## Exercise sheet 7

**1.** (i) Let $F : \mathcal{A} \to \mathcal{A}'$ be an exact functor between abelian categories. Show that $\mathrm{Ker}(F) \subseteq \mathcal{A}$, the full subcategory spanned by objects $A$ such that $F(A) \simeq 0$, is a Serre subcategory.

(ii) Let $F : \mathcal{A} \to \mathcal{A}'$ be an exact functor between abelian categories. Suppose that $F$ admits a right adjoint $G$ such that the co-unit transformation $FG \to \mathrm{id}$ is invertible (equivalently, $G$ is fully faithful). Show that there is a canonical equivalence

$$\mathcal{A}/\mathrm{Ker}(F) \to \mathcal{A}'.$$

(iii) Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subseteq \mathcal{A}$ a Serre subcategory. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a full subabelian subcategory such that if $A \in \mathcal{A}_0$ and $B \in \mathcal{B}$ is a subobject or quotient of $A$ then also $B \in \mathcal{A}_0$. Show that the canonical functor

$$\mathcal{A}_0/(\mathcal{B} \cap \mathcal{A}_0) \to \mathcal{A}/\mathcal{B}$$

is fully faithful.

(i) Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence in $\mathcal{A}$. Since $F$ is exact,

$$0 \to F(X') \to F(X) \to F(X'') \to 0$$

is still exact. Thus $F(X) \simeq 0$ iff $F(X') \simeq 0$ and $F(X'') \simeq 0$.

(ii) We show that the functor $F : \mathcal{A} \to \mathcal{A}'$ satisfies the universal property of the quotient $\gamma : \mathcal{A} \to \mathcal{A}/\mathrm{Ker}(F)$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} \\ \downarrow{\scriptstyle F} & \nearrow{\scriptstyle \beta} & \\ \mathcal{A}' & & \end{array}$$

Thus, let $\alpha : \mathcal{A} \to \mathcal{B}$ be a functor such that $\alpha(\mathrm{Ker}(F)) = 0$, i.e., $\mathrm{Ker}(F) \subseteq \mathrm{Ker}(\alpha)$. We first note that if $\beta$ exists (making the diagram commute), then we have a canonical isomorphism

$$\beta \simeq \beta FG \simeq \alpha G,$$

where the first isomorphism is induced by the co-unit $\mathrm{id} \simeq FG$. Thus $\beta$ is unique up to isomorphism if it exists. For existence, it will suffice to show that the only possible candidate $\beta := \alpha G$ does satisfy $\beta F \simeq \alpha$. For an object $X \in \mathcal{A}$, consider the unit morphism $\eta_X : X \to GF(X)$ and let $K$ be its kernel. The triangle identities for the adjunction $(F, G)$ imply that the composite

$$F(X) \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{\varepsilon_{F(X)}} F(X)$$

is the identity, where $\varepsilon$ is the co-unit. Since the latter is invertible by assumption, so is $F(\eta_X) : F(X) \to FGF(X)$. In particular $F(K) \simeq 0$, hence also $\alpha(K) \simeq 0$ by the assumption on $\alpha$. The same argument applies to the cokernel so we find that $\alpha(\eta_X)$ is an isomorphism $\beta F(X) = \alpha GF(X) \simeq \alpha(X)$. The argument is natural in $X$ so we get an isomorphism of functors $\beta F \simeq \alpha$ as desired.

(iii) [Thanks to V. Sosnilo for this argument.] The existence of the functor comes from the universal property: the inclusion functor $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ clearly sends $\mathcal{B} \cap \mathcal{A}_0$ to $\mathcal{B}$. For objects X and Y of $\mathcal{A}_0$, we need to show that the map

$$\mathrm{Hom}_{\mathcal{A}_0/(\mathcal{B}\cap\mathcal{A}_0)}(X, Y) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$$

is bijective. An element of the target can be represented by a zig-zag in $\mathcal{A}$

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & & \searrow^{g} & \\ X & & & & Y \end{array}$$

where $f$ is a $\mathcal{B}$-isomorphism, i.e., $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are contained in $\mathcal{B}$. This represents the morphism $g \circ f^{-1} : X \to Y$ in $\mathcal{A}/\mathcal{B}$.

