Exercise sheet 8

1. Let k be a field and $\mathrm{A} = k[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ the polynomial algebra on n generators.

(i) If k is algebraically closed, show that the closed points in $|\mathrm{Spec}(\mathrm{A})|$ are in bijection with tuples $(x_1, \ldots, x_n) \in k^n$.

(ii) In general, let $\overline{k}$ be an algebraic closure of k and consider the automorphism group $\mathrm{G} = \mathrm{Aut}(\overline{k}/k)$. Show that there is a canonical action of G on the set of closed points of $|\mathrm{Spec}(\overline{k}[\mathrm{T}_1, \ldots, \mathrm{T}_n])|$.

(iii) Show that the closed points in $|\mathrm{Spec}(\mathrm{A})|$ are in bijection with the G-orbits of the closed points of $|\mathrm{Spec}(\overline{k}[\mathrm{T}_1, \ldots, \mathrm{T}_n])|$.

This follows from the Nullstellensatz. See [Bourbaki, Commutative algebra, Chap. V, §3.3, Prop. 2] for the strong version that is useful here.

2. Let A be a noetherian local ring. Recall that $|\mathrm{Spec}(\mathrm{A})|$ has a unique closed point x.

(i) Show that $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$ is supported on $\mathrm{V}(\mathfrak{m}) \simeq \{x\}$ iff it is of finite length.

(ii) Show that the dévissage isomorphism $\mathrm{G}_0^{\{x\}}(\mathrm{A}) \simeq \mathbf{Z}$ sends $[\mathrm{M}] \mapsto \ell_{\mathrm{A}}(\mathrm{M})$, where $\ell_{\mathrm{A}}(\mathrm{M})$ denotes the length of M.

(iii) If A is regular, show that the intersection multiplicity is computed by the formula

$$\chi_{\mathrm{A}}(\mathrm{M}, \mathrm{N}) = \sum_i (-1)^i \, \ell_{\mathrm{A}}(\mathrm{Tor}_i^{\mathrm{A}}(\mathrm{M}, \mathrm{N}))$$

where M and N are A-modules with $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) \cap \mathrm{Supp}_{\mathrm{A}}(\mathrm{N}) = \{x\}$ (x being the closed point of $|\mathrm{Spec}(\mathrm{A})|$).

(i) This follows from Sheet 2, Exercise 4.

(ii) Let M be a f.g. A-module which is supported on $\mathrm{V}(\mathfrak{m})$. To describe the image of [M] through the dévissage isomorphism $\mathrm{G}_0^{\{x\}}(\mathrm{A}) \simeq \mathrm{G}_0(\kappa(x))$, we are free to choose any finite filtration $(\mathrm{M}_i)_i$ of M, where the successive quotients are $\mathrm{A}/\mathfrak{m}$-modules, and take the sum

$$\sum_i [\mathrm{M}_i/\mathrm{M}_{i-1}].$$

Since M is of finite length by (i), say, $n := \ell_{\mathrm{A}}(\mathrm{M})$, it admits a composition series: that is, we can choose such a filtration of length n where $\mathrm{M}_i/\mathrm{M}_{i-1}$ are simple modules, hence each isomorphic to $\mathrm{A}/\mathfrak{m} = \kappa(x)$ (see Lemma below). Thus [M] corresponds under dévissage to $n \cdot [\kappa(x)] \in \mathrm{G}_0(\kappa(x))$. The isomorphism $\mathrm{G}_0(\kappa(x)) \simeq \mathbf{Z}$ sends the class of a $\kappa(x)$-vector space to its dimension, hence [M] is sent to $n \in \mathbf{Z}$.

