is *flat*, then there is a pull-back map $f^* : Z^p(Y) \to Z^p(X)$, preserving codimension. If $Z \subset X$ is an closed integral subscheme, then:

$$f^* : [Z] \mapsto [\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Z] ,$$

which is then extended to the full group of cycles by linearity.

If f is *proper*, then there is a push-forward map on cycles; if Z is a k-dimensional closed integral subscheme, then:

$$f_*([Z]) = \begin{cases} [k(Z) : k(f(Z))]\,[f(Z)] & \text{if } \dim(f(Z)) = \dim(Z) \\ 0 & \text{if } \dim(f(Z)) < \dim(Z) . \end{cases}$$

Push-forward preserves dimension, or dimension relative to a fixed base S, by Theorem 7.

Proposition 20 Both flat pull-back and proper push-forward are compatible with rational equivalence, and therefore induce maps on Chow groups. *I.e.*, if $f : X \to Y$ is a morphism of schemes, we have maps: 20

$$f^* : \mathrm{CH}^p(Y) \to \mathrm{CH}^p(X)$$

for f flat, and

$$f_* : \mathrm{CH}_q(X) \to \mathrm{CH}_q(Y)$$

for f proper.

See [17], for proofs, at least for maps between varieties over fields. For general schemes, there is a proof in [24] using algebraic K-theory.

Intersection Multiplicities and the Moving Lemma

The product structure on the Chow groups of a smooth quasi-projective variety over a field was first constructed in the 1950's. See Séminaire Chevalley ([1]), exposés 2 and 3. Two key steps in the construction are:

- Defining intersection multiplicities.
- The Moving Lemma.

Suppose that X is a noetherian scheme. Two closed subsets Y and Z of X, of codimensions p and q, respectively, are said to *intersect properly* if every irreducible component W of $Y \cap Z$ has codimension $p + q$ – note that this is vacuously true if Y and Z do not intersect. Two cycles $\eta = \sum_i m_i[Y_i]$ and $\zeta = \sum_j n_j[Z_j]$ are said to *meet properly* if their supports intersect properly. If two prime cycles $[Y]$ and $[Z]$ meet properly, then the problem of *intersection multiplicities* is to assign an integer $\mu_W(Y,Z)$ to each irreducible component W of $Y \cap Z$. These multiplicities should have the property that if one defines the product of two prime cycles $[Y]$ and $[Z]$ to be $[Y].[Z] = \sum_W \mu_W(Y,Z)[W]$, then one gets a ring structure on the Chow groups.

To get a well defined product, one would certainly require that if η and ζ are two cycles which meet properly, and η (resp. ζ) is rationally equivalent to a cycle η' (resp. ζ'), such that η' and ζ' meet properly, then $\eta.\zeta$ and $\eta'.\zeta'$ are rationally equivalent, and that every pair of cycles η and ζ is rationally equivalent to a pair η' and ζ' which meet properly. Finally, one requires, that the product with divisors be consistent with intersection with divisors.

Since $\eta.\zeta$ is defined when η is a divisor, then it will also be defined when $\eta = \alpha_1.\dots.\alpha_p$ is the successive intersection product of divisors. (Of course one should worry whether this product is independent of the choice of the α_i.) Thus one will be able to define to define $[Y].\zeta$ when $Y \subset X$ is a closed subvariety which is globally a complete intersection in X. If Y is only a local complete intersection, and if W is an irreducible component of $Y \cap Z$ with generic point w, then one defines the intersection multiplicity $\mu_W(Y,Z)$ by using the fact that Y is a complete intersection in a Zariski open neighborhood of w.

The original definition of the intersection multiplicities of the components of the intersection of two closed integral subschemes $Y \subset X$ and $Z \subset X$ which meet properly on a smooth variety X over a field, was given by Samuel [59], when one of them, Y say, is a local complete intersection subscheme of X. One then defines the multiplicities for general integral subschemes Y and Z of a smooth variety X, by observing that $Y \cap Z = \Delta_X \cap (Y \times_k Z)$, where $\Delta_X \subset X \times_k X$ is the diagonal, and then setting the $\mu_W(Y,Z) = \mu_{\Delta_W}(\Delta_X, Y \times_k Z)$. Note that Δ_X is an l.c.i. subvariety if and only if X is smooth.

Once given a definition of multiplicity, one has an intersection product for cycles which meet properly. The next step is:

21 **Theorem 21:** *Chow's Moving Lemma* Suppose that X is a smooth quasi-projective variety over a field k, and that Y and Z are closed integral subschemes of X. Then the cycle $[Y]$ is rationally equivalent to a cycle η which meets Z properly.

Proof See [1] and [55].

22 **Theorem 22** Let X be a smooth quasi-projective variety over a field k. Then one has:
- Given elements α and β in the Chow ring, let η and ζ be cycles representing them which meet properly (these exist by the moving lemma). Then the class in $\mathrm{CH}^*(X)$ of $\eta.\zeta$ is independent of the choice of representatives η and ζ, and depends only on α and β.

- The product on $\mathrm{CH}^*(X)$ that this defines is commutative and associative.
- Given an arbitrary (*i.e.*, not necessarily flat) morphism $f : X \to Y$ between smooth projective varieties, there is a pull back map $\mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$, making $X \mapsto \mathrm{CH}^*(X)$ is a contravariant functor from the category of quasi-projective smooth varieties to the category of commutative rings.

Proof See Séminaire Chevalley, exposés 2 and 3 in [1].

There are several drawbacks to this method of constructing the product on the Chow ring:
- It only works for X smooth and quasi-projective over a field.
- It does not respect supports. It is reasonable to expect that the intersection product of two cycles should be a cycle supported on the set-theoretic intersection of the support of the two cycles, but the Moving Lemma does not "respect supports".
- It requires a substantial amount of work to check that this product is well defined and has all the properties that one requires.

In the next section, we shall see an alternative geometric approach to this problem.

Intersection via Deformation to the Normal Cone

Let us recall the goal: one wishes to put a ring structure on $\mathrm{CH}^*(X)$, for X a smooth quasi-projective variety, and one wants this ring structure to have various properties, including compatibility with intersections with Cartier divisors. The approach of Fulton, [17], as it applies to smooth varieties over a field, can be summarized in the following theorem:

Theorem 23 On the category of smooth, not necessarily quasi-projective, varieties over a field, there is a *unique* contravariant graded ring structure on CH^* such that: 23

1. It agrees with flat pull-back of cycles when $f : X \to Y$ is flat.
2. It agrees with the product $\mathrm{CH}^1(X) \times \mathrm{CH}^p(X) \to \mathrm{CH}^{p+1}(X)$ induced by intersection with Cartier divisors, for all X and p.
3. If V and W are arbitrary integral closed subschemes a smooth variety X, then we have an equality of cycles on $X \times_k X$:

$$[V \times_k X].[X \times_k W] = [V \times_k W]$$

4. If $f : X \to Y$ is a proper map between nonsingular varieties, and $\alpha \in \mathrm{CH}^*(X), \beta \in \mathrm{CH}^*(Y)$, then

$$f_*\left(\alpha . f^*(\beta)\right) = f_*(\alpha).\beta\ .$$

   (The projection formula)

5. If $p : V \to X$ is a vector bundle over a variety X, then the flat pull-back map $p^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(V)$ is an isomorphism. (Homotopy Invariance).

Sketch of Proof
Any map $f : Y \to X$ of smooth varieties can be factored into $Y \xrightarrow{\Gamma_f} Y \times X \xrightarrow{\pi_X} X$, with Γ_f the graph of f, and π_X the projection map. Since π_X is flat, to define $f^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ we need only define Γ_f^*.

Therefore we need only construct the pull-back map for a general regular immersion $f : Y \to X$. First we need (see [17], sects. 2.3 and 2.4, especially corollary 2.4.1):

24 **Lemma 24:** *Specialization* Let $D \subset S$ be a principal divisor in the scheme S. Since $\mathcal{O}_S(D)|_D$ is trivial, intersection with D, $\cap[D] : \mathrm{CH}^*(D) \to \mathrm{CH}^{*-1}(D)$, is zero. It follows that $\cap[D] : \mathrm{CH}^*(S) \to \mathrm{CH}^*(D)$ factors through $\mathrm{CH}^*(S - D)$.

Let $W_{Y/X}$ be the associated deformation to the normal bundle space (see [3] and appendix 2.7). Since the special fiber $W_0 \subset W_{Y/X}$ is a principal divisor, there is an associated specialization map

$$\sigma : \mathrm{CH}^*\left((W_{Y/X} - N_{Y/X}) \simeq X \times \mathbb{G}_m\right) \to \mathrm{CH}^*(W_0)\,.$$

Composing with the flat pull-back:

$$\mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times \mathbb{G}_m)\,,$$

we get a map

$$\mathrm{CH}^*(X) \to \mathrm{CH}^*(N_{Y/X}) \simeq \mathrm{CH}^*(Y)\,,$$

where $\mathrm{CH}^*(N_{Y/X}) \simeq \mathrm{CH}^*(Y)$ by homotopy invariance.

It is not difficult to show, using homotopy invariance, that this must agree with the pull-back map.

Finally, to get the product, one simply composes the pull-back along the inclusion of the diagonal with the external product

$$\mathrm{CH}^*(X) \times \mathrm{CH}^*(X) \to \mathrm{CH}^*(X \times X)\,.$$

This geometric construction avoids any need to give a definition of intersection multiplicity, and also shows that for any two cycles Y and Z, $Y.Z$ is naturally a cycle on the intersection $\mathrm{supp}(Y) \cap \mathrm{supp}(Z)$.

Corresponding to this cohomology theory on the category of non-singular varieties over a field k, we also have *Chow Homology* groups, defined for *all* varieties over k:

Definition 25 If X is a (possibly singular) variety over a field, let $Z_p(X)$ be the group of *dimension* p cycles on X, and $\mathrm{CH}_p(X)$ the corresponding quotient by rational equivalence. These groups are *covariant* functors with respect to *proper* morphisms between varieties, and contravariant with respect to flat maps (but with a degree shift by the relative dimension). 25

K-Theory and Intersection Multiplicities 2.3

Serre's tor Formula 2.3.1

While deformation to the normal cone tells us that intersection theory is unique, given a collection of reasonable axioms, one can ask if there is an intrinsic, purely algebraic, description of intersection multiplicities, and in particular a definition that is valid on any regular scheme. A solution to this problem was given by Serre in his book [61].

If R is a noetherian local ring, an R-module has finite length if and only if it is supported at the closed point of $\mathrm{Spec}(R)$, and K_0 of the category of modules of finite length is isomorphic, by dévissage, to K_0 of the category of vector spaces over the residue field k of R, *i.e.*, to $\mathbb{Z}$. Given an R-module M of finite length, we write $\ell(M)$, for its length.

Definition 26 Suppose that R is a regular local ring, and that M and N are finitely generated R-modules, the supports of which intersect only at the closed point of $\mathrm{Spec}(R)$, then Serre defines their intersection multiplicity: 26

$$\chi(M,N) := \sum_{i\geq 0}(-1)^i \ell\left(\mathrm{Tor}_i^R(M,N)\right) \ .$$

In his book, Serre proved:

Theorem 27 The multiplicity defined above agrees with Samuel's multiplicity, when that is defined. 27

Theorem 28 If R is essentially of finite type over a field, and if the codimensions of the supports of M and N sum to more than the dimension of R, then the intersection multiplicity vanishes, while if the sum is *equal* to the dimension of R, the intersection multiplicity is (strictly) positive. 28

Idea of Proof

There are two key points:

- **Reduction to the diagonal.** If R is a k-algebra, and M and N are R-modules, then $M \otimes_R N \simeq R \otimes_{R\otimes_k R} (M \otimes_k N)$. Thus if M and N are flat k-modules, as is the case when k is a field, to understand $Tor_*^R(M,N)$, it is enough to understand $Tor_*^{R\otimes_k R}(R, \_)$.
- **Koszul Complexes.** If R is regular local ring which is a localization of an algebra which is smooth over a field k, then a choice of system of parameters for R determines a finite free resolution of R as an $R \otimes_k R$-algebra by a Koszul complex. One proves positivity, first for intersections with principal effective Cartier divisors, and then using induction on the number of parameters.

Serre conjectured:

29 **Conjecture 29** The conclusion of the theorem holds for any regular local ring.

The vanishing conjecture was proved in 1985 by Roberts [56] and independently by Gillet and Soulé [21]. Non-negativity (but not strict positivity) was proved by Gabber in 1996. Gabber's proof uses, in an essential fashion, de Jong's theorem [14] on the existence of non-singular alterations of varieties over discrete valuation rings. Gabber did not publish his proof, but there are various expositions of it, for example by Berthelot in his Bourbaki exposé on the work of de Jong [4] and by Roberts [57].

2.3.2 K_0 with Supports

Serre's definition of intersection multiplicity can be rephrased using K_0 with supports. (We shall discuss higher K-theory with supports later).

Let X be a noetherian scheme. Then if $Y \subset X$ is a closed subset, we define $K_0^Y(X)$ to be the quotient of the Grothendieck group of bounded complexes of locally free coherent sheaves of $\mathcal{O}_X$-modules, having cohomology with supports in Y, by the subgroup of classes of acyclic complexes.

There is a natural map

$$\begin{aligned} K_0^Y(X) &\to G_0(Y) \\ [\mathcal{E}^*] &\mapsto \sum_i (-1)^i \left[\mathcal{H}^i(\mathcal{E}^*)\right] \ . \end{aligned}$$

If X is a regular noetherian scheme this is map is an isomorphism, because every coherent sheaf has a resolution by locally free sheaves.

