Exercise sheet 4

1. Let A be a commutative ring and $M_\bullet$ a chain complex of A-modules. Show that if $M_\bullet$ is acyclic, then it is perfect.

2. Let A be a commutative ring and $M_\bullet$ a chain complex of A-modules. Suppose that $M_\bullet$ is *n-connective* for some integer n, i.e., $H_i(M_\bullet) = 0$ for $i < n$. Then there is a diagram of chain complexes

$$M_\bullet \xleftarrow{\text{qis}} \tau_{\geqslant n}(M_\bullet) \to H_n(M_\bullet)[n].$$

Here $\tau_{\geqslant n}(M_\bullet)$ denotes the truncated complex

$$\cdots \to M_{n+2} \xrightarrow{d_{n+2}} M_{n+1} \to \mathrm{Ker}(d_n) \to 0,$$

where $\mathrm{Ker}(d_n)$ is in degree n (and the differential $M_{n+1} \to \mathrm{Ker}(d_n)$ factors through $\mathrm{Im}(d_{n+1}) \subseteq \mathrm{Ker}(d_n)$).

3. Let A be a commutative ring and $M_\bullet$ a chain complex of A-modules. Show that the following conditions are equivalent:

(a) $H_i(M_\bullet) \neq 0$ for exactly one $i \in \mathbf{Z}$.

(b) $M_\bullet$ is quasi-isomorphic to $H_i(M_\bullet)[i]$, via a zig-zag $M_\bullet \leftarrow ? \to H_i(M_\bullet)[i]$, where both arrows are quasi-isomorphisms.

4. (i) Give an example of a perfect complex $P_\bullet$ over some ring A which is unbounded ($P_i \neq 0$ for infinitely many $i \in \mathbf{Z}$).

(ii) Give an example of a perfect complex $Q_\bullet$ over some ring A which has $H_i(Q_\bullet) \neq 0$ for at least two $i \in \mathbf{Z}$, and which is not a bounded complex of f.g. projective modules.