Lemma 1. *Let* A *be a ring and* N *an* A*-module. Then* N *is simple iff it is isomorphic to* $\mathrm{A}/\mathfrak{m}$ *for some maximal ideal* $\mathfrak{m}$*.*

Proof. Recall that N is simple if $\ell_{\mathrm{A}}(\mathrm{N}) = 1$, i.e., if it admits exactly two submodules, 0 and N. Since N is nonzero we can choose a nonzero element $n \in \mathrm{N}$. The multiplication map $n : \mathrm{A} \to \mathrm{N}$ has image a submodule $n\mathrm{N} \subseteq \mathrm{N}$. We cannot have $n\mathrm{N} = 0$ since at least $n = n \cdot 1 \in n\mathrm{N}$. Thus $n\mathrm{N} = \mathrm{N}$. In other words, N is generated by the element n, and $\mathrm{N} \simeq \mathrm{A}/\mathrm{I}$ where $\mathrm{I} = \mathrm{Ann}(n)$.

It remains to show that I is a maximal ideal. Since I is a proper ideal (as $\mathrm{N} \neq 0$), we at least have $\mathrm{I} \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Since ideals of A containing I are in bijection with ideals of A/I (i.e., submodules of N), there are exactly two of them, namely I and the unit ideal. The claim follows.

(iii) Given point (ii), this follows immediately from the construction of χ_{A}. □

3. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{T}, \mathrm{U}]$. Let I and J be prime ideals of A defining *distinct* integral closed subsets $\mathrm{Y} = \mathrm{V}(\mathrm{I})$ and $\mathrm{Z} = \mathrm{V}(\mathrm{J})$ of codimension 1. Let p be a closed point of $\left|\mathrm{Spec}(\mathrm{A})\right|$ which lies in the intersection $\mathrm{Y} \cap \mathrm{Z}$, and let $\mathfrak{m}$ be the corresponding maximal ideal of A. Show that

$$\chi_{\mathrm{A}_\mathfrak{m}}(\mathrm{A}_\mathfrak{m}/\mathrm{I}\mathrm{A}_\mathfrak{m}, \mathrm{A}_\mathfrak{m}/\mathrm{J}\mathrm{A}_\mathfrak{m}) = \dim_k(\mathrm{A}_\mathfrak{m}/(\mathrm{I}+\mathrm{J})\mathrm{A}_\mathfrak{m}).$$

The following commutative algebra fact shows that I and J are principal ideals. (This condition actually characterizes factoriality of noetherian integral domains.)

Lemma 2. *Let* A *be a factorial ring. Then for every integral subset* $\mathrm{V}(\mathfrak{p}) \subset \left|\mathrm{Spec}(\mathrm{A})\right|$ *of codimension 1, the prime ideal* $\mathfrak{p}$ *is principal.*

Proof. Let $\mathfrak{p}$ be a prime ideal such that $\mathrm{V}(\mathfrak{p})$ is of codimension 1. Given a nonzero $f \in \mathfrak{p}$, choose a factorization $f = g_1 \cdots g_n$ with the g_i irreducible (hence prime). Since $\mathfrak{p}$ is prime, we have $g_i \in \mathfrak{p}$ for some i. But then we have an inclusion of prime ideals $\langle g_i \rangle \subseteq \mathfrak{p}$, hence of integral subsets $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{V}(\langle g_i \rangle)$. But since $\mathrm{V}(\mathfrak{p})$ is of codimension 1, it follows that $\mathfrak{p} = \langle g_i \rangle$. □

Let f be a generator of I. By assumption $\mathrm{V}(\mathrm{I})$ and $\mathrm{V}(\mathrm{J})$ are distinct, in particular $\mathrm{V}(\mathrm{J}) \not\subseteq \mathrm{V}(\mathrm{I})$ and therefore $f \notin \mathrm{J}$. Thus f is a non-zero-divisor both in A and A/J (both integral domains). Therefore

$$\mathrm{A}/\mathrm{I} \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}/\mathrm{J} \simeq \mathrm{Kosz}_{\mathrm{A}}(f) \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{J} \simeq \left[\mathrm{A}/\mathrm{J} \xrightarrow{f} \mathrm{A}/\mathrm{J}\right]$$