If Y and Z are closed subsets of X, then there is a natural product

$$K_0^Y(X) \otimes K_0^Z(X) \to K_0^{Y\cap Z}(X)$$

given by

$$[\mathcal{E}^*] \otimes [\mathcal{F}^*] \mapsto [\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{F}^*] \,.$$

(The definition of the tensor product of two complexes may be found, for example, in [70]).

If X is regular then this induces a pairing:

$$\begin{aligned} G_0(Y) \otimes G_0(Z) &\to G_0(Y \cap Z) \\ [\mathcal{E}] \otimes [\mathcal{F}] &\mapsto \sum_i (-1)^i \left[\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})\right] \,. \end{aligned}$$

Therefore we see that Serre's intersection multiplicity is a special case of the product in K-theory with supports. *I.e.*, if $X = \operatorname{Spec}(R)$, with R a regular local ring, and if M and N are finitely generated R-modules with supports $Y \subset X$ and $Z \subset X$ respectively, such that $Y \cap Z = \{x\}$, with $x \in X$ the closed point, then $\chi(M, N) = [M].[N] \in K_0^{\{x\}}(X) \simeq \mathbb{Z}$, where $[M] \in K_0^Y(X)$ is the class of any projective resolution of M, and similarly for $[N]$.

The Filtration by Codimension of Supports

Definition 30 A family of supports on a topological space X is a collection Φ of closed subsets of X which is closed under finite unions, and such that any closed subset of a member of Φ is also in Φ. Given two families of supports Φ and Ψ we set $\Phi \wedge \Psi$ equal to the family generated by the intersections $Y \cap Z$ with $Y \in \Phi$ and $Z \in \Psi$. **30**

Definition 31 Let X be a scheme, and let Φ be a family of supports on X. Then **31**

$$K_0^{\Phi}(X) := \lim_{\substack{\to \\ Y\in\Phi}} K_0^Y(X)$$

Clearly there is a product

$$K_0^{\Phi}(X) \otimes K_0^{\Psi}(X) \to K_0^{\Phi\wedge\Psi}(X) \,.$$

For intersection theory, the most important families of supports are $X^{\geq i}$, the closed subsets of codimension at least i, and $X_{\leq j}$, the subsets of dimension at most j.

32 **Definition 32** The *filtration by codimension of supports*, or *coniveau filtration*, is the decreasing filtration, for $i \geq 0$:

$$F^i_{\mathrm{cod}}(K_0(X)) := \mathrm{Image}\left(K_0^{X^{\geq i}}(X) \to K_0(X)\right) .$$

Similarly, we can consider the coniveau filtration on $G_0(X)$ where $F^i(G_0(X))$ is the subgroup of $G_0(X)$ generated by the classes of those $\mathcal{O}_X$-modules $[\mathcal{M}]$ for which $\mathrm{codim}(\mathrm{Supp}(\mathcal{M})) \geq i$.

We shall write $\mathrm{Gr}^{\bullet}_{\mathrm{cod}}(K_0(X))$ and $\mathrm{Gr}^{\bullet}_{\mathrm{cod}}(G_0(X))$ for the associated graded groups.

The Coniveau Filtration and Chow Groups

If $Y \subset X$ is a codimension p subscheme of a Noetherian scheme, then $[\mathcal{O}_X] \in F^p(G_0(X))$. Thus we have a map:

$$Z^p(X) \to F^p_{\mathrm{cod}}(G_0(X)) .$$

By dévissage, *i.e.*, the fact that every coherent sheaf has a filtration with quotients which are coherent sheaves on, and have supports equal to, closed integral subschemes, ([2] appendix to exp. 0, prop. 2.6.), we have:

33 **Lemma 33** If X is a noetherian scheme, the induced map

$$Z^p(X) \to \mathrm{Gr}^p_{\mathrm{cod}}(G_0(X))$$

is *surjective.*

34 **Theorem 34** For an arbitrary noetherian scheme, this map factors through $\mathrm{CH}^p(X)$.

Proof The original proof due to Grothendieck, is in *op. cit.*, appendix to exp. 0, Theorem 2.12. One can also observe that the homomorphism of Lemma 33 is simply an edge homomorphism from $Z^p(X) = E_1^{p,-p}$ to $\mathrm{Gr}^p_{\mathrm{cod}}(G_0(X)) = E_\infty^{p,-p}$ in the Quillen spectral sequence (Sect. 2.5.4 below) which factors through $E_2^{p,-p} \simeq \mathrm{CH}^p(X)$.

Because there is such a close relationship between the Chow groups and $\mathrm{Gr}^{\bullet}_{\mathrm{cod}}K_0$, including the fact that Serre's definition of intersection multiplicities is via the product in K-theory, it is reasonable to ask whether the product structure on K-theory is compatible with the coniveau filtration, *i.e.*, whether $F^p_{\mathrm{cod}}(K_0(X)).F^q_{\mathrm{cod}}(K_0(X)) \subset F^{p+q}_{\mathrm{cod}}(K_0(X))$.

Observe that this is not true at the level of modules. *I.e.*, if $\mathcal{E}^*$ and $\mathcal{F}^*$ are complexes of locally free sheaves of $\mathcal{O}_X$-modules which have their cohomology sheaves supported in codimensions p and q respectively, then it is *not* in general true that $\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{F}^*$ has its cohomology sheaves in codimension at least $p + q$.

The following theorem was proved by Grothendieck, using Chow's moving lemma; see [2], appendix to exp. 0, §4, corollary 1 to theorem 2.12.

Theorem 35 If X is a smooth quasi-projective variety over a field, then the product structure on $K_0(X)$ is compatible with the coniveau filtration. 35

Using the Riemann–Roch theorem for a closed immersion between smooth (not necessarily quasi-projective) varieties, one can extend Grothendieck's result to all smooth varieties. See [32] for details. Later, in Sect. 2.5.11, we shall prove the analogous result for $K_p(X)$ for all $p \geq 0$, again for general smooth varieties over a field, using deformation to the normal cone rather than the moving lemma. There is also another proof of this more general result, which uses Quillen's theorem that Gersten's conjecture is true for non-singular varieties, together with the homotopy theory of sheaves of spectra, in Sect. 2.5.6 below.

In general, one conjectures:

Conjecture 36: *Multiplicativity of the coniveau filtration* If X is a regular noetherian scheme, the product on K-theory respects the filtration by codimension of supports, and $*$ is the product on K-theory: 36

$$F^i_{\mathrm{cod}}\left(K_0^Y(X)\right) * F^j_{\mathrm{cod}}\left(K_0^Z(X)\right) \subset F^{i+j}_{\mathrm{cod}}\left(K_0^{Y\cap Z}(X)\right)$$

Note that:

Proposition 37 Conjecture 36 implies Serre's vanishing conjecture. 37

Proof Suppose that R is a regular local ring of dimension n, and that M and N are finitely generated R-modules, supported on closed subsets Y (of codimension p) and Z (of codimension q) of $X = \mathrm{Spec}(R)$. Suppose also that $Y \cap Z = \{x\}$, where $x \in X$ is the closed point. Then $[M] \in F^p_{\mathrm{cod}}(K_0^Y(X))$, $[N] \in F^q_{\mathrm{cod}}(K_0^Z(X))$, and

$$\chi(M,N) = [M] \cup [N] \in F^{p+q}_{\mathrm{cod}}\left(K_0^{\{x\}}(X)\right) .$$

Now $K_0^{\{x\}}(X) \simeq \mathbb{Z}[\mathbf{k}(x)]$, with $[\mathbf{k}(x)] \in F^n_{\mathrm{cod}}(K_0^{\{x\}}(X)) \setminus F^{n+1}_{\mathrm{cod}}(K_0^{\{x\}}(X))$. Therefore if $p + q > n$, we have

$$\chi(M,N) = [M] \cup [N] \in F^{n+1}_{\mathrm{cod}}\left(K_0^{\{x\}}(X)\right) \simeq 0 .$$

In the next section, we shall sketch how Conjecture 36 can be proved, *after tensoring with* $\mathbb{Q}$, following the method of [21].

2.3.3
The Coniveau Filtration and the γ-Filtration

In [2] Grothendieck constructed a product on $\mathrm{CH}^*(X)_{\mathbb{Q}}$, for X a regular scheme, by constructing a *multiplicative* filtration $F^\cdot_\gamma(K_0(X))$ on K_0, such that there are Chern classes with values in the graded ring $\mathrm{Gr}^\cdot_\gamma(K_0(X))_{\mathbb{Q}}$. He then used the Chern classes to define an isomorphism:

$$\mathrm{CH}^*(X)_{\mathbb{Q}} \simeq \mathrm{Gr}^\cdot_\gamma K_0(X)_{\mathbb{Q}} \ ,$$

and hence a product on $\mathrm{CH}*(X)_{\mathbb{Q}}$.

For any scheme, there are operations $\lambda^i : K_0(X) \to K_0(X)$, defined by taking exterior powers: $\lambda^i([\mathcal{E}]) = [\bigwedge^i(\mathcal{E})]$. Note that these are not group homomorphisms, but rather $\lambda^n(x+y) = \sum_{i=0}^n \lambda^i(x)\lambda^{(n-i)}(y)$.

38 **Definition 38** The γ-operations are defined by:

$$\gamma^n : K_0(X) \to K_0(X)$$
$$\gamma^n : x \mapsto \lambda^n\left(x + (n-1)[\mathcal{O}_X]\right) \ .$$

The γ-filtration $F^\cdot_\gamma(K_0(X))$ is defined by setting $F^1_\gamma(K_0(X))$ equal to the subgroup generated by classes that are (locally) of rank zero, and then requiring that if $x \in F^1_\gamma(K_0(X))$ then $\gamma^i(x) \in F^i_\gamma(K_0(X))$, and that the filtration be multiplicative, i.e., $F^i_\gamma(K_0(X)).F^j_\gamma(K_0(X)) \subset F^{i+j}_\gamma(K_0(X))$, so that the associated graded object $\mathrm{Gr}^\cdot_\gamma K_0(X)$ is a commutative ring, which is contravariant with respect to X.

Recall, following [38], that a theory of Chern classes with values in a cohomology theory A^* associates to every locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on a scheme X, classes $C_k(\mathcal{E}) \in A^k(X)$, for $k \geq 0$ such that

1. $C_0(\mathcal{E}) = 1$.
2. The map $\mathcal{L} \to C_1(\mathcal{L}) \in A^1(X)$ defines a natural transformation $\mathrm{Pic} \to A^1$.
3. If

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is an exact sequence of locally free sheaves, then, for all $n \geq 0$, we have the *Whitney sum formula*:

$$C_n(\mathcal{F}) = \sum_{i=0}^{n} C_i(\mathcal{E})C_{n-i}(\mathcal{G}) \ .$$

39 **Proposition 39:** *Grothendieck, [2]* The natural transformations which assign to a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules, the elements, for $k \geq 1$,

$$C_k(\mathcal{E}) := \gamma^k([\mathcal{E}] - \mathrm{rk}(\mathcal{E})) \in \mathrm{Gr}^k_\gamma K_0(X) \ ,$$

satisfy the axioms for Chern classes.

Corresponding to these classes there is a Chern character which is a natural transformation:

$$ch : K_0(X) \to \mathrm{Gr}^*_\gamma(K_0(X))_\mathbb{Q}$$

The following theorem is proved in [2], theorem VII 4.11, where there is the extra assumption that X possesses an ample sheaf. However as explained in [32], the same proof, with only minor modifications, works for general regular schemes. This theorem also follows from the results of Soulé ([64]), where the result is also proved for K_1 and K_2, by studying the action of the Adams operations on the Quillen spectral sequence; see Theorem 78 below.

Theorem 40 If X is a regular scheme, the Chern character induces an isomorphism: 40

$$\mathrm{Gr}^*_{cod}(K_0(X))_\mathbb{Q} \to \mathrm{Gr}^*_\gamma(K_0(X))_\mathbb{Q}$$

Furthermore, there is an isomorphism, for each $k \geq 0$

$$\begin{aligned} \mathrm{CH}^k(X)_\mathbb{Q} &\to \mathrm{Gr}^k_\gamma(K_0(X))_\mathbb{Q} \\ [Y] &\mapsto ch_k([\mathcal{O}_Y]) \end{aligned}$$

Corollary 41 If X is a regular Noetherian scheme, the coniveau filtration on $K_0(X)_\mathbb{Q}$ is multiplicative. 41

In [21], it shown that one can also construct lambda operations on K-theory with supports, and that, after tensoring with $\mathbb{Q}$, the coniveau filtration and γ-filtrations on the K_0, with supports in a closed subset, of a finite dimensional regular noetherian scheme are isomorphic, and hence the coniveau filtration (tensor $\mathbb{Q}$) is multiplicative. An immediate consequence of this result is Serre's vanishing conjecture for general regular local rings. Robert's proof in [56] used Fulton's operational Chow groups, which give an alternative method of constructing the product on $\mathrm{CH}^*(X)_\mathbb{Q}$ for a general regular Noetherian scheme.

Complexes Computing Chow Groups 2.4

Higher Rational Equivalence and Milnor K-Theory 2.4.1

Suppose that X is a noetherian scheme, and that $Y \subset X$ is a closed subset with complement $U = X - Y$. Then we have short exact sequences (note that $Z(Y)$ and $R(Y)$ are independent of the particular subscheme structure we put on Y):

$$0 \to R(Y) \to R(X) \to R(U) \to 0$$

and

$$0 \to Z(Y) \to Z(X) \to Z(U) \to 0$$

and hence an exact sequence:

$$\mathrm{Ker}\left(\mathrm{div} : R(U) \to Z(U)\right) \to \mathrm{CH}(Y) \to \mathrm{CH}(X) \to \mathrm{CH}(U) \to 0 \, .$$

It is natural to ask if this sequence can be extended to the left, and whether there is a natural notion of rational equivalence between K_1-chains. In particular are there elements in the kernel $\mathrm{Ker}(\mathrm{div} : R(U) \to Z(U))$ that obviously have trivial divisor, and so will map to zero in $\mathrm{CH}(Y)$?

