Exercise sheet 5

1. Let A be a commutative ring.

(a) Show that A is a filtered colimit (inductive limit) of its finitely generated subrings (i.e., subrings that are finitely generated as **Z**-algebras).

(b) Suppose given a diagram of commutative rings $(\mathrm{A}_\lambda)_{\lambda\in\Lambda}$ indexed by a poset (partially ordered set) Λ. Let A denote the filtered colimit $\varinjlim_\lambda \mathrm{A}_\lambda$ (in the category of commutative rings). Show that there is a canonical isomorphism

$$\mathrm{K}_0(\mathrm{A}) \simeq \varinjlim_{\lambda\in\Lambda} \mathrm{K}_0(\mathrm{A}_\lambda).$$

(c) Deduce from (a) and (b) that, for every commutative ring A, there is a canonical isomorphism $\mathrm{K}_0(\mathrm{A}) \simeq \varinjlim_{\lambda\in\Lambda} \mathrm{K}_0(\mathrm{A}_\lambda)$ where Λ is a poset and A_λ are noetherian.

(a) The set Λ of finitely generated subrings of A becomes a poset with the inclusion order. Any poset can be regarded as a filtered category whose objects are the elements of the poset, and there is at most one morphism between any two objects. We then have a canonical functor $\Lambda \to \mathrm{CRing}$, sending a f.g. subring $\mathrm{A}_\lambda \in \Lambda$ to itself. Since A is the union of these subrings, it is easy to see that A is the colimit of this diagram.

(b) We first show that the canonical homomorphism of monoids

$$\varinjlim_{\lambda} \mathcal{M}(\mathrm{A}_\lambda) \to \mathcal{M}(\mathrm{A}),$$

induced by the extension of scalars maps $\mathcal{M}(\mathrm{A}_\lambda) \to \mathcal{M}(\mathrm{A})$, is invertible. For surjectivity, we need to show that every f.g. projective A-module M descends to a f.g. projective A_α-module M_α, for some sufficiently large $\alpha \in \Lambda$. Note that M is a direct summand of some $\mathrm{A}^{\oplus n}$, so it is the image of some projector (idempotent endomorphism) $\phi : \mathrm{A}^{\oplus n} \to \mathrm{A}^{\oplus n}$. This corresponds to a matrix $(\phi_{i,j})_{i,j}$ with entries $\phi_{i,j} \in \mathrm{A}$. Since A is the colimit of $(\mathrm{A}_\lambda)_\lambda$, there exists some $\alpha \in \Lambda$ and a matrix $(\psi_{i,j})_{i,j}$ with entries in A_α, such that $\psi_{i,j} \mapsto \phi_{i,j}$ for all i, j. We let M_α be the image of the corresponding projector $\mathrm{A}_\alpha^{\oplus n} \to \mathrm{A}_\alpha^{\oplus n}$. Then we have $\mathrm{M}_\alpha \otimes_{\mathrm{A}_\alpha} \mathrm{A} \simeq \mathrm{M}$ by construction.

For injectivity, we need to show that if M and N are A_λ modules, then any A-module isomorphism $\mathrm{M} \otimes_{\mathrm{A}_\lambda} \mathrm{A} \simeq \mathrm{N} \otimes_{\mathrm{A}_\lambda} \mathrm{A}$ descends to an A_α-module isomorphism $\mathrm{M} \otimes_{\mathrm{A}_\lambda} \mathrm{A}_\alpha \simeq \mathrm{N} \otimes_{\mathrm{A}_\lambda} \mathrm{A}_\alpha$ for some $\alpha \geqslant \lambda$. The argument is similar to above, noting that such an A-module isomorphism is determined by (finitely many) elements of A, which must come from some A_α.

Now, apply group completion $(-)^{\mathrm{gp}}$ to the above monoid isomorphism. Since $(-)^{\mathrm{gp}}$ is left adjoint to the functor from abelian groups to commutative monoids, it commutes with colimits. Thus the claim follows.

(c) We only need to show that finitely generated **Z**-algebras are noetherian. This follows from the Hilbert basis theorem.

2. (a) Let A be a commutative ring and $\phi : \mathrm{M}_\bullet \to \mathrm{N}_\bullet$ a quasi-isomorphism of chain complexes over A. Let $\mathrm{P}_\bullet$ be a chain complex of projective A-modules. Suppose that $\mathrm{M}_\bullet$, $\mathrm{N}_\bullet$ and $\mathrm{P}_\bullet$ are all bounded below. Show that any morphism $\beta : \mathrm{P}_\bullet \to \mathrm{N}_\bullet$ lifts to a morphism $\alpha : \mathrm{P}_\bullet \to \mathrm{M}_\bullet$, such that the diagram

$$\begin{tikzcd} & \mathrm{M}_\bullet \arrow[d, "\phi"] \\ \mathrm{P}_\bullet \arrow[ur, dashed, "\alpha"] \arrow[r, "\beta"] & \mathrm{N}_\bullet \end{tikzcd}$$

commutes up to homotopy (i.e., the morphisms β and $\phi \circ \alpha$ are homotopic).