is acyclic in positive degrees (where $\simeq$ means quasi-isomorphism). The same holds after localizing at $\mathfrak{m}$, i.e.,

$$\mathrm{A}_\mathfrak{m}/\mathrm{I}\mathrm{A}_\mathfrak{m} \otimes^{\mathbf{L}}_{\mathrm{A}_\mathfrak{m}} \mathrm{A}_\mathfrak{m}/\mathrm{J}\mathrm{A}_\mathfrak{m} \simeq \left[\mathrm{A}_\mathfrak{m}/\mathrm{J}\mathrm{A}_\mathfrak{m} \xrightarrow{f} \mathrm{A}_\mathfrak{m}/\mathrm{J}\mathrm{A}_\mathfrak{m}\right]$$

is acyclic in positive degrees, since $(-)_{\mathfrak{m}}$ is an exact functor. Thus we get:

$$\begin{aligned}\chi_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/IA_{\mathfrak{m}}, A_{\mathfrak{m}}/JA_{\mathfrak{m}}) &= \sum_i (-1)^i \ell_{A_{\mathfrak{m}}}\left(\mathrm{Tor}_i^{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/IA_{\mathfrak{m}}, A_{\mathfrak{m}}/JA_{\mathfrak{m}})\right) \\ &= \ell_{A_{\mathfrak{m}}}\left(A_{\mathfrak{m}}/(I+J)A_{\mathfrak{m}}\right) \\ &= \dim_k\left(A_{\mathfrak{m}}/(I+J)A_{\mathfrak{m}}\right)\end{aligned}$$

as desired. (For the last equality, note that we can view $A_{\mathfrak{m}}/(I+J)A_{\mathfrak{m}}$ as a module over k, and its length doesn't change when we do so.) Note the algebraic closedness assumption on k was irrelevant.

4. Let k be an algebraically closed field and $A = k[T_1, T_2, T_3, T_4]$. Consider the ideals

$$\begin{aligned} I &= \langle T_1, T_2\rangle \cap \langle T_3, T_4\rangle = \langle T_1T_3, T_1T_4, T_2T_3, T_2T_4\rangle \\ J &= \langle T_1 - T_3, T_2 - T_4\rangle, \end{aligned}$$

which define closed subsets $Y = V(I)$ and $Z = V(J)$ of $X = |\mathrm{Spec}(A)|$.

(i) Show that Y has two irreducible components, each of codimension 2 in X.

(ii) Show that each component of Y intersects Z at exactly one closed point p in X.

(iii) Let $\mathfrak{m}$ be the maximal ideal of A corresponding to p. Compute the integers

$$\ell_A(A/(I+J)), \qquad \ell_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/(I+J)A_{\mathfrak{m}}).$$

(iv) Compute the intersection number

$$\chi_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/IA_{\mathfrak{m}}, A_{\mathfrak{m}}/JA_{\mathfrak{m}}).$$

(i) Let $I_1 = \langle T_1, T_2\rangle$ and $I_2 = \langle T_3, T_4\rangle$. Since A/I_1 and A/I_2 are integral domains, these are prime ideals of A which define integral closed subsets $Y_1 = V(I_1)$ and $Y_2 = V(I_2)$ of X. As $Y = V(I) = V(I_1 \cap I_2) = Y_1 \cup Y_2$, it follows that Y_1 and Y_2 are the irreducible components of Y. It is clear that $Y_1 = V(T_1, T_2) \subsetneq V(T_1) \subsetneq V(0) = X$ is a maximal chain of integral closed subsets of X, so Y_1 is of codimension 2 and similarly for Y_2.

(ii) We have $Y_1 \cap Z = V(\langle T_1, T_2, T_1 - T_3, T_2 - T_4\rangle) = V(\langle T_1, T_2, T_3, T_4\rangle)$, which consists of the single closed point p corresponding to the maximal ideal $\mathfrak{m} = \mathfrak{p}(p) = \langle T_1, T_2, T_3, T_4\rangle$. Same for $Y_2 \cap Z$.