Let us start by asking, given a scheme X, whether there are elements in $R(X)$ which obviously have divisor 0. First of all, any $f \in k(x)^*$ which has valuation zero for all discrete valuations of the field $k(x)$ is in the kernel of the divisor map. However since we are dealing with general schemes, the only elements of $k(x)$ which we can be sure are of this form are ± 1.

Suppose now that X is an integral scheme, and that f and g are two rational functions on X, *i.e.*, elements of $\mathbf{k}(X)^*$, such that the Weil divisors $\mathrm{div}(f)$ and $\mathrm{div}(g)$ have no components in common. Writing $\{f\}$ and $\{g\}$ for the two (principal) Cartier divisors defined by these rational functions, we can consider the two cap products:

$$\{f\} \cap \mathrm{div}(g) = \mathrm{div}\left\{f|_{\mathrm{div}(g)}\right\}$$

and

$$\{g\} \cap \mathrm{div}(f) = \mathrm{div}\left\{g|_{\mathrm{div}(f)}\right\} \, .$$

Here $\{f\}|_{\sum_i [Y_i]} := \sum_i \{f\}|_{[Y_i]}$, where $\{f\}|_{[Y_i]}$ denotes, equivalently, the restriction of f either as a Cartier divisor (Definition 14), or simply as a rational function which is regular at the generic point of Y_i. If one supposes that the cap product between Chow cohomology and Chow homology is to be associative, and that the product in Chow cohomology is to be commutative, then we should have:

$$\begin{aligned} \{f\} \cap \left(\{g\} \cap [X]\right) &= \left(\{f\} * \{g\}\right) \cap [X] \\ &= \left(\{g\} * \{f\}\right) \cap [X] \\ &= \{g\} \cap \left(\{f\} \cap [X]\right) \, . \end{aligned}$$

I.e. the K_1-chain

$$g|_{\mathrm{div}(f)} - f|_{\mathrm{div}(g)}$$

should have divisor zero.

That this is indeed the case follows from the following general result, in which $\{f\}$ and $\{g\}$ are replaced by general Cartier divisors.

Proposition 42: *Commutativity of Intersections of Cartier Divisors* Let X be an integral scheme, and suppose that ϕ and ψ are two Cartier divisors, with $div(\phi) = \sum_i n_i[Y_i]$ and $div(\psi) = \sum_j m_j[Z_j]$ their associated Weil Divisors. Then **42**

$$\sum_i n_i \mathrm{div}\big(\psi|_{Y_i}\big) = \sum_j m_j \mathrm{div}\big(\phi|_{Z_j}\big) .$$

The original proof of this result, in [24], used higher algebraic K-theory, and depended on the properties of the coniveau spectral sequence for K-theory ([53]). However if one wants to avoid proofs using K-theory, then for varieties over fields this is proved in Fulton's book ([17], theorem 2.4), and there is a purely algebraic proof of the general case by Kresch in [46].

Thus every pair of rational functions (f, g), as above, gives rise to a K_1-chain with trivial divisor. This suggests that one could view such K_1-chains as being rationally equivalent to zero, *i.e.* that one should extend the complex $R(X) \to Z(X)$ to the left by $\bigoplus_x k(x)^* \otimes k(x)^*$, with "div" $(f \otimes g) = f|_{\mathrm{div}(g)} - g|_{\mathrm{div}(f)}$. Since

$$\text{“div”}\ (f \otimes g) = -\text{“div”}\ (g \otimes f) ,$$

it seems reasonable to impose the relation $f \otimes g + g \otimes f = 0$. Again it is natural to ask what elements are obviously in the kernel of this map. An element $f \otimes g \in k(x)^* \otimes k(x)^*$ such that $f \equiv 1(\mathrm{mod} g)$, and $g \equiv 1(\mathrm{mod} f)$, will be in the kernel of the map "div", and the elements of $k(x)^* \otimes k(x)^*$ that we can be sure are of this type are those of the form $f \otimes g$ with $f + g = 1$.

This leads naturally to the quotient of the exterior algebra $\bigwedge^*(F) = \bigoplus_n \bigwedge^n_{\mathbb{Z}} F^*$ (of F^* viewed as a $\mathbb{Z}$-module) by the two-sided ideal I generated by elements of the form $f \otimes g$ with $f + g = 1$:

Definition 43 If F is a field, its Milnor K-theory is defined to be: **43**

$$K^M_*(F) := \bigwedge\nolimits^*(F)/I .$$

Note that the relation $\{f, g\} = -\{g, f\} \in K^M_2(F)$ can be deduced from the relation $\{f, 1 - f\} = 0$, and hence one can equally write:

$$K^M_*(F) := T^*(F)/I ,$$

where $T^*(F)$ is the tensor algebra of F.

Rost's Axiomatics 2.4.2

The fact that Milnor K-theory appears so naturally when trying to construct a complex that computes the Chow groups is fully explored in the paper [58] of Rost, where he proves that one need impose no more relations, or add any more generators, to get a theory which has very nice properties.

Rost considers a more general structure, which includes Milnor K-theory as a special case.

44 **Definition 44** A (graded) cycle module is a covariant functor M from the category of fields (over some fixed base scheme S) to the category of $\mathbb{Z}$-graded (or $\mathbb{Z}/2$-graded) Abelian groups, together with:

1. Transfers $\mathrm{tr}_{E/F} : M(E) \to M(F)$, of degree zero, for every finite extension $F \subset E$.
2. For every discrete valuation v of a field F a residue or boundary map $\partial_v : M(F) \to M(k(v))$ of degree -1.
3. A pairing, for every F, $F^* \times M(F) \to M(F)$ of degree 1, which extends to a pairing $K^M_*(F) \times M(F) \to M(F)$ which makes $M(F)$ a graded module over the Milnor-K-theory ring.

These data are required to satisfy axioms which may be found in *op. cit.*, Definitions 1.1 and 2.1.

A cycle module M is said to be a cycle module with *ring structure* if there is a pairing $M \times M \to M$, respecting the grading, which is compatible with the cycle module structure; see *op. cit.* Definition 1.2.

Milnor K-theory itself is a cycle module with ring structure. This follows from results of Bass and Tate, of Kato, and of Milnor; see [58], theorem 1.4. We shall see later (Theorem 65) that the same holds for the Quillen K-theory of fields.

45 **Definition 45** Let X be a variety over a field. Then $C^*(X, M, q)$ is the complex:

$$C^p(X, M, q) := \bigoplus_{x \in X^{(p)}} M_{q-p}(k(x))$$

with the differential $C^p(X, M, q) \to C^{p+1}(X, M, q)$ induced by the maps $\partial_v : M_{q-p}(k(x)) \to M_{q-p-1}(k(v))$ for each discrete valuation v of $k(x)$ which is trivial on the ground field. (Here, if k is a field, $M_n(k)$ is the degree n component of $M(k)$.)

Similarly, one defines the homological complex:

$$C_p(X, M, q) := \bigoplus_{x \in X^{(p)}} M_{q-p}(k(x)) \ .$$

These complexes are the "cycle complexes" associated to the cycle module M.

One then defines:

46 **Definition 46**

$$\mathrm{A}_p(X, M, q) := H_p\left(C_*(X, M, q)\right)$$

and

$$\mathrm{A}^p(X, M, q) := H^p\left(C^*(X, M, q)\right) \ .$$

It is easy to prove that the cohomological complex for Milnor K-theory is contravariant with respect to flat maps. To prove that the corresponding homological complex is covariant with respect to proper maps, one uses Weil reciprocity for curves, see [58], proposition 4.6.; a similar argument for Quillen K-theory is also in [24]. Therefore the groups $\mathrm{A}^*(X, M, q)$ are contravariant with respect to flat morphisms, while the groups $\mathrm{A}_*(X, M, q)$ are covariant with respect to proper morphisms.

Remark 47 One can consider bases more general than a field. In [58] Rost fixes a base B which is a scheme over a field, and then considers schemes X of finite type over B. More generally, it is easy to see that the homological theory can be defined for schemes of finite type over a fixed excellent base B, so long as one grades the complexes by $\dim_B$ (see Definition 9). 47

Since, if F is a field, $K_1(F) = K_1^M(F) = F^*$, and $K_0(F) = K_0^M(F) = \mathbb{Z}$, we see that the last two terms in $C_*(X, M, p)$ are the groups $R_{p-1}(X)$ and $Z_p(X)$ of dimension $p-1$ K_1-chains and dimension p cycles on X. Therefore we have ([58], remark 5.1):

Proposition 48 If M_* is Milnor K-theory (or Quillen K-theory): 48

$$A_p(X, K^M, -p) \simeq \mathrm{CH}_p(X)$$

$$A^p(X, K^M, p) \simeq \mathrm{CH}^p(X) .$$

Rost shows:

Theorem 49 49

1. For any M_*, the cohomology groups $A^*(X, M, *)$ are homotopy invariant, *i.e.*, for any flat morphism $\pi : E \to X$ with fibres affine spaces, $\pi^* : A^*(X, M, *) \to A^*(E, M, *)$ is an isomorphism.
2. For any M_*, if $f : X \to S$ is a flat morphism with S the spectrum of a Dedekind domain Λ, and t is a regular element of Λ, there is a specialization map $\sigma_t : A^*(X_t, M, *) \to A^*(X_0, M, *)$, which preserves the bigrading. Here $X_t = X \times_S \mathrm{Spec}(\Lambda[1/t])$ and $X_0 = X \times_S \mathrm{Spec}(\Lambda/(t))$.
3. If M_* is a cycle module with ring structure, and $f : Y \to X$ is a regular immersion, then there is a Gysin homomorphism $f^* : A^*(X, M, *) \to A^*(Y, M, *)$. This Gysin homomorphism is compatible with flat pull-back:
   a) If $p : Z \to X$ is flat, and $i : Y \to X$ is a regular immersion, then $p_X^* \cdot i^* = i_Z^* \cdot p^*$, where $p_X : X \times_Y Z \to X$ and $i_Z : X \times_Y Z \to Z$ are the projections in the fiber product over Y.
   b) If $p : X \to Y$ is flat, and $i : Y \to X$ is a section of p which is a regular immersion, then $i^* \cdot p^* = 1_Y^*$.
4. If M_* is a cycle module with ring structure, and if X is a smooth variety over a field, then there is a product structure on $A^*(X, M, *)$.

The map σ_t is the composition of the cup-product by $\{t\} \in H^1(X_t, \mathcal{O}_X^*)$ (which is defined since M_* is a module over Milnor K-theory) with the boundary map in the localization sequence for the open subset $X_t \subset X$ with complement X_0. See [58] sect. 11, as well as [24], where a similar construction is used for the case when M_* is Quillen K-theory. The construction of the Gysin map uses deformation to the normal cone, specialization, and homotopy invariance. The product is constructed as the composition of the external product $A^*(X, M, *) \times A^*(X, M, *) \to A^*(X \times X, M, *)$ with the Gysin morphism associated to the diagonal map $\Delta : X \to X \times X$.

50 **Corollary 50** For all $p \geq 0$ and $q \geq 0$, $X \mapsto A^p(X, M, q)$ is a contravariant abelian group valued functor on the category of smooth varieties over k.

Sketch of Proof
If $f : X \to Y$ is a map of smooth varieties over k, we can factor $f = p \cdot \gamma_f$ with $p : X \times_k Y \to Y$ the projection, and $\gamma_f : X \to X \times_k Y$ the graph of f. We then define $f^* = (\gamma_f)^* \cdot p^*$, where p^* is defined since p is flat, and $(\gamma_f)^*$ is defined since γ_f is a regular immersion. To prove that this is compatible with composition, one uses parts a) and b) part 3 of the theorem.

More generally f^* can be defined for any local complete intersection morphism between (not necessarily regular) varieties over k, using the methods of [20].

Let us write $\mathcal{M}_q$ for the sheaf $X \mapsto A^0(X, M, q)$ on the big Zariski site of regular varieties over k. Rost also shows

51 **Theorem 51** If X is the spectrum of regular semi-local ring, which is a localization of an algebra of finite type over the ground field, then for all p, the complex $C^*(X, M, p)$ only has cohomology in degrees $i = 0$.

The proof is variation on the proofs of Gersten's conjecture by Quillen [53] and Gabber [19].

52 **Corollary 52** If X is a regular variety over k, then $H^p(C^*(X, M, q)) \simeq H^p(X, \mathcal{M}_q)$.

We then get immediately, the following variation on Bloch's formula:

53 **Corollary 53** For a variety X as above: $\mathrm{CH}^p(X) \simeq H^p(X, \mathcal{M}_p)$.

Thus Rost's paper shows us that one can construct intersection theory, together with higher "rational equivalence", *i.e.* the higher homology of the cycle complexes, building just on properties of divisors, and that Milnor K-theory arises naturally in this process.