(b) Let $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ be bounded below complexes over A. If $\mathrm{N}_\bullet$ is projective, show that any quasi-isomorphism $\phi : \mathrm{M}_\bullet \to \mathrm{N}_\bullet$ admits a section up to homotopy, which is also a quasi-isomorphism.

(c) Let $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ be bounded below complexes over A. Suppose they are quasi-isomorphic (in the sense that there exists a zig-zag of quasi-isomorphisms between them). Show that if $\mathrm{M}_\bullet$ is projective, then there exists a quasi-isomorphism $\mathrm{M}_\bullet \to \mathrm{N}_\bullet$. Give an example to show that this is not true if $\mathrm{M}_\bullet$ is not projective.

(a) We use the following observation:

Lemma 1. *Let $\mathrm{P}_\bullet$ be a complex of projectives and $\mathrm{Q}_\bullet$ an acyclic complex. Let α and β be morphisms $\mathrm{P}_\bullet \to \mathrm{Q}_\bullet$. Suppose given, for some integer n, a collection of morphisms $h_i : \mathrm{P}_i \to \mathrm{Q}_{i+1}$ for $i \leqslant n$, satisfying*

$$\alpha_i - \beta_i = h_{i-1} d_i + d_{i+1} h_i$$

for each $i \leqslant n-1$. Then the h_i extend to a homotopy $\alpha \simeq \beta$.

Proof. Let $\phi_n : \mathrm{P}_n \to \mathrm{Q}_n$ be the morphism $(\alpha_n - \beta_n) - h_{n-1} d_n$. Note that ϕ_n lands in $\mathrm{Ker}(d_n)$. Indeed from the relation

$$\alpha_{n-1} - \beta_{n-1} = d_n h_{n-1} + h_{n-2} d_{n-1}$$

we derive, by composing with d_n on the right (and using the fact that α and β commute with d and that $d^2 = 0$),

$$d_n(\alpha_n - \beta_n) = d_n h_{n-1} d_n.$$

In other words, $d_n \phi_n = 0$. Now since $\mathrm{Q}_\bullet$ is acyclic and in particular $\mathrm{H}_n(\mathrm{Q}_\bullet) = 0$, it follows that $\phi_n \in \mathrm{Im}(d_{n+1})$. Since $\mathrm{P}_\bullet$, we can find a lift in the diagram

$$\begin{tikzcd} & \mathrm{Q}_{n+1} \arrow[d, two heads, "d_{n+1}"] \\ \mathrm{P}_n \arrow[ur, dashed, "h_n"] \arrow[r, "\phi_n"] & \mathrm{Im}(d_{n+1}). \end{tikzcd}$$

The relation

$$d_{n+1} h_n = \phi_n = (\alpha_n - \beta_n) - h_{n-1} d_n$$

translates to

$$\alpha_n - \beta_n = h_{n-1} d_n + d_{n+1} h_n.$$

The claim follows by induction. □

Let $\mathrm{K}_\bullet = \mathrm{Cone}(\phi)_\bullet$. Since ϕ is a quasi-isomorphism, $\mathrm{K}_\bullet$ is acyclic. Applying the Lemma (we can take $h_i = 0$ for $i \ll 0$ since the complexes are bounded below), we deduce that the morphism

$$\mathrm{P}_\bullet \xrightarrow{\beta} \mathrm{N}_\bullet \to \mathrm{K}_\bullet$$

is homotopic to zero through a homotopy $(h_i)_i$. The morphisms $h_i : \mathrm{P}_i \to \mathrm{K}_{i+1} = \mathrm{N}_{i+1} \oplus \mathrm{M}_i$ give rise in particular to morphisms $\alpha_i : \mathrm{P}_i \to \mathrm{M}_i$. From the homotopy relation satisfied by h_i, one reads off that α defines a morphism of chain complexes and that β is homotopic to $\phi \circ \alpha$.

(b) Apply (a) to the diagram

$$\begin{tikzcd} & \mathrm{M}_\bullet \arrow[d, "\phi"] \\ \mathrm{N}_\bullet \arrow[ur, dashed] \arrow[r, equal] & \mathrm{N}_\bullet . \end{tikzcd}$$

(c) If $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ are quasi-isomorphic, there exists a zig-zag

$$\mathrm{M}_\bullet = \mathrm{L}^0_\bullet \leftarrow \mathrm{L}^1_\bullet \to \mathrm{L}^2_\bullet \leftarrow \cdots \to \mathrm{L}^n_\bullet = \mathrm{N}_\bullet \tag{0.1}$$

where all arrows are quasi-isomorphisms. Note that the $\mathrm{L}^i_\bullet$ are not necessarily bounded below (and not necessarily projective). However, since $\mathrm{M}_\bullet$ is bounded below, it is n-connective for some n, hence so are all $\mathrm{L}^i_\bullet$. For each diagram

$$\mathrm{L}^{i-1}_\bullet \leftarrow \mathrm{L}^i_\bullet \to \mathrm{L}^{i+1}_\bullet,$$