Consider the short exact sequence

$$0 \to \mathrm{Im}(f) \to X \to \mathrm{Coker}(f) \to 0.$$

Since $X \in \mathcal{A}_0$ and $\mathrm{Coker}(f) \in \mathcal{B}$, the assumption implies $\mathrm{Coker}(f) \in \mathcal{A}_0$ and hence also $Z/\mathrm{Ker}(f) = \mathrm{Im}(f) \in \mathcal{A}_0$. Consider the commutative diagram of short exact sequences

$$\begin{CD} 0 @>>> \mathrm{Ker}(f) @>>> Z @>p>> Z/\mathrm{Ker}(f) @>>> 0\\ @. @VVV @VVgV @VV\overline{g}V @.\\ 0 @>>> g(\mathrm{Ker}(f)) @>>> Y @>q>> Y/g(\mathrm{Ker}(f)) @>>> 0 \end{CD}$$

Since $\mathrm{Ker}(f) \in \mathcal{B}$, also $g(\mathrm{Ker}(f)) \in \mathcal{B}$ since $\mathcal{B}$ is a Serre subcategory. Since $Y \in \mathcal{A}_0$ it also follows by the assumption that $g(\mathrm{Ker}(f)) \in \mathcal{A}_0$. Since $\mathcal{A}_0$ is abelian, then $Y' := Y/g(\mathrm{Ker}(f))$ is also in $\mathcal{A}_0$. In particular, $q$ is a $(\mathcal{B} \cap \mathcal{A}_0)$-isomorphism between objects of $\mathcal{A}_0$. The commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & & \searrow^{qg} & \\ X & & \downarrow{\scriptstyle p} & & Y' \\ & \nwarrow_{\overline{f}} & & \nearrow_{\overline{g}} & \\ & & Z/\mathrm{Ker}(f) & & \end{array}$$

exhibits an equivalence between the two zig-zags $X \leftarrow Z \to Y'$ and $X \leftarrow Z/\mathrm{Ker}(f) \to Y'$. In particular, $qgf^{-1}$ and $\overline{g}\overline{f}^{-1}$ represent the same morphism $X \to Y'$ in $\mathcal{A}/\mathcal{B}$. It follows that $q^{-1}qgf^{-1} = gf^{-1}$ and $q^{-1}\overline{g}\overline{f}^{-1}$ represent the same morphism $X \to Y$ in $\mathcal{A}/\mathcal{B}$. In other words, the zig-zags $X \leftarrow Z/\mathrm{Ker}(f) \to Y'$ and $Y' \leftarrow Y \to Y$ both represent morphisms in $\mathcal{A}_0/(\mathcal{A}_0 \cap \mathcal{B})$ which compose to a morphism whose image in $\mathcal{A}/\mathcal{B}$ is equivalent to our original morphism $g \circ f^{-1}$.

**2.** Let A be a ring and $f \in A$ an element.

(i) Let $\mathrm{Mod}_{\mathrm{A}}(f^{\infty}) \subseteq \mathrm{Mod}_{\mathrm{A}}$ denote the full subcategory of A-modules M that are $f^{\infty}$-torsion (i.e., for every $x \in \mathrm{M}$, $f^k x = 0$ for $k \gg 0$). Show that this is a Serre subcategory and that the canonical functor

$$\mathrm{Mod}_{\mathrm{A}}/(\mathrm{Mod}_{\mathrm{A}}(f^{\infty})) \to \mathrm{Mod}_{\mathrm{A}[f^{-1}]}$$

is an equivalence.

(ii) Assume that A is noetherian. Show that the canonical functor

$$\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}/(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(f^{\infty})) \to \mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}[f^{-1}]}$$

is fully faithful.

(iii) Let $\mathrm{B} = \mathrm{A}[f^{-1}]$. Show that every f.g. B-module N lifts to a f.g. A-module M such that $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B} \simeq \mathrm{N}$. Deduce that the canonical functor

$$\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}/(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(f^{\infty})) \to \mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}[f^{-1}]}$$

is an equivalence. (Hint: consider $\mathrm{N}_{[\mathrm{A}]} \in \mathrm{Mod}_{\mathrm{A}}$, which may not be f.g. However you can find a surjection $\mathrm{A}^{\oplus(\mathrm{I})} \to \mathrm{N}_{[\mathrm{A}]}$ from a free A-module indexed on a (possibly infinite) set I...)

(i) Consider the exact functor

$$(-) \otimes_{\mathrm{A}} \mathrm{A}[f^{-1}] : \mathrm{Mod}_{\mathrm{A}} \to \mathrm{Mod}_{\mathrm{A}[f^{-1}]}.$$

Its kernel consists of A-modules M such that $\mathrm{M}[f^{-1}] = 0$, or equivalently, M is $f^{\infty}$-torsion. In other words, this is the full subcategory $\mathrm{Mod}_{\mathrm{A}}(f^{\infty})$. Thus by Exercise 1(i), the latter is a Serre subcategory. Recall that $(-) \otimes_{\mathrm{A}} \mathrm{A}[f^{-1}]$ is left adjoint to the restriction of scalars functor $(-)_{[\mathrm{A}]}$. The latter is fully faithful (note that $\mathrm{A}[f^{-1}] \otimes_{\mathrm{A}} \mathrm{A}[f^{-1}] \simeq \mathrm{A}[f^{-1}]$ and then argue as in the proof that restriction of scalars along $\mathrm{A} \twoheadrightarrow \mathrm{A}/\mathrm{I}$ is fully faithful, §1.2). Now the claim follows from Exercise 1(ii).