To construct Chern classes, we can follow the method of [24]. Start by observing that since M_* is a K_*^M-module, there are products

$$H^1(X, \mathcal{O}_X^*) \otimes A^p(X, M, q) \to A^{p+1}(X, M, q+1)\ .$$

Proposition 54 Let M_* be a cycle module. Then if X is a variety over k, and $\pi : E \to X$ is a vector bundle of constant rank n, there is an isomorphism $A^p(\mathbb{P}(E), M, q) \simeq \bigoplus_{i=0}^{n-1} A^{p-i}(\mathbb{P}(E), M, q-i)\xi^i$, where $\xi \in H^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}^*)$ is the class of $\mathcal{O}_{\mathbb{P}(E)}(1)$. 54

Proof By a standard spectral sequence argument, this may be reduced to the case when X is a point, and the bundle is trivial, so that $\mathbb{P}(E) \simeq \mathbb{P}^n$. Let $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ be the hyperplane at infinity. It is easy to see that there is a short exact sequence:

$$0 \to C^*(\mathbb{P}^{n-1}, M, q-1)[1] \to C^*(\mathbb{P}^n, M, q) \to C^*(\mathbb{A}^n, M, q) \to 0\ ,$$

which gives rise to a long exact sequence:

$$\ldots \to A^{p-1}(\mathbb{P}^{n-1}, M, q-1) \xrightarrow{j_*} A^p(\mathbb{P}^n, M, q) \xrightarrow{i^*} A^p(\mathbb{A}^n, M, q) \to \ldots\ ,$$

in which j_* is the Gysin homomorphism, and i^* is the pull back map associated to the inclusion of the open subset $i : \mathbb{A}^n \to \mathbb{P}^n$. Let $\pi : \mathbb{P}^n \to \mathrm{Spec}(k)$ be the projection. By homotopy invariance, the map

$$(\pi \cdot i)^* : A^p(\mathrm{Spec}(k), M, q) \to A^p(\mathbb{A}^n, M, q)$$

is an isomorphism, and so i^* is a split monomorphism, while j^* is a split epimorphism. Next, one may show that $i_* \cdot i^* : A^p(\mathbb{P}^n, M, q) \to A^{p+1}(\mathbb{P}^n, M, q+1)$ is the same as cap product by ξ. Since i^* is defined using deformation to the normal cone this is not completely tautologous. The proof then finishes by induction on n.

One may now apply the axiomatic framework of [24], to obtain:

Theorem 55 There is a theory of Chern classes for vector bundles, and also for higher algebraic K-theory, on the category of regular varieties over k, with values in Zariski cohomology with coefficients in the Milnor K-theory sheaf: 55

$$C_n : K_p(X) \to H^{n-p}(X, \mathcal{K}_n^M)$$

which satisfies the properties of *op. cit.*

These classes seem not to be in the literature, though they are known to the experts, and they induce homomorphisms $K_n(F) \to K_n^M(F)$ which are presumably the same as the homomorphisms defined by Suslin in [65].

56 **Remark 56** One can also construct the universal Chern classes $C_p \in H^p(B.\mathbb{G}L_n, \mathcal{K}_p^M)$ by explicitly computing $H^p(B.\mathbb{G}L_n, \mathcal{K}_q^M)$ for all p and all q. To do this one first computes $H^p(\mathbb{G}L_n, \mathcal{K}_q^M)$, using the cellular decomposition of the general linear groups, and then applies a standard spectral sequence argument, to get:

$$H^*(B.\mathbb{G}L_n, \mathcal{K}_*^M) \simeq K_*^M(k)[C_1, C_2, \ldots] .$$

2.5 Higher Algebraic *K*-Theory and Chow Groups

The connection between higher K-theory and Chow groups has at its root the relationship between two different filtrations on the K-theory spectrum of a regular scheme. One of these, the Brown filtration, is intrinsically functorial and compatible with the product structure on K-theory. The other is the coniveau filtration, or filtration by codimension, which is directly related to the Chow groups.

Gersten's conjecture implies that there should be an isomorphism of the corresponding spectral sequences, and hence that these two different filtrations of the K-theory spectrum should induce the same filtration on the K-theory groups. At the E_2 level, this isomorphism of spectral sequences includes Bloch's formula:

$$\mathrm{CH}^p(X) \simeq H^p\left(X, K_p(\mathcal{O}_X)\right) .$$

The equality of these two filtrations on the K-theory groups tells us that, if Gersten's conjecture holds, then the product on the K-theory of a regular scheme is compatible with the coniveau filtration. Recall that this compatibility implies Serre's conjecture on the vanishing of intersection multiplicities.

At the moment Gersten's conjecture is only known for regular varieties over a field. As we saw in the last section, one can also develop intersection theory for smooth varieties over a field, using deformation to the normal cone. At the end of this section, we use deformation to the normal cone to give a new proof, which does not depend on Gersten's conjecture, that the product on the K-theory of a smooth variety is compatible with the coniveau filtration.

2.5.1 Stable Homotopy Theory

Before discussing higher algebraic K-theory, we should fix some basic ideas of stable homotopy theory and of the homotopy theory of presheaves of spectra.

There are various versions of the stable homotopy category available, such as the category of symmetric spectra of [40] and the category of S-modules of [16]. It is shown in [60] that these are essentially equivalent.

For us, spectra have two advantages. The first is that cofibration sequences and fibration sequences are equivalent (see Theorem 3.1.14 of [40]), and the second is that the product in K-theory can be described via smash products of spectra. In particular we will need the following lemma which gives information about the stable homotopy groups of smash products:

Lemma 57 Suppose that $\mathbf{E}$ and $\mathbf{F}$ are spectra with $\pi_i(\mathbf{E}) = 0$ if $i < p$ and $\pi_i(\mathbf{F}) = 0$ if $i < q$. Then $\pi_i(\mathbf{E} \wedge \mathbf{F}) = 0$ if $i < p + q$. 57

Proof This follows from the spectral sequence

$$\mathrm{Tor}^{\pi_*(\mathbf{S})}\left(\pi_*(\mathbf{E}), \pi_*(\mathbf{F})\right) \Rightarrow \pi_*(\mathbf{E} \wedge \mathbf{F}) ,$$

see [16], Chapt. II, Theorem 4.5.

Following the paper [41] of Jardine, the category of presheaves of spectra on (the Zariski topology of) a scheme X may be given a closed model structure in the sense of [54], in which the weak equivalences are the maps of presheaves which induce weak equivalences stalkwise. See also the papers [10] of Brown and [11] of Brown and Gersten, as well as [23].

If $\mathbf{E}$ is a presheaf of spectra on X, we define $R\Gamma(X, \mathbf{E})$ to be $\Gamma(X, \widetilde{\mathbf{E}})$, where $i : \mathbf{E} \to \widetilde{\mathbf{E}}$ is a *fibrant resolution* of $\mathbf{E}$, *i.e.*, i is a trivial cofibration and $\widetilde{\mathbf{E}}$ is fibrant. If $Y \subset X$ is a closed subset, or if Φ is a family of supports, we define $R\Gamma_Y(X, \mathbf{E})$ and $R\Gamma_\Phi(X, \mathbf{E})$ similarly. By a standard argument, one can show that $R\Gamma(X, \mathbf{E})$ is a fibrant spectrum which is, up to weak equivalence, independent of the choice of fibrant resolution.

One also defines

$$H^n(X, F) := \pi_{-n}\left(R\Gamma(X, F)\right) ,$$

and

$$H^n_\Phi(X, F) := \pi_{-n}\left(R\Gamma_\Phi(X, F)\right) .$$

Note that these are abelian groups.

The reason for this notation is that if $\mathcal{A}$ is a sheaf of abelian groups on X and $\Pi(\mathcal{A}, n)$ is the corresponding sheaf of Eilenberg-Maclane spectra, which has $\pi_i(\Pi(\mathcal{A}, n)) = \mathcal{A}$ if $i = n$, and equal to 0 otherwise, we have:

$$H^p\left(X, \Pi(\mathcal{A}, n)\right) \simeq H^{n-p}(X, \mathcal{A})$$

for $n \geq p$.

2.5.2 Filtrations on the Cohomology of Simplicial Sheaves

If $\mathbf{E}$ is a spectrum, and $F^{\cdot}\mathbf{E}$ is a decreasing filtration of $\mathbf{E}$ by subspectra, there is an associated spectral sequence with

$$E_1^{p,q} = \pi_{-p-q}(F^p\mathbf{E}/F^{p+1}\mathbf{E}) .$$

See [23] for a detailed construction of this spectral sequence, and the associated exact couple.

Given a scheme X satisfying our standard assumptions, and a presheaf of spectra $\mathbf{E}$ on X, one can consider two different filtrations on $R\Gamma(X, \mathbf{E})$.

The first is the "Brown" or hypercohomology filtration:

58 **Definition 58** Let $\mathbf{E}$ be a fibrant simplicial presheaf on the scheme X. Let $\mathbf{E}(\infty, k) \subset \mathbf{E}$ be the sub-presheaf with sections over an open $U \subset X$ consisting of those simplices which have all of their faces of dimension less than k trivial. Since the stalks of $\mathbf{E}$ are fibrant (*i.e.* are Kan simplicial sets), the stalk of $\mathbf{E}(\infty, k)$ at $x \in X$ is the fibre of the map from $\mathbf{E}_x$ to the k-th stage of its Postnikov tower.

If $\mathbf{E} = (E_i)_{i\in\mathbb{N}}$ is a fibrant presheaf of spectra, then we can define similarly its Postnikov tower:

$$\mathbf{E}(\infty, k)_i := E_i(\infty, k + i) .$$

In either case, we set

$$F_B^k R\Gamma(\mathbf{E}) := R\Gamma\left(X, \mathbf{E}(\infty, k)\right) ,$$

and, if Φ is a family of supports on X,

$$F_B^k R\Gamma_{\Phi}(\mathbf{E}) := R\Gamma_{\Phi}\left(X, \mathbf{E}(\infty, k)\right) .$$

Associated to this filtration we have a spectral sequence:

59 **Proposition 59** If X is a scheme (which as usual, we assume to be finite dimensional), and if $\mathbf{E}$ is a presheaf of connective spectra on X, there is a hypercohomology spectral sequence:

$$E_1^{p,q} = \pi_{-p-q}\left(R\Gamma(X, \mathbf{E}(p))\right) \simeq H^q\left(X, \pi_{-p}(\mathbf{E})\right) \Rightarrow H^{p+q}(X, \mathbf{E}) .$$

Here $\mathbf{E}(p)$ denotes the cofiber of $\mathbf{E}(\infty, p+1) \to \mathbf{E}(\infty, p)$, which is weakly equivalent to the presheaf of Eilenberg-Maclane spectra with homotopy groups $\pi_k(\mathbf{E}(p)) = \pi_p(\mathbf{E})$ if $k = p$ and 0 otherwise. This spectral sequence is concentrated in degrees $q \leq 0$ and $0 \leq p \leq \dim(X)$.

More generally, if Φ is a family of supports on X, then we have

$$E_1^{p,q} = \pi_{-p-q}\left(R\Gamma_{\Phi}(X, \mathbf{E}(p))\right) \simeq H^q_{\Phi}\left(X, \pi_{-p}(\mathbf{E})\right) \Rightarrow H^{p+q}_{\Phi}(X, \mathbf{E}) .$$

Proof See [11] and [23].

Definition 60 We shall refer to the spectral sequence of the previous proposition as the *Brown spectral sequence*, and the corresponding filtration on the groups $H^*(X, \mathbf{E})$ and $H^*_\Phi(X, \mathbf{E})$ as the Brown filtration. 60

The second filtration on $R\Gamma(X, \mathbf{E})$ is the coniveau filtration:

Definition 61 Recall that $\Gamma_{X^{\geq k}}$ denotes sections with support of codimension at least k; then we set, for $\mathbf{E}$ a presheaf either of connective spectra, 61

$$F^k R\Gamma(\mathbf{E}) := R\Gamma_{X^{\geq k}}(X, \mathbf{E}) \ .$$

The resulting spectral sequence

$$E_1^{p,q} = E_{1,\mathrm{cod}}^{p,q}(X, \mathbf{E}) \simeq \pi_{-p-q}(R\Gamma_{X^{(p)}}(X, \mathbf{E})) \Rightarrow H^{p+q}(X, \mathbf{E})$$

converges to the coniveau filtration on $H^*(X, \mathbf{E})$.

We can also require everything to have supports in a family of supports Φ:

$$F^k_{\mathrm{cod}} R\Gamma_\Phi(\mathbf{E}) := R\Gamma_{X^{\geq k} \cap \Phi}(X, \mathbf{E}) \ ,$$

to obtain a spectral sequence:

$$E_1^{p,q} = E_{1,\mathrm{cod},\Phi}^{p,q}(X, \mathbf{E}) \simeq \pi_{-p-q}\left(R\Gamma_{X^{(p)} \cap \Phi}(X, \mathbf{E})\right) \Rightarrow H_\Phi^{p+q}(X, \mathbf{E}) \ .$$

It was shown in [23] that these spectral sequences are related. First note that we can renumber the Brown spectral sequence so that it starts at E_2:

$$\widehat{E}_2^{p,q}(X, \mathbf{E}) := \pi_{-p-q}\left(R\Gamma(X, \mathbf{E}(-q))\right) \ .$$

Theorem 62 With X and $\mathbf{E}$ as above, There is a map of spectral sequences, for $r \geq 2$, 62

$$\widehat{E}_r^{p,q}(X, \mathbf{E}) \to E_{r,\mathrm{cod}}^{p,q}(X, \mathbf{E}) \ ,$$

and more generally, given a family of supports Φ,

$$\widehat{E}_{r,\Phi}^{p,q}(X, \mathbf{E}) \to E_{r,\mathrm{cod},\Phi}^{p,q}(X, \mathbf{E}) \ ,$$

Proof See [23], Theorem 2, Sect. 2.2.4. The proof there uses a generalization to sheaves of simplicial groups of the techniques that Deligne used in a unpublished proof of the analogous result for complexes of sheaves of abelian groups (see [9]). There is a discussion of Deligne's result in [51]. A key point in the proof is that

$X - X^{\geq p}$ has dimension $p-1$, so that $H^i_{X-X^{\geq p}}(X, \mathcal{A}) = 0$ for all $i \geq q$ and any sheaf of abelian groups $\mathcal{A}$.