(iii) We have

$$A/(I+J) \simeq k[T_1, T_2]/\langle T_1^2, T_1T_2, T_2^2\rangle,$$

which is a 3-dimensional vector space over k with basis $\{1, T_1, T_2\}$. Thus

$$\ell_A(A/(I+J)) = \dim_k(k[T_1, T_2]/\langle T_1^2, T_1T_2, T_2^2\rangle) = 3,$$

and the same after localizing.

(iv) For any two ideals I_1, I_2 in a ring A, we have a short exact sequence

$$0 \to A/(I_1 \cap I_2) \to A/I_1 \oplus A/I_2 \to A/(I_1 + I_2) \to 0$$

from which we derive the formula

$$[A/(I_1 \cap I_2)] = [A/I_1] + [A/I_2] - [A/(I_1 + I_2)]$$

in $G_0(A)$ or even in $G_0^{V(I_1)\cup V(I_2)}(A) \simeq G_0(A/(I_1 \cap I_2))$.

In our case, with $I = I_1 \cap I_2$, we get

$$[A/I] = [A/I_1] + [A/I_2] - [A/(I_1 + I_2)]$$

in $G_0(A/I)$. The same holds after localizing at the ideal $\mathfrak{m}$. We have then

$$\chi_{A_\mathfrak{m}}(A_\mathfrak{m}/IA_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m}) = \chi_{A_\mathfrak{m}}(A_\mathfrak{m}/I_1A_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m}) + \chi_{A_\mathfrak{m}}(A_\mathfrak{m}/I_2A_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m}) - \chi_{A_\mathfrak{m}}(A_\mathfrak{m}/(I_1 + I_2)A_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m}).$$

since χ factors through $G_0(A/I)$ by definition and is linear.

To compute the last term (we ignore the localization at $\mathfrak{m}$, which has no effect on the computation), use the Koszul complex on the regular sequence $(T_1 - T_3, T_2 - T_4)$ to resolve A/J; after tensoring with $A/(I_1 + I_2) \simeq k$ the differentials vanish and we get the complex

$$k \otimes_A^{\mathbf{L}} A/J \simeq \left[k \xrightarrow{0} k\right] \otimes_k \left[k \xrightarrow{0} k\right] \simeq \left[k \xrightarrow{0} k \oplus k \xrightarrow{0} k\right].$$

The alternating sum of the dimensions of the terms is $1 - 2 + 1 = 0$.

To compute the term $\chi_{A_\mathfrak{m}}(A_\mathfrak{m}/I_1A_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m})$ (again we'll ignore the localization) we can use the same resolution of A/J to get

$$\begin{aligned} A/I_1 \otimes_A^{\mathbf{L}} A/J &\simeq \left[A/\langle T_1, T_2\rangle \xrightarrow{-T_3} A/\langle T_1, T_2\rangle\right] \otimes_A \left[A/\langle T_1, T_2\rangle \xrightarrow{-T_4} A/\langle T_1, T_2\rangle\right] \\ &\simeq \mathrm{Kosz}_{k[T_3,T_4]}(-T_3, -T_4) \\ &\simeq k \end{aligned}$$

where the last quasi-isomorphism is because $(-T_3, -T_4)$ is a regular sequence in $k[T_3, T_4]$. Thus this term has a contribution

$$\chi_{A_\mathfrak{m}}(A_\mathfrak{m}/I_1A_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m}) = 1$$

and similarly

$$\chi_{A_\mathfrak{m}}(A_\mathfrak{m}/I_2A_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m}) = 1.$$

We end up with

$$\chi_{A_\mathfrak{m}}(A_\mathfrak{m}/IA_\mathfrak{m}, A_\mathfrak{m}/JA_\mathfrak{m}) = 1 + 1 - 0 = 2.$$