(ii) We want to apply Exercise 1(iii) to the Serre subcategory $\mathrm{Mod}_{\mathrm{A}}(f^{\infty}) \subseteq \mathrm{Mod}_{\mathrm{A}}$ and the subcategory $\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}} \subseteq \mathrm{Mod}_{\mathrm{A}}$. The condition is that if $\mathrm{M} \in \mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}$ and $\mathrm{N} \in \mathrm{Mod}_{\mathrm{A}}(f^{\infty})$ is a subobject or quotient of M, then N is also f.g. This is clear since A is noetherian. Thus Exercise 1(iii) yields that

$$\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}/(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(f^{\infty})) \to \mathrm{Mod}_{\mathrm{A}}/(\mathrm{Mod}_{\mathrm{A}}(f^{\infty}))$$

is fully faithful. By (i) the target is equivalent to $\mathrm{Mod}_{\mathrm{A}[f^{-1}]}$, so we have shown that

$$\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}/(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(f^{\infty})) \to \mathrm{Mod}_{\mathrm{A}[f^{-1}]}$$

is fully faithful. But this functor lands in the full subcategory $\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}[f^{-1}]}$ and the induced functor

$$\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}/(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(f^{\infty})) \to \mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}[f^{-1}]}$$

must then also be fully faithful.

(iii) Consider the A-module $\mathrm{N}_{[\mathrm{A}]}$. We can find a surjection $\phi : \mathrm{A}^{\oplus(\mathrm{I})} \to \mathrm{N}_{[\mathrm{A}]}$ from a free A-module indexed on a (possibly infinite) set I (for example, take I to be the set of elements of N). These correspond to elements $\phi_i \in \mathrm{N}$ for $i \in \mathrm{I}$. Since

$N_{[A]} \otimes_A B \simeq N$ is f.g., we know that there exists a finite subset $J \subset I$ such that the induced map $B^{\oplus(J)} \to N$ is surjective. Let $M \subset N_{[A]}$ be the image of the map $A^{\oplus(J)} \to N_{[A]}$. It is then f.g. and satisfies $M \otimes_A B \simeq N$ by construction. This shows that the functor in question is essentially surjective, and it was already shown to be fully faithful in part (ii).

**3.** Let A be a noetherian ring.

(i) Show that $\phi : A \to A[T]$ induces is an injective homomorphism

$$\phi^* : G_0(A) \to G_0(A[T]).$$

(Hint: Note that $\phi$ admits a retraction in the category of commutative rings...)

(ii) If A is a field $k$, show that $\phi^* : G_0(k) \to G_0(k[T])$ is an isomorphism.

(i) Note that $\phi : A \to A[T]$ is flat and in particular of finite Tor-amplitude. Therefore there is a well-defined homomorphism $\phi^* : G_0(A) \to G_0(A[T])$ (see §6.3). Let $\sigma : A[T] \to A$ be the ring homomorphism $T \mapsto 0$. Since $\sigma \circ \phi = \mathrm{id}$, we have (see §6.3)

$$\sigma^* \phi^* = \mathrm{id} : G_0(A) \to G_0(A).$$

In particular, $\phi^*$ is injective.

(ii) It remains to show that $\phi^*$ is surjective. Note that we have a commutative square

$$\begin{array}{ccc} K_0(k) & \xrightarrow{\phi^*} & K_0(k[T]) \\ \downarrow & & \downarrow \\ G_0(k) & \xrightarrow{\phi^*} & G_0(k[T]). \end{array}$$

Since $k$ and $k[T]$ are regular rings (see §2.3 in the lecture), the vertical arrows are invertible. The upper horizontal arrow is also invertible: for both $k$ and $k[T]$, every f.g. projective module is free, so the map is identified with the identity $\mathrm{id} : \mathbf{Z} \to \mathbf{Z}$. It follows that the lower horizontal arrow is also invertible.

**4.** Let A be an integral domain. Given an element $f \in A$ and a point $p \in |\mathrm{Spec}(A)|$, the *value* of $f$ at $p$, denoted $f(p)$, is the image of $f$ by the homomorphism $\phi : A \to \kappa(p)$. (Elements of A are thought of as "algebraic functions" on Spec(A).)

(i) Show that if an element $f$ vanishes at the generic point $\eta$ then $f = 0$.

(ii) Give an example to show that if A is not an integral domain, then an element $f \in A$ can vanish at every point without being zero.

(Use the definition of $|\mathrm{Spec}(A)|$ given in the lecture, not the one using prime ideals.)

(i) Since $A \hookrightarrow \kappa(\eta) = \mathrm{Frac}(A)$ is injective, we have $f(\eta) = 0$ iff $f = 0$.

(ii) Consider the ring of dual numbers $\mathrm{A} = k[\varepsilon]/\langle \varepsilon^2 \rangle$ over a field $k$. Recall that A has a single point $p = [\mathrm{A} \to k]$. The element $\varepsilon \in \mathrm{A}$ has value $\varepsilon(p) = 0$ at this point, but $\varepsilon \neq 0$.