This theorem may be viewed as an analog, in the homotopy theory of simplicial presheaves, of a result of Maunder [50], in which he compared the two different ways of defining the Atiyah-Hirzebruch spectral sequence for the generalized cohomology of a CW complex.

Looking at the map on E_∞ terms, we get:

63 **Corollary 63** With X and E as above,

$$F^k H^*(X, \mathbf{E}) \subset F^k_{\mathrm{cod}} H^*(X, \mathbf{E}) ,$$

for all $k \geq 0$. If Φ is a family of supports on X, then:

$$F^k H^*_\Phi(X, \mathbf{E}) \subset F^k_{\mathrm{cod}} H^*_\Phi(X, \mathbf{E}) .$$

We shall see that for the algebraic K-theory of regular schemes over a field, as well as other cohomology theories for which one can prove Gersten's conjecture, that these two spectral sequences are isomorphic, and hence the filtrations that they converge to are equal.

2.5.3 Review of Basic Notions of K-Theory

Recall that if $\mathcal{E}$ is an *exact category* the K-theory groups $K_p(\mathcal{E})$, for $p \geq 0$ of $\mathcal{E}$ were originally defined by Quillen in [53], to be the homotopy groups $\pi_{p+1}(BQ\mathcal{E})$ of the classifying space of the category $Q\mathcal{E}$ defined in *op.cit.* (Here one takes the zero object of the category as a base point.)

An alternative construction, which gives a space which can be shown to be a deformation retract of $BQ\mathcal{E}$, is Waldhausen's $S_\cdot$-construction. This associates to the exact category $\mathcal{E}$ a simplicial set $S_\cdot\mathcal{E}$. The iterates of $S_\cdot$-construction then give a sequence of deloopings $S^k_\cdot\mathcal{E}$ of $S_\cdot\mathcal{E}$. See Sect. 1.4 of the article of Carlsson in this volume for details. It is straightforward to check that these deloopings may used to define a symmetric spectrum which we will denote $\mathbf{K}(\mathcal{E})$, with $K_*(\mathcal{E}) \simeq \pi_*(\mathbf{K}(\mathcal{E}))$. We may then think of K-theory as a functor from the category of exact categories and exact functors to the category of spectra.

Any bi-exact functor $\Phi : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces pairings

$$S^p_\cdot(\mathcal{A}) \wedge S^q_\cdot(\mathcal{B}) \to S^{p+q}_\cdot(\mathcal{C}) ,$$

which are compatible with the actions of the relevant symmetric groups, and hence induce a pairing:

$$\mathbf{K}(\Phi) : \mathbf{K}(\mathcal{A}) \wedge \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{C}) .$$

If X is a scheme we can consider the abelian category $\mathcal{M}(X)$ of all coherent sheaves on X, and the exact subcategory $\mathcal{P}(X) \subset \mathcal{M}(X)$ of locally free coherent sheaves on X. We shall denote the K-theory spectra of these categories by $\mathbf{G}(X)$ and $\mathbf{K}(X)$ respectively, and the K-theory groups by $G_*(X)$ and $K_*(X)$ respectively. (Note that in [53], $G_*(X)$ is written $K'_*(X)$.) These groups have the following basic properties:

- $X \mapsto K_*(X)$ is a contravariant functor from schemes to graded (anti-)commutative rings. See [53]. The product is induced by the bi-exact functor
$$\mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$
$$(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} ,$$
which induces a pairing of spectra:
$$\mathbf{K}(X) \wedge \mathbf{K}(X) \to \mathbf{K}(X) .$$
- $X \mapsto G_*(X)$ is a covariant functor from the category of proper morphisms between schemes to the category of graded abelian groups. The covariance of $G_*(X)$ is proved in [53] for projective morphisms, while for general proper morphisms it is proved in [27] and [66].
- G_* is also contravariant for flat maps: if $f : X \to Y$ is a flat morphism, then the pull-back functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is exact. More generally, $G_*(X)$ is contravariant with respect any morphism of schemes $f : X \to Y$ which is of finite tor-dimension. This is proven in [53] when Y has an ample line bundle. The general case may deduced from this case by using the fact that the pull back exists locally on Y, since all affine schemes have an ample line bundle, together with the weak equivalence $\mathbf{G}(Y) \simeq R\Gamma(Y, \mathbf{G}_Y)$ discussed in Theorem 66 below. Alternatively, one may show that the pull-back for general f (of finite tor-dimension) exists by the methods of Thomason [66].
- There is a "cap product" $K_*(X) \otimes G_*(X) \to G_*(X)$, which makes $G_*(X)$ a graded $K_*(X)$-module. If $f : X \to Y$ is a proper morphism of schemes, then $f_* : G_*(X) \to G_*(Y)$ is a homomorphism of $K_*(Y)$-modules, where $G_*(X)$ is a $K_*(Y)$-module via the ring homomorphism $f^* : K_*(Y) \to K_*(X)$. This fact is known as the projection formula. The cap product is induced by the bi-exact functor:
$$\mathcal{P}(X) \times \mathcal{M}(X) \to \mathcal{M}(X)$$
$$(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} ,$$
which induces a pairing of spectra:
$$\mathbf{K}(X) \wedge \mathbf{G}(X) \to \mathbf{G}(X) .$$
- If X is regular, then the inclusion $\mathcal{P}(X) \subset \mathcal{M}(X)$ induces an isomorphism on K-theory, $K_*(X) \simeq G_*(X)$.

While $X \mapsto K_*(X)$ is a functor, the operation $X \mapsto \mathcal{P}(X)$ is *not* a functor, since given maps $f : X \to Y$ and $g : Y \to Z$, the functors $(g \cdot f)^*$ and $f^* \cdot g^*$ are only isomorphic, rather than equal. There are standard ways of replacing such a 'pseudo'-functor or 'lax'-functor by an equivalent strict functor; in this case we can replace the category $\mathcal{P}(X)$ of locally free sheaves on X by the equivalent category $\mathcal{P}_{\mathrm{Big}}(X)$ of locally free sheaves on the *Big Zariski Site* over X. An object in this category consists of giving, for every morphism $f : U \to X$, a locally free sheaf $\mathcal{F}_f$ of $\mathcal{O}_U$-modules, and for every triple $(f : U \to X, g : V \to X, h : U \to V)$ such that $g \cdot h = f$, an isomorphism $g^* : g^*(\mathcal{F}_h) \to \mathcal{F}_f$. These data are required to satisfy the obvious compatibilities. Then if $\mathcal{F} \in \mathcal{P}_{\mathrm{Big}}(X)$, and $f : Y \to X$ is a morphism, $(f^*\mathcal{F}_g)$, for $g : U \to Y$ is set equal to $\mathcal{F}_{f \cdot g}$. It is a straightforward exercise to show that $X \mapsto \mathcal{P}_{\mathrm{Big}}(X)$ is a strict functor.

We may then view $X \mapsto \mathbf{K}(X)$ as a contravariant functor from schemes to spectra. When we restrict this functor to a single scheme X, we get a presheaf of spectra which we denote $\mathbf{K}_X$. Similarly we have the presheaf $\mathbf{G}_X$ associated to G theory, together with pairings of presheaves:

$$\mathbf{K}_X \wedge \mathbf{K}_X \to \mathbf{K}_X$$

$$\mathbf{K}_X \wedge \mathbf{G}_X \to \mathbf{G}_X .$$

64 **Definition 64** Let X be a scheme. Given an open subset $U \subset X$, we define $\mathbf{K}(X, U)$ to be the homotopy fibre of the restriction $\mathbf{K}(X) \to \mathbf{K}(U)$. If $Y \subset X$ is a closed subset, then we also write $\mathbf{K}^Y(X) = \mathbf{K}(X, X - Y)$, and $K_*(X, U)$, $K_*^Y(X)$, for the corresponding groups.

We can perform similar constructions for G-theory. However by Quillen's localization and dévissage theorems ([53]) if $i : Y \to X$ is the inclusion of a closed subset of a scheme X, with its structure as a closed reduced subscheme, the exact functor $i_* : \mathcal{M}(Y) \to \mathcal{M}(X)$ induces a map $i_* : \mathbf{G}(Y) \to \mathbf{G}_Y(X)$ which is a homotopy equivalence. More generally, if Φ is any family of supports on X, then $\mathbf{G}_\Phi(X) \simeq \mathbf{K}(\mathcal{M}_\Phi(X))$, the K-theory of the category of coherent sheaves of $\mathcal{O}_X$-modules with support belonging to Φ.

2.5.4 Quillen's Spectral Sequence

For a general noetherian scheme X, the exact category $\mathcal{M}(X)$ of coherent sheaves of $\mathcal{O}_X$-modules has a decreasing filtration

$$\mathcal{M}(X) = \mathcal{M}^{\geq 0}(X) \supset \ldots \supset \mathcal{M}^{\geq i}(X) \supset \mathcal{M}^{\geq i+1}(X) \supset \ldots ,$$

in which $\mathcal{M}^{\geq i}(X)$ is the Serre subcategory consisting of those sheaves which have supports of codimension at least i. Applying the K-theory functor, we get a filtration of the G-theory spectrum by

$$\ldots \subset \mathbf{G}^{X^{\geq i+1}}(X) \subset \mathbf{G}^{X^{\geq i}}(X) \subset \ldots \subset \mathbf{G}(X) .$$

We shall refer to the corresponding spectral sequence

$$\begin{aligned} E_1^{p,q}(X) &= \pi_{-p-q}\left(\mathbf{G}^{X^{\geq p}}(X)/\mathbf{G}^{X^{\geq p+1}}(X)\right) \\ &\simeq \bigoplus_{x\in X^{(p)}} K_{-p-q}(\mathbf{k}(x)) \Rightarrow F^{\cdot}_{\mathrm{cod}}\left(K_{-p-q}(X)\right) . \end{aligned}$$

as the *Quillen spectral sequence*. The identification of the $E_1^{p,q}$-term follows from a combination of localization and dévissage; see [53] for details.

Observe that $E_1^{p,-p}(X) \simeq Z^p(X)$ and $E_1^{p-1,-p}(X) \simeq R^p(X)$. One may also prove that the differential $E_1^{p-1,-p}(X) \to E_1^{p,-p}(X)$ is simply the divisor map. Hence $E_2^{p,-p}(X) \simeq \mathrm{CH}^p(X)$.

Thus for each $p \geq 0$, we get a complex $R_q^*(X)$, which we shall call the Gersten complex:

$$R_q^p(X) := E_1^{p,-q}(X) = \bigoplus_{x\in X^{(p)}} K_{q-p}(\mathbf{k}(x)) .$$

We may also filter the spectrum $\mathbf{G}(X)$ by *dimension* of supports:

$$\ldots \mathbf{G}^{X_{\leq p-1}}(X) \subset \mathbf{G}^{X_{\leq p}}(X) \subset \ldots \subset \mathbf{G}(X) .$$

The corresponding spectral sequence is:

$$\begin{aligned} E^1_{p,q}(X) &= \pi_{p+q}\left(\mathbf{G}^{X_{\leq p}}(X)/\mathbf{G}^{X_{\leq p-1}}(X)\right) \\ &\simeq \bigoplus_{x\in X_{(p)}} K_{p+q}(\mathbf{k}(x)) \Rightarrow F^{\cdot}_{\mathrm{dim}} K_{p+q}(X) . \end{aligned}$$

We also have the corresponding homological Gersten complex:

$$R_{p,q}(X) := E^1_{p,q}(X) .$$

Again, we have that $E^2_{p,p} \simeq CH_p(X)$.

If $f : X \to Y$ is a flat morphism, and $Z \subset Y$ has codimension p, then $f^{-1}(Z)$ has codimension p in X and hence $f^{-1}(Y^{\geq p}) \subset X^{\geq p}$. If f is proper, and $W \subset X$ has dimension q, then $f(W)$ has dimension at most q and hence $f(X_{\leq q}) \subset Y_{\leq q}$. It follows that if f is flat, flat pull-back induces a map of coniveau spectral sequences, and hence of Gersten complexes. If f is proper, then push-forward induces a map of spectral sequences, and hence of Gersten complexes:

$$f_* : R_{*,q}(X) \to R_{*,q}(Y) .$$

Notice that this automatically gives the covariance of the Chow groups with respect to proper maps.

One can extend these results to prove:

65 **Theorem 65** Quillen K-theory of fields is a cycle module, and the Gersten complexes are the associated cycle complexes.

Proof See [53], [62] and [24].

2.5.5 K-Theory as Sheaf Hypercohomology

Recall that if X is a scheme, $\mathbf{G}_X$ denotes the presheaf of G-theory spectra on X.

66 **Theorem 66** If $Y \subset X$ is a closed subset, the natural map

$$\mathbf{G}(Y) \simeq \mathbf{G}^Y(X) \to R\Gamma_Y(X, \mathbf{G}_X) ,$$

is a weak homotopy equivalence.

Proof It is enough to prove that this is true for $Y = X$. The general result then follows by comparing the fibration sequences:

$$\mathbf{G}_Y(X) \to \mathbf{G}(X) \to \mathbf{G}(X - Y)$$

and

$$R\Gamma_Y(X, \mathbf{G}_X) \to R\Gamma(X, \mathbf{G}_X) \to R\Gamma(X - Y, \mathbf{G}_X) \simeq R\Gamma(X - Y, \mathbf{G}_{X-Y}) .$$

The result for X is a consequence of the Mayer-Vietoris property of G-theory. See [11].