we can expand this to a diagram

$$\begin{tikzcd} \mathrm{L}^{i-1}_\bullet & \mathrm{L}^i_\bullet \arrow[l] \arrow[r] & \mathrm{L}^{i+1}_\bullet \\ & \tau_{\geqslant n}(\mathrm{L}^i_\bullet) \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

where all arrows are still quasi-isomorphisms. For every odd $i \geqslant 1$, replace $\mathrm{L}^i_\bullet$ in (0.1) by $\tau_{\geqslant n}\mathrm{L}^i_\bullet \to \mathrm{L}^{i+1}_\bullet$. For every odd $i \geqslant 1$, we can also find a quasi-isomorphism from a bounded below complex $\mathrm{P}^i_\bullet$ of projectives, and replace every $\mathrm{L}^i_\bullet$ by $\mathrm{P}^i_\bullet$. Applying (b), we can now reverse the wrong-way arrows in the zigzag (0.1).

3. Let A be a commutative ring and let $\mathrm{M}_\bullet \to \mathrm{N}_\bullet \to \mathrm{K}_\bullet$ be an exact triangle of chain complexes of A-modules. Show that if any two of the terms is perfect, so is the third.

Given the technology we have, the proof is much easier in the case where A is noetherian, which we assume. From the long exact sequence in homology we see that if any two of the terms is coherent, then so is the third. Thus it will suffice to show that if any two of the terms is of finite Tor-amplitude, then so is the third. This follows by inspecting the long exact sequence associated to the exact triangle $\mathrm{M}_\bullet \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{E} \to \mathrm{N}_\bullet \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{E} \to \mathrm{K}_\bullet \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{E}$, for every A-module E.

Here is another argument which works if we assume that the complexes are bounded below. By rotating the triangle, we may assume that $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ are perfect (since shifting has no effect on perfectness). We may assume that $\mathrm{K}_\bullet$ is the cone of $\phi : \mathrm{M}_\bullet \to \mathrm{N}_\bullet$. Let $\mathrm{P}_\bullet \to \mathrm{M}_\bullet$ and $\mathrm{Q}_\bullet \to \mathrm{N}_\bullet$ be quasi-isomorphisms with $\mathrm{P}_\bullet, \mathrm{Q}_\bullet \in \mathrm{Proj}^b_\mathrm{A}$. Then by Exercise 2 (a), we can find a morphism ψ making the left-hand square below commute up to homotopy:

$$\begin{tikzcd} \mathrm{P}_\bullet \arrow[r, dashed, "\psi"] \arrow[d] & \mathrm{Q}_\bullet \arrow[r] \arrow[d] & \mathrm{Cone}(\psi)_\bullet \arrow[d, dashed] \\ \mathrm{M}_\bullet \arrow[r, "\phi"] & \mathrm{N}_\bullet \arrow[r] & \mathrm{Cone}(\phi)_\bullet \end{tikzcd}$$

Then there is an induced quasi-isomorphism between the cones. Since P_i and Q_i are all projectives, we also have $\mathrm{Cone}(\psi) \in \mathrm{Proj}^b_\mathrm{A}$. Thus $\mathrm{K}_\bullet = \mathrm{Cone}(\phi)_\bullet$ is perfect.

4. Let A be a commutative ring. A chain complex $\mathrm{M}_\bullet$ is called *connective* if it is 0-connective, i.e., $\mathrm{H}_i(\mathrm{M}_\bullet) = 0$ for $i < 0$. Imitate the construction of $\mathrm{K}_0(\mathrm{Perf}_\mathrm{A})$ to define a variant $\mathrm{K}_0(\mathrm{Perf}^{\mathrm{cn}}_\mathrm{A})$ using (quasi-isomorphism classes of) connective perfect complexes. Show that there is a canonical isomorphism

$$\mathrm{K}_0(\mathrm{Perf}^{\mathrm{cn}}_\mathrm{A}) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{Perf}_\mathrm{A}).$$

Note that there is a commutative diagram

$$\begin{tikzcd} \mathrm{K}_0(\mathrm{A}) \arrow[rr, "\sim"] \arrow[dr, "(\dagger)"] & & \mathrm{K}_0(\mathrm{Perf}_\mathrm{A}) \\ & \mathrm{K}_0(\mathrm{Perf}^{\mathrm{cn}}_\mathrm{A}) \arrow[ur, "(\star)"] & \end{tikzcd}$$

where the upper arrow is known to be an isomorphism from Lecture 4. This already implies that the arrow in question is surjective. There are various ways to proceed. One is to show that the map $(\dagger)$ is surjective, repeating the proof for the upper arrow. Another is to write down a retraction $\mathrm{K}_0(\mathrm{Perf}_\mathrm{A}) \to \mathrm{K}_0(\mathrm{Perf}^{\mathrm{cn}}_\mathrm{A})$. Given a perfect complex $\mathrm{M}_\bullet \in \mathrm{Perf}_\mathrm{A}$, send $[\mathrm{M}_\bullet] \mapsto [\mathrm{M}_\bullet[k]]$ where k is an arbitrary even shift large enough so that $\mathrm{M}_\bullet[k]$ is connective.