67 **Corollary 67** The Quillen spectral sequence for $G_*(X)$ is the same as the coniveau spectral sequence for the sheaf of spectra $\mathbf{G}_X$, and both converge to the coniveau filtration on G-theory.

Let X be a scheme, and suppose that $Y \subset X$ and $Z \subset X$ are closed subsets. Then, using the fact that smash products preserve cofibration sequences, one may easily check that the K-theory product respects supports:

$$\mathbf{K}^Y(X) \wedge \mathbf{K}^Z(X) \to \mathbf{K}^{Y \cap Z}(X) .$$

When X is regular, then this may be identified with the pairing on generalized sheaf cohomology:

$$R\Gamma_Y(X, \mathbf{K}_X) \wedge R\Gamma_Z(X, \mathbf{K}_X) \to R\Gamma_{Y \cap Z}(X, \mathbf{K}_X) .$$

Corollary 68 Let X be a scheme. Then the Brown spectral sequence 68

$$E_2^{p,-q}(X,\mathbf{G}) = H^p(X, G_q(\mathcal{O}_X)) \Rightarrow G_{-p-q}(X)$$

determines a filtration $F^{\cdot}(G_*(X))$. By Corollary 63, we have an inclusion of filtrations $F^{\cdot}(G_i(X)) \subset F^{\cdot}_{\mathrm{cod}}(G_i(X))$.

When X is regular, we then get a filtration $F^k K_*(X)$, which we will still call the Brown filtration, and which has nice properties:

Theorem 69 Let X be a regular scheme. Then the Brown filtration on $\mathbf{K}(X)$ is 69
compatible with the product on K-theory, and is (contravariant) functorial in X. *I.e.*, if $*$ denotes the K-theory product,

$$F^i K_p(X) * F^j K_q(X) \subset F^{i+j} K_{p+q}(X)$$

and if Y and Z are closed subsets of X, then

$$F^i K_{Y,p}(X) * F^j K_{Z,q}(X) \subset F^{i+j} K_{Y\cap Z, p+q}(X)\ .$$

If $f: X \to Y$ is a map of regular schemes, and $W \subset Y$ is a closed subset then

$$f^*(F^i K_{W,p}(Y)) \subset F^i K_{f^{-1}(W),p}(X)\ .$$

Proof We have $F^i K_p(X) = \mathrm{Image}(H^{-p}(X, \mathbf{K}_X(\infty, i)) \to H^{-p}(X, \mathbf{K}_X))$. Hence it suffices to know that the map

$$\mathbf{K}_X(\infty, i) \wedge \mathbf{K}_X(\infty, j) \to \mathbf{K}_X$$

induced by the K-theory product factors, up to homotopy, through $\mathbf{K}_X(\infty, i+j)$, and this is a straightforward consequence of the universal coefficient theorem 57.

The compatibility of the filtration with pull-backs is a consequence of the functoriality of the Postnikov tower.

As we will see below Gersten's conjecture implies that for a regular scheme X, the Brown and coniveau filtrations coincide, and hence Gersten's conjecture implies that the coniveau filtration is multiplicative.

Gersten's Conjecture, Bloch's Formula and the Comparison of Spectral Sequences 2.5.6

We have seen on that on a nonsingular variety, a divisor corresponds to an element of $H^1(X, K_1(\mathcal{O}_X))$, which is determined by the local equations of the divisor. In the seminal paper [8], Bloch showed that on a smooth algebraic surface, the fact that a point is given locally by a *pair* of equations could be used to provide an isomorphism $\mathrm{CH}^2(X) \simeq H^2(X, K_2(\mathcal{O}_X))$.

Quillen's generalization of Bloch's formula to all codimensions, starts from:

70 **Conjecture 70:** *Gersten's conjecture* Suppose that R is a regular local ring. Then for all $i > 0$, the map

$$\mathcal{M}^{(i)}(\mathrm{Spec}(R)) \subset \mathcal{M}^{(i-1)}(\mathrm{Spec}(R))$$

induces zero on K-theory.

71 **Proposition 71** If Gersten's conjecture holds for a given regular local ring R, then, for all p, the following complex is exact:

$$0 \to K_p(R) \to K_p(F) \to \bigoplus_{x \in X^{(1)}} K_{p-1}(\mathbf{k}(x)) \to \ldots \to K_{p-\dim(R)}(k) \to 0$$

Here $X := \mathrm{Spec}(R)$, while F and k are the fraction and residue fields of R respectively.

Note that this implies that $\mathrm{CH}^p(\mathrm{Spec}(R)) \simeq 0$, if $p > 0$, for R a regular local ring; this is a conjecture of Fossum, [12].

72 **Corollary 72** If X is regular scheme, and Gersten's conjecture holds for all the local rings on X, then the augmentation:

$$K_p(\mathcal{O}_X) \to \mathcal{R}_{p,X}$$

is a quasi-isomorphism, where $\mathcal{R}_{p,X}$ is the sheaf of Gersten complexes $U \mapsto R_p^*(U)$. Hence, since the $\mathcal{R}_{p,X}$ are flasque, we have *Bloch's formula*:

$$H^p\left(X, K_p(\mathcal{O}_X)\right) \simeq H^p\left(R_p^*(X)\right) \simeq \mathrm{CH}^p(X) \; .$$

If $Y \subset X$ is a subset of pure codimension r, then:

$$H_Y^p\left(X, K_p(\mathcal{O}_X)\right) \simeq H_Y^p\left(X, \mathcal{R}_{p,X}\right) \simeq H^{p-r}\left(R_{p-r}^*(Y)\right) \simeq \mathrm{CH}^{p-r}(Y) \; .$$

Finally,

$$H_Y^p\left(X, K_q(\mathcal{O}_X)\right) \simeq 0$$

if $p > q$.

73 **Theorem 73:** *Quillen, [53]* If X is a regular variety over a field, then Gersten's conjecture is true for all the local rings on X.

Gersten's conjecture may be viewed as a "local" version of the moving lemma, and versions of it play a key role elsewhere, such as in proving the local acyclicity of the motivic cohomology complexes.

The key point in Quillen's proof of Gersten's conjecture is that if X is a regular affine variety over a field k, then given a divisor $D \subset X$, and a point $x \in D$, the map $i : D \hookrightarrow X$ is "homotopic" to zero in a neighborhood of x. Quillen uses a variant of Noether normalization to show that there is a map $U \to \mathbb{A}_k^{d-1}$, with domain an affine open $U \subset X$ neighborhood of x, which is smooth and which has finite restriction to $D \cap U$. A variation on Quillen's proof may be found in the paper [19] of Gabber, where he proves Gersten's conjecture for Milnor K-theory. See also [13].

Corollary 74 If X is a regular variety of finite type over a field, and Φ is a family of supports on X, then the map of Theorem 62 from the Brown spectral sequence to the coniveau spectral sequence is an isomorphism from E_2 onward, and hence the Brown and the coniveau filtrations on the groups $K_*^{\Phi}(X)$ agree. **74**

Proof See [27] and [23]. The key point is that the map on E_2-terms: $\widehat{E}_{2,\Phi}^{p,q}(X) = H_{\Phi}^p(X, K_{-q}(\mathcal{O}_X)) \to E_{2,\mathrm{cod},\Phi}^{p,q}(X) \simeq H_{\Phi}^p(X, \mathcal{R}_{-q,X})$ is the same as the map induced by the augmentation $K_{-q}(\mathcal{O}_X) \to \mathcal{R}_{-q,X}$.

Since, by Theorem 69, the Brown filtration is multiplicative, we get:

Corollary 75 If X is a regular variety over a field, the coniveau filtration on K-theory with supports is multiplicative. **75**

We shall give another proof of this result in Sect. 2.5.11, using deformation to the normal cone.

The Coniveau Spectral Sequence for Other Cohomology Theories **2.5.7**

We can replace the presheaf of spectra F in the previous section by a complex of sheaves of abelian groups $\mathcal{F}_X^*$. Then the Brown spectral sequence is the standard hypercohomology spectral sequence, and one also has the coniveau spectral sequence. In [9], Bloch and Ogus considered (graded) cohomology theories $X \mapsto \mathcal{F}_X^*(*)$, satisfying suitable axioms, and showed that the analog of Gersten's conjecture holds in these cases. Examples of theories satisfying the Bloch-Ogus axioms are étale cohomology (in which case the Brown spectral sequence is the Leray spectral sequence for the map form the étale site to the Zariski site) and Deligne-Beilinson cohomology ([5], see also [26]).

If $X \mapsto H^*(X,*) = \mathbb{H}^*(X, \mathcal{F}_X^*(*))$ is a theory satisfying the axioms of Bloch and Ogus, then the analog of Gersten's conjecture implies that the E_2 term of the coniveau spectral sequence is isomorphic to $H^p(X, \mathcal{H}^q(*))$, where $\mathcal{H}^*(*)$ is the Zariski sheaf associated to $X \mapsto H^q(X,*)$. Deligne then showed, in an unpublished note:

76 **Theorem 76** The coniveau and hypercohomology spectral sequences agree from E^2 on.

Proof A version of Deligne's proof may be found in the paper [51]. In addition, the proof in the paper [23] of the analogous result for K-theory, is based on the methods of Deligne.

2.5.8 Compatibility with Products and Localized Intersections

One of the great virtues of Bloch's formula is that the K-cohomology groups have a product structure, induced by the K-theory product.

Let us write $\eta : \mathrm{CH}^p(X) \to H^p(X, K_p(\mathcal{O}_X))$ for the isomorphism induced by the Gersten resolution of the K-theory sheaf. Grayson proved in [34],

77 **Theorem 77** Let X be a smooth variety over a field k. If $\alpha \in Z^p(X)$ and $\beta \in Z^q(X)$ are two cycles which intersect properly, then

$$\eta(\alpha)\eta(\beta) = (-1)^{\frac{p(p-1)}{2}\frac{q(q-1)}{2}}\eta(\alpha.\beta)$$

where $\alpha.\beta$ is the product defined by using the intersection multiplicities of Serre, (see Definition 26), and hence with the intersection product defined by Samuel.

Proof By additivity, one can reduce to the case in which $\alpha = [Y]$ and $\beta = [Z]$, where Y and Z are two integral subschemes of X which meet properly. From Quillen's proof of Gersten's conjecture, we have:

$$H^p_Y\left(X, K_p(\mathcal{O}_X)\right) \simeq \mathrm{CH}^0(Y) = \mathbb{Z}[Y]\ ,$$

$$H^q_Z\left(X, K_q(\mathcal{O}_X)\right) \simeq \mathrm{CH}^0(Z) = \mathbb{Z}[Z]\ ,$$

and

$$H^{p+q}_{Y\cap Z}\left(X, K_{p+q}(\mathcal{O}_X)\right) \simeq \mathrm{CH}^0(Y \cap Z) = \bigoplus_S \mathbb{Z}\ ,$$

where the direct sum is over the irreducible components S of $Y \cap Z$.

Using the equality of the Brown and coniveau spectral sequences, Theorem 74, we can identify these isomorphisms with the edge homomorphisms

$$H^p_Y\left(X, K_p(\mathcal{O}_X)\right) \to Gr^p K^Y_0(X) = \mathbb{Z}[\mathcal{O}_Y] ,$$
$$H^q_Z\left(X, K_q(\mathcal{O}_X)\right) \to Gr^q K^Z_0(X) = \mathbb{Z}[\mathcal{O}_Z] ,$$

and

$$H^{p+q}_{Y\cap Z}\left(X, K_{p+q}(\mathcal{O}_X)\right) \to Gr^{p+q} K^{Y\cap Z}_0(X) = \bigoplus_S \mathbb{Z}[\mathcal{O}_S] ,$$

in the Brown spectral sequences for K-theory with supports in Y, Z, and $Y \cap Z$ respectively. By the multiplicativity of the Brown spectral sequence, these edge homomorphisms are compatible with products, and so the product of the cycles associated to the classes $[\mathcal{O}_Y]$ and $[\mathcal{O}_Z]$ maps to the cycle associated to the K-theory product $[\mathcal{O}_Y].[\mathcal{O}_Z]$, which is non other than the cycle defined using Serre's definition of intersection multiplicities.

The sign comes from the fact that the isomorphism

$$\widehat{E}^{p,p}_{2,Y}(X) \simeq H^{2p}\left(X, K_p(\mathcal{O}_X)[p]\right)$$

preserves products, while the isomorphism

$$H^{2p}\left(X, K_p(\mathcal{O}_X)[p]\right) \simeq H^p\left(X, K_p(\mathcal{O}_X)\right)$$

only preserves products up to the factor $(-1)^{\frac{p(p-1)}{2}\frac{q(q-1)}{2}}$.

By Quillen's proof of Bloch's formula, if $Y \subset X$ is a closed set, and X is equidimensional of dimension n, then

$$\mathrm{CH}_{n-p}(Y) \simeq H^p_Y\left(X, K_p(\mathcal{O}_X)\right) .$$

It follows that purely by the formalism of cohomology with supports, that we get a product, for $Y \subset X$ and $Z \subset X$ closed subsets,

$$\mathrm{CH}_k(Y) \times \mathrm{CH}_l(Z) \to \mathrm{CH}_{k+l-n}(Y \cap Z) ,$$

which may be shown to agree with the product with supports on Chow homology constructed by Fulton and MacPherson ([17]). See [24] and [30].

Other Cases of Gersten's Conjecture 2.5.9

For non-geometric regular local rings, the only case for which Gersten's conjecture is known is that of henselian discrete valuation rings Λ with finite residue field k, a result due to Sherman ([63]). The idea of Sherman's proof is that since the general linear group of a finite field is finite, one can use Brauer lifting to show that the restriction map

$$K_*(\Lambda) \to K_*(k)$$

is surjective. A variation of this is result is that if Λ is a discrete valuation ring, the conjecture is true for K-theory with coefficients $\mathbb{Z}/n$ of order prime to the characteristic of k ([29]). The proof depends on the result of Gabber ([18]), and of Gillet and Thomason ([33]), that if R is a Henselian discrete valuation ring then the restriction map

$$K_*(\Lambda, \mathbb{Z}/n) \to K_*(k, \mathbb{Z}/n)$$

is an isomorphism.

If R is a regular local ring which is smooth over a discrete valuation ring Λ with maximal ideal $\pi\Lambda \subset \Lambda$, then one can consider relative versions of Gersten's conjecture, in which one considers not all R-modules, but only those which are flat over Λ. See [7], and [31], where it is shown that this "relative" version of Gersten's conjecture implies that Gersten's conjecture is true for R if it is true for the discrete valuation ring associated to the ideal πR.

2.5.10 Operations on the Quillen Spectral Sequence

One can show that the λ-operations on K_0 of Sect. 2.3.3 can be extended to the higher K-theory of rings and of regular schemes (see the papers of Kratzer ([45]) and Soulé ([64])). Of particular use are the Adams operations ψ^p for $p \in \mathbb{N}$. These are defined as follows. For $x \in K_*(X)$, consider the formal power series $\lambda_t(x) := \sum_i t^i\lambda^i(x) \in K_*(X)[[t]]$. Then the ψ^i are defined by

$$\frac{d}{dt}\left(\lambda_t(x)\right)/\lambda_t(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\psi^k(x)t^{k-1} .$$

One can show (*op. cit.*) that, if X is a regular scheme satisfying our standing assumptions, then the action of the Adams operations on $K_*(X)_{\mathbb{Q}}$ can be diagonalized, so that ψ^p acts with eigenvalue k^p on a subspace which is isomorphic to $\mathrm{Gr}^k_\gamma(K_*(X)_{\mathbb{Q}})$.

Note that if X is a variety over the finite field $\mathbb{F}_p$, then the action of ψ^p on K-theory is the same as the action induced by the Frobenius endomorphism of X. The Adams operation act compatibly with supports and hence act on the Quillen spectral sequence. Using a variant of the Riemann–Roch theorem for higher K-theory of [24], Soulé (*op. cit.*) identified the action on the E^2-term, and could thereby deduce by weight considerations, that for a regular scheme, the differentials into the $E^2_{p,-p}$ and $E^2_{p,1-p}$ terms are torsion, and therefore:

78 **Theorem 78** If X is a regular scheme, there are isomorphisms, for all $p \geq 0$:

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \simeq \mathrm{Gr}^p_{cod}(K_0(X))_{\mathbb{Q}} \simeq \mathrm{Gr}^p_\gamma(K_0(X))_{\mathbb{Q}} .$$

This extends a result that was proved in SGA6 [2]; the first isomorphism was proved in in *op. cit.* for smooth varieties over a field, see *op. cit.* Sect. 4.2 of expose é XIV, while the second was proved with the (unnecessary) assumption that there is an ample invertible sheaf on X; see Theorem 40.

It is natural to ask what the relationship between the γ-filtration and the other filtrations on higher K-theory is. In [23], we prove:

Theorem 79 Let X be a variety over a field k. Then 79

$$F^p_\gamma K_m(X) \subset F^{p-m}_{\mathrm{cod}} K_m(X) \; .$$

Note that in [2], exposé X, Jussila proved:

Theorem 80 Let X be a noetherian scheme. Then 80

$$F^p_\gamma K_0(X) \subset F^p_{\mathrm{cod}} K_0(X) \; .$$

It is therefore natural to ask:

Question 81 Can one extend Theorem 79 to the case of general noetherian schemes? 81

While this would follow from Gersten's conjecture, there may be other ways to approach this problem, such as the construction of a filtration on the K-theory of a regular ring constructed by Grayson ([35]) using commuting automorphisms, which is conjecturally related to the γ-filtration.

The Multiplicativity of the Coniveau Filtration: a Proof Using Deformation to the Normal Cone. 2.5.11

In this section we shall give a proof of the multiplicativity of the coniveau filtration on higher K-theory for arbitrary smooth varieties over a field which uses deformation to the normal cone, rather than hypercohomology of sheaves.

Let X be a smooth variety over a field k. We can decompose the product on $K_*(X)$ into the composition of the external product:

$$\boxdot : K_*(X) \otimes K_*(X) \to K_*(X \times X)$$

and pull-back via the diagonal map $\Delta : X \to X \times X$:

$$\Delta^* : K_*(X \times X) \to K_*(X) \; .$$

Lemma 82 The coniveau filtration is multiplicative with respect to the external product. 82

Proof The external product is induced by the bi-exact functor:

$$\begin{aligned}\mathcal{M}(X) \times \mathcal{M}(X) &\rightarrow \mathcal{M}(X \times X)\\ (\mathcal{F}, \mathcal{G}) &\mapsto \mathcal{F} \otimes_k \mathcal{G}\end{aligned}$$

Let $\pi_i : X \times X \rightarrow X$, for $i = 1, 2$ be the two projections. Suppose that $Y \subset X$ and $Z \subset X$ are closed subsets of X of codimensions p and q respectively. If $\mathcal{F}$ is a coherent sheaf supported on Y, and $\mathcal{G}$ is a coherent sheaf supported on Z, then $\mathcal{F} \otimes_k \mathcal{G}$ is supported on $\pi_1^{-1}(Y) \cap \pi_2^{-1}(Z)$, which has codimension $p + q$ in $X \times X$. Hence the product $\boxdot : (G_i(Y) \otimes G_j(Z)) \rightarrow K_*(X \times X)$ factors through $G_{i+j}(Y \times Z)$, and hence its image lies in $F^{p+q}K_{i+j}(X \times X)$.

Therefore we need only show that pull back by the diagonal preserves the coniveau filtration. More generally, we have:

83 **Theorem 83** If $f : Y \hookrightarrow X$ is a regular immersion of schemes satisfying our standing hypotheses, then $f^*(F^i_{\text{cod}}(G_p(X))) \subset F^i_{\text{cod}}(G_p(Y))$.

Proof The pull back map $f^* : G_*(X) \rightarrow G_*(Y)$ is defined because f is a morphism of finite Tor-dimension. However, it can also be constructed using deformation to the normal bundle.

First we need two lemmas:

84 **Lemma 84** Let $f : X \rightarrow Y$ be a flat morphism. Then

$$f^*\left(F^p(G_q(Y))\right) \subset F^p_{\text{cod}}(G_q(X)) .$$

Proof The pull-back map f^* is induced by the exact functor $f^* : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$, and if a coherent sheaf $\mathcal{F}$ on Y is supported on closed subset $Z \subset Y$ of codimension p, then $f^*(\mathcal{F})$ is supported on $f^{-1}(Z)$ which has codimension p in X.

85 **Lemma 85** Let $p : N \rightarrow Y$ be a vector bundle. Then the map $p^* : F^p_{\text{cod}}(G_q(Y)) \rightarrow F^p_{\text{cod}}(G_q(N))$ is an isomorphism for all p and for all q.

Proof Since p is flat, the functor $p^* : \mathcal{M}(X) \rightarrow \mathcal{M}(N)$ preserves codimension of supports, and so we get a map of filtered spectra $F^*\mathbf{G}(X) \rightarrow F^*\mathbf{G}(E)$ which induces a map

$$p^* : E_r^{p,q}(X) \rightarrow E_r^{p,q}(X)$$

of coniveau spectral sequences. However it is shown in [62] (and also [24]) that the cohomology of the E^2 term of the Quillen spectral sequence is homotopy invariant, and hence the coniveau filtration is homotopy invariant.

Consider the deformation to the normal cone space $W := W_{Y/X}$. Then we have maps:

Pull Back — $\pi^* : K_*(X) \to K_*(W \setminus W_0 \simeq X \times \mathbb{G}_m)$.

Specialization — If t is the parameter on $\mathbb{G}_m$, *i.e.*, the equation of the principal divisor W_0, then we have maps, for all $p \geq 0$:

$$\begin{aligned} \sigma_t : K_p(W \setminus W_0) &\to K_p(W_0 \simeq N_{Y/X}) \\ \alpha &\mapsto \partial(\alpha * \{t\}) \end{aligned}$$

Here ∂ is the boundary map in the localization sequence

$$\ldots \to K_{p+1}(W) \to K_{p+1}(W \setminus W_0) \xrightarrow{\partial} K_p(W_0) \to \ldots \ .$$

Homotopy Invariance — If $p : N_{Y/X} \to Y$ is the projection, the pull back map $p^* : K_*(Y) \to K_*(N_{Y/X})$ is an isomorphism.

Proposition 86 With the notation above, we have **86**

$$f^* = (p^*)^{-1} \cdot \sigma_t \cdot \pi^* \ .$$

Proof See [24].

By Lemmas 84 and 85, π^* and $(p^*)^{-1}$ preserve the coniveau filtration. It remains to show that the coniveau filtration is preserved by specialization.

Suppose that $Z \subset X$ is a codimension p closed, reduced, subscheme. If we are given an element α in $G_p(X)$ which is supported on Z – *i.e.*, its restriction to $X - Z$ vanishes, we know from the localization sequence that it is the image of an element $\gamma \in G_p(Z)$. It will be enough to show that $\sigma_t(\alpha)$ is supported on a closed subset of codimension p in $N_{Y/X}$.

By Lemma 90 below, we know that the Zariski closure in W of $Z \times \mathbb{G}_m \subset X \times \mathbb{G}_m \subset W$ is isomorphic to the deformation to the normal cone space $W_{Z/(Y\cap Z)}$ associated to the subscheme $Y \cap Z \subset Z$. Furthermore, the special fibre of $W_{Z/(Y\cap Z)}$ is the normal cone $C_{Z/(Y\cap Z)} \subset N_{Y/X}$, which has codimension p in the normal bundle.

Since $C_{Z/Y\cap Z} = W_{Z/Y\cap Z} \cap N_{Y/X}$, if we write $j : W_{Z/Y\cap Z} \to W_{Y/X}$ for the inclusion, j_* induces a map of localization sequences, and in particular a commutative square:

$$\begin{tikzcd} G_{p+1}(Z \times \mathbb{G}_m) \arrow[r, "\partial"] \arrow[d, "j_*"'] & G_p(C_{Z/Y\cap Z}) \arrow[d, "j_*"'] \\ G_{p+1}(X \times \mathbb{G}_m) \arrow[r, "\partial"] & G_p(N_{Y/X}) \ . \end{tikzcd}$$

By the projection formula, we also have a commutative square:

$$\begin{tikzcd} G_p(Z \times \mathbb{G}_m) \arrow[r, "*\{t\}"] \arrow[d, "j_*"'] & G_{p+1}(Z \times \mathbb{G}_m) \arrow[d, "j_*"'] \\ G_p(X \times \mathbb{G}_m) \arrow[r, "*\{t\}"] & G_{p+1}(X \times \mathbb{G}_m)\ . \end{tikzcd}$$

Putting these together, we get a commutative diagram:

$$\begin{tikzcd} G_p(Z \times \mathbb{G}_m) \arrow[r, "\sigma_t"] \arrow[d, "j_*"'] & G_p(C_{Z/Y\cap Z}) \arrow[d, "j_*"'] \\ G_p(X \times \mathbb{G}_m) \arrow[r, "\sigma_t"] & G_p(N_{Y/X})\ , \end{tikzcd}$$

as desired. Hence $\sigma_t(\alpha) \in j_*(G_p(C_{Z/Y\cap Z})) \subset F^p_{\mathrm{cod}}(G_p(N_{Y/X}))$.

This completes the proof of the proposition, and hence of Theorem 35.

Since every regular variety over a field is a localization of smooth variety over the prime field, it also follows that the theorem is true for regular varieties.

A similar reduction to the diagonal argument may also be used to prove the theorem for schemes which are smooth over the spectrum of a discrete valuation ring.

2.6 Bloch's Formula and Singular Varieties

2.6.1 Cohomology Versus Homology

If X is a general CW complex which is not a manifold, it will no longer be the case that there is a Poincaré duality isomorphism $H^*(X,\mathbb{Z}) \simeq H_*(X,\mathbb{Z})$. If X is a singular variety, the Chow groups of cycles modulo rational equivalence are analogous (even when graded by codimension) to the singular homology of a CW complex. It is natural to ask if there is appropriate theory of *Chow cohomology*.

One answer to this question is given by Fulton in his book [17], in which he defines the cohomology groups to be the *operational Chow groups* $\mathrm{CH}^*_{\mathrm{op}}(X)$. An element $\alpha \in \mathrm{CH}^p_{\mathrm{op}}(X)$ consists, essentially, of giving homomorphisms, for every map $f : Y \to X$ of varieties and every $q \geq 0$, $\cap\alpha : \mathrm{CH}_q(Y) \to \mathrm{CH}_{q-p}(Y)$, which satisfy various compatibilities. Fulton's operational groups are the target of a theory of Chern classes for vector bundles, and any regularly immersed codimension p

closed subscheme $Y \subset X$ has a cycle class $[Y] \in \mathrm{CH}^p_{\mathrm{op}}(X)$. (The existence of this class uses deformation to the normal bundle.) Any Chow cohomology theory that has reasonable properties, in particular that contains Chern classes for vector bundles, and which has cap products with Chow homology for which the projection formula holds, will map to the operational groups. However the operational groups, while they have many virtues, also miss some information. For example, if X is a nodal cubic curve over a field k, one can prove that $\mathrm{CH}^1_{\mathrm{op}}(X) \simeq \mathbb{Z}$. However the group of Cartier divisors is isomorphic to $\mathbb{Z} \oplus k^*$, which carries more information about the motive of X. Even if $\mathrm{Pic}(X)$ gave a "good" definition of codimension 1 Chow cohomology, it is not clear what should happen in higher codimensions – *i.e.* is there a theory of codimension p "Cartier Cycles"?

It is natural to consider the groups $H^p(X, K_p(\mathcal{O}_X))$, because of Bloch's formula and by analogy with $\mathrm{Pic}(X) \simeq H^1(X, K_1(\mathcal{O}_X))$.

Arguments in favor of this choice are:

1. The functor $X \mapsto \bigoplus_p H^p(X, K_p(\mathcal{O}_X))$ is a contravariant functor from varieties over a given field k to commutative graded rings with unit.
2. There are cap products $H^p(X, K_p(\mathcal{O}_X)) \otimes \mathrm{CH}_q(X) \to \mathrm{CH}_{q-p}(X)$, which satisfy the projection formula.
3. There are Chern classes for vector bundles: $C_p(E) \in H^p(X, K_p(\mathcal{O}_X))$, for E is a vector bundle on X, which are functorial and satisfy the Whitney sum formula for exact sequences of bundles.
4. Any codimension p subscheme $Y \subset X$ of a which is a local complete intersection has a fundamental class $[Y] \in H^p(X, K_p(\mathcal{O}_X))$.

This approach to Chow cohomology is developed in [24] and [25].

A strong argument against this choice is that these groups (including $\mathrm{Pic}(X)$, see [69]) are not homotopy invariant.

One way to get a homotopy invariant theory, if the ground field k has characteristic zero, is given a singular variety X, to take a nonsingular simplicial hyperenvelope $\widetilde{X}. \to X$ (see Appendix 2.8) and take $H^p(\widetilde{X}., K_p(\mathcal{O}_{\widetilde{X}.}))$ (or $H^p(\widetilde{X}., \mathcal{K}^M_p)$) as the definition of Chow cohomology. One can show (at least for the usual K-cohomology, see *op. cit.*) that these groups are independent of the choice of hyperenvelope, are homotopy invariant, will have cap products with the homology of the Gersten complexes, and will be the target of a theory of Chern classes.

Given a variety X and a nonsingular simplicial hyperenvelope $\widetilde{X}. \to X$ there will be, for each $q \geq 0$, a spectral sequence:

$$E_1^{i,j} = H^i\big(\widetilde{X}_j, K_q(\mathcal{O}_{\widetilde{X}_j})\big) \Rightarrow H^{i+j}\left(\widetilde{X}., K_q(\mathcal{O}_{\widetilde{X}.})\right) \ .$$

One can show, by the method of [27], that the E_2 term of this spectral sequence is independent of the choice of hyperenvelope, and hence gives a filtration on the groups $H^p(\widetilde{X}., K_q(\mathcal{O}_{\widetilde{X}.}))$. This filtration is a K-cohomology version of the weight filtration of mixed Hodge theory.

Note that if X is a (projective) nodal cubic, and $\widetilde{X}. \to X$ is a hyperenvelope, then

$$H^1\left(\widetilde{X}., K_1(\mathcal{O}_{\widetilde{X}.})\right) \simeq \mathrm{Pic}(X) \ ,$$

while if X is a cuspidal cubic, then

$$H^1\left(\widetilde{X}_\cdot, K_1(\mathcal{O}_{\widetilde{X}_\cdot})\right) \simeq \mathbb{Z} \neq \mathrm{Pic}(X)\ .$$

It follows from the following theorem that the weight zero part of these groups are Fulton's operational Chow groups:

87 **Theorem 87:** *Kimura, [42]* Let X be variety over a field. Given nonsingular envelopes $p_0 : \widetilde{X}_0 \to X$ and $p_1 : \widetilde{X}_1 \to \widetilde{X}_0 \times_X \widetilde{X}_0$ we have an exact sequence:

$$0 \to \mathrm{CH}^p_{\mathrm{op}}(X) \xrightarrow{p_0^*} \mathrm{CH}^p(\widetilde{X}_0) \xrightarrow{\delta} \mathrm{CH}^p(\widetilde{X}_1)\ ,$$

where $\delta = p_1^*(\pi_1^* - \pi_0^*)$, with $\pi_i : \widetilde{X}_0 \times_X \widetilde{X}_0 \to \widetilde{X}_0$ the projections.

2.6.2 Local Complete Intersection Subschemes and Other Cocyles

If X is a scheme, we know that any subscheme the ideal of which is generated locally by nonzero divisors defines a Cartier divisor and is a "codimension 1" cocycle. What about higher codimension?

Let $Y \subset X$ be a codimension p regularly immersed subscheme. Recall that there is an operational class $[Y]_{\mathrm{op}} \in \mathrm{CH}^p_{\mathrm{op}}$ corresponding to the "pull-back" operation constructed using deformation to the normal cone.

88 **Theorem 88:** *[25]* Let X be a variety over a field k, and suppose that $Y \subset X$ is closed subscheme which is a codimension p local complete intersection. Then there is a natural class $[Y] \in H^p(X, K_p(\mathcal{O}_X))$, such that cap product with $[Y]$ induces the operational product by $[Y]_{\mathrm{op}}$.

Idea of Proof

While the theorem is stated using K-cohomology, it really holds for almost *any* cohomology theory constructed using sheaf cohomology, that has a theory of cycle classes with supports. The key point is that for a given $p > 0$, there is a pair of simplicial schemes $V_\cdot \hookrightarrow U_\cdot$ smooth over the base, which is a "universal" codimension p local complete intersection. That is, given $Y \subset X$, a codimension p local complete intersection, there is a Zariski open cover $\mathcal{W}$ of X, and a map of simplicial schemes $\eta : N_\cdot(\mathcal{W}) \to U_\cdot$ such that $\eta^{-1}(V_\cdot) = Y \cap N_\cdot(\mathcal{W})$. Then one may construct a universal class in $\alpha \in H^p_{V_\cdot}(U_\cdot, K_p(\mathcal{O}_{V_\cdot}))$, and define $[Y] := \eta^*(\alpha)$.

The same principle also is true for codimension 2 subschemes $Y \subset X$ for which the sheaf of ideals $\mathcal{I}_{Y/X}$ is locally of projective dimension 2. Such ideals are determinantal, *i.e.*, locally they are generated by the maximal minors of an $n \times (n-1)$ matrix.

Chow Groups of Singular Surfaces 2.6.3

Bloch's formula for codimension two cycles on a singular surface over a field, at least if it has isolated singularities, is fairly well understood, thanks to the work of Collino, Levine, Pedrini, Srinivas, Weibel, and others.

In particular, if X is a reduced quasi-projective surface X over an algebraically closed field k, Biswas and Srinivas, [6] have constructed a Chow ring

$$\mathrm{CH}^*(X) = \mathrm{CH}^0(X) \oplus \mathrm{CH}^1(X) \oplus \mathrm{CH}^2(X)$$

satisfying the usual properties of intersection theory for smooth varieties. In particular, there are Chern class maps $C_i\colon K_0(X) \to \mathrm{CH}^i(X)$ satisfying the Riemann–Roch formula such that, if $F_0K_0(X)$ denotes the subgroup generated by the classes of the structure sheaves of smooth points of X, then $C_2\colon F_0K_0(X) \to \mathrm{CH}^2(X)$ is an isomorphism, inverse (up to sign) to the cycle map $\mathrm{CH}^2(X) \to K_0(X)$.

The definition of the group of 0-cycles modulo rational equivalence for a singular variety X follows the one given by Levine and Weibel ([49]), i.e. as the Chow group $\mathrm{CH}_0(X, Y)$ of X relative to its singular locus Y. This is the group generated by closed points on $X - Y$, with rational equivalence defined using rational functions on Cartier curves, *i.e.* every point of $Z \cap Y$ lies in an open neighborhood U where $Z \cap U$ is defined by a regular sequence. See also [52], [48].

Intersection Theory on Stacks 2.6.4

If X is a smooth Deligne Mumford stack over a field, then one can define Chow groups $\mathrm{CH}^p(X)$, where $Z^p(X)$ is the free abelian group on the reduced irreducible substacks, and rational equivalence is defined using rational functions on substacks. Bloch's formula remains true, though one is forced to take rational coefficients, and to replace the Zariski topology with the étale topology:

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \simeq H^p_{\text{ét}}\left(X, K_p(\mathcal{O}_X)\right)_{\mathbb{Q}} \ .$$

This leads to a K-theoretic construction of an intersection product on X. See [28]. One should note that there are other approaches intersection theory on stacks, using operational Chow groups, by Vistoli ([68]), Kresch ([47])and others.

If $\overline{X}$ is the coarse moduli space, or quotient, of the stack, then one can show that the quotient map $\pi : X \to \overline{X}$ induces an isomorphism:

$$\pi_* : \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(\overline{X})_{\mathbb{Q}} \ ,$$

and hence a product structure on the Chow groups, with rational coefficients of the singular variety $\overline{X}$. This is analogous to the construction of the rational cohomology ring of an orbifold.

2.7 Deformation to the Normal Cone

This section is based on the expositions in [17] and [67].

89 **Definition 89** Let X be a scheme, satisfying our standing assumptions, and suppose that $Y \hookrightarrow X$, is a closed subscheme, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Let $W_{Y/X}$ be the scheme obtained by blowing up $\mathbb{A}^1_X = \mathbf{Spec}(\mathcal{O}_X[t])$ with respect to the sheaf of ideals $(\mathcal{I}, t) \subset \mathcal{O}_{\mathbb{A}^1_X}$ (*i.e.*, along the subscheme $Y \times \{0\}$), and then deleting the divisor (isomorphic to the blow up of X along Y) which is the strict transform of $X \times \{0\}$.

Observe that $t \in \Gamma(W_{Y/X}, \mathcal{O}_{W_{Y/X}})$ is a regular element and so defines a (principal effective Cartier) divisor $W_{Y/X,0} \subset W_{Y/X}$, which is isomorphic to the normal cone $C_{Y/X} = \mathbf{Spec}(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1})$. Also note that $W_{Y/X} \setminus W_{Y/X,0} \simeq \mathbb{G}_{m,X} = \mathbf{Spec}(\mathcal{O}_X[t, t^{-1}])$; we shall write $\pi : \mathbb{G}_{m,X} \to X$ for the natural projection.

We write $p : W_{Y/X,0} = C_{Y/X} \to Y$ for the natural projection.

If $Y \hookrightarrow X$ is a regular immersion, in the sense of EGA IV ([37]) 16.9.2, then $W_{Y/X,0} = C_{Y/X} \simeq N_{Y/X}$ is a vector bundle over Y.

There is a natural inclusion $\mathbb{A}^1_Y \hookrightarrow W$, because $Y \times \{0\}$ is principal divisor in $\mathbb{A}^1_Y$.

$$\begin{tikzcd}
W_0 \simeq C_{Y/X} \arrow[r, hook] & W_{Y/X} & \mathbb{G}_{m,X} \arrow[l, hook'] \\
Y \arrow[r, hook] \arrow[u, hook] & \mathbb{A}^1_Y \arrow[u, hook] & \mathbb{G}_{m,Y} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

We will need the following lemma, which is a straightforward consequence of the basic properties of blow-ups (see [39], II.7):

90 **Lemma 90** Suppose that $Z \subset X$ is a closed subscheme. Then $W_{Y\cap Z/Z}$ is a closed subscheme of $W_{Y/X}$, indeed it is the strict transform of $\mathbb{A}^1_Z \subset \mathbb{A}^1_Z$ with respect to the blow up, and $W_{Y\cap Z/Z} \cap W_{Y/X,0} = W_{Y\cap Z/Z,0}$

2.8 Envelopes and Hyperenvelopes

91 **Definition 91** A map $f : X \to Y$ is said to be an *envelope* if it is proper and if for every field F, $X(F) \to Y(F)$ is surjective – or equivalently, for every integral subscheme $Z \subset Y$, there is an integral subscheme $\widetilde{Z} \subset X$ such that $f(\widetilde{Z}) = Z$, and $f|_{\widetilde{Z}} \to Z$ is birational.

Theorem 92 Suppose that resolution of singularities holds for the category of varieties over the field k, Then, for every variety X over k, there is an envelope $p : \widetilde{X} \to X$ with $\widetilde{X}$ nonsingular. 92

Proof The proof is by induction on the dimension of X. If $\dim(X) = 0$, then X is already nonsingular. Suppose the theorem is true for all varieties of dimension less than $d > 0$. If $\dim(X) = d$, let $p_1 : \widetilde{X}_1 \to X$ be a resolution of singularities of X such that there is a subvariety $Y \subset X$ such that p_1 is an isomorphism over $X - Y$, with $\dim(X) < d$. By the induction hypothesis there is an envelope $q : \widetilde{Y} \to Y$. Now set $\widetilde{X} := \widetilde{X}_1 \sqcup Y$, and $p := p_1 \sqcup q$.

Definition 93 We say that a map of simplicial schemes $f. : X. \to Y.$ is *hyperenvelope* if for all fields F, $f.(F) : X.(F) \to Y.(F)$ is a trivial Kan fibration between simplicial sets. Alternatively, $f.$ is a hypercovering in the topology for which envelopes are the coverings. See [22] for more details. 93

It follows from Theorem 92 that if X is a variety over a field of characteristic zero, then there is a non-singular hyperenvelope $\widetilde{X}. \to X$.

Notice that this argument also works for schemes of dimension d over a base S, if resolution of singularities holds for such schemes.

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