Exercise sheet 4

1. Let A be a commutative ring and $M_\bullet$ a chain complex of A-modules. Show that if $M_\bullet$ is acyclic, then it is perfect.

Note that the acyclicity of $M_\bullet$ means that the unique morphism $0 \to M_\bullet$ is a quasi-isomorphism (where 0 is the zero complex). Since 0 is a bounded complex of f.g. projectives, this means that $M_\bullet$ is perfect.

2. Let A be a commutative ring and $M_\bullet$ a chain complex of A-modules. Suppose that $M_\bullet$ is *n-connective* for some integer n, i.e., $H_i(M_\bullet) = 0$ for $i < n$. Then there is a diagram of chain complexes

$$M_\bullet \xleftarrow{\text{qis}} \tau_{\geqslant n}(M_\bullet) \to H_n(M_\bullet)[n].$$

Here $\tau_{\geqslant n}(M_\bullet)$ denotes the truncated complex

$$\cdots \to M_{n+2} \xrightarrow{d_{n+2}} M_{n+1} \to \mathrm{Ker}(d_n) \to 0,$$

where $\mathrm{Ker}(d_n)$ is in degree n (and the differential $M_{n+1} \to \mathrm{Ker}(d_n)$ factors through $\mathrm{Im}(d_{n+1}) \subseteq \mathrm{Ker}(d_n)$).

Note that replacing $M_\bullet$ by $M_\bullet[n]$ has the effect of replacing $H_n(M_\bullet)[n]$ by $H_n(M_\bullet[n])[n] = H_0(M_\bullet)[n]$, and $\tau_{\leqslant n}(M_\bullet)$ by $\tau_{\leqslant n}(M_\bullet[n]) = \tau_{\leqslant 0}(M_\bullet)$. Therefore, we may as well assume that $M_\bullet$ is 0-connective. (This simplifies nothing except the notation.)

The morphisms $M_\bullet \leftarrow \tau_{\geqslant 0}(M_\bullet) \to H_0(M_\bullet)[0]$ are defined as

$$\begin{array}{ccccccccccc}
 & & & & 0 & \longrightarrow & H_0(M_\bullet) & \longrightarrow & 0 & & \\
 & & & & \uparrow & & \uparrow\!\uparrow & & \uparrow & & \\
\cdots & \longrightarrow & M_2 & \xrightarrow{d_2} & M_1 & \xrightarrow{d_1} & \mathrm{Ker}(d_0) & \longrightarrow & 0 & & \\
 & & \| & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M_2 & \xrightarrow{d_2} & M_1 & \xrightarrow{d_1} & M_0 & \xrightarrow{d_0} & M_{-1} & \longrightarrow & \cdots
\end{array}$$

It is clear that $M_\bullet \leftarrow \tau_{\geqslant 0}(M_\bullet)$ is a quasi-isomorphism.

3. Let A be a commutative ring and $M_\bullet$ a chain complex of A-modules. Show that the following conditions are equivalent:

(a) $H_i(M_\bullet) \neq 0$ for exactly one $i \in \mathbf{Z}$.

(b) $M_\bullet$ is quasi-isomorphic to $H_i(M_\bullet)[i]$, via a zig-zag $M_\bullet \leftarrow ? \to H_i(M_\bullet)[i]$, where both arrows are quasi-isomorphisms.

Since the complex $\mathrm{H}_i(\mathrm{M}_\bullet)[i]$ is concentrated in degree i, it is clear that it has exactly one non-vanishing homology group. Since the condition in (a) is preserved by quasi-isomorphisms, it follows that (b) implies (a).

Suppose (a), and let $\mathrm{H}_i(\mathrm{M}_\bullet)$ be the only non-vanishing homology group. Then $\mathrm{M}_\bullet$ is in particular i-connective, so by Exercise 1 there exists a zig-zag

$$\mathrm{M}_\bullet \xleftarrow{\text{qis}} \tau_{\geqslant i}(\mathrm{M}_\bullet) \to \mathrm{H}_i(\mathrm{M}_\bullet)[i].$$

By definition, $\tau_{\geqslant i}(\mathrm{M}_\bullet)$ and $\mathrm{H}_i(\mathrm{M}_\bullet)[i]$ are both bounded on the right by i (below i, all their terms vanish). At i, the map clearly induces an isomorphism on H_i. To the left (above i), $\tau_{\geqslant i}(\mathrm{M}_\bullet)$ has the same homology groups as $\mathrm{M}_\bullet$ and is therefore acyclic. Thus we see that $\tau_{\geqslant i}(\mathrm{M}_\bullet) \to \mathrm{H}_i(\mathrm{M}_\bullet)[i]$ is also a quasi-isomorphism.

4. (i) Give an example of a perfect complex $\mathrm{P}_\bullet$ over some ring A which is unbounded ($\mathrm{P}_i \neq 0$ for infinitely many $i \in \mathbf{Z}$).

(ii) Give an example of a perfect complex $\mathrm{Q}_\bullet$ over some ring A which has $\mathrm{H}_i(\mathrm{Q}_\bullet) \neq 0$ for at least two $i \in \mathbf{Z}$, and which is not a bounded complex of f.g. projective modules.

(i) For example,

$$\mathrm{P}_\bullet = \left(\cdots \xrightarrow{0} \mathrm{A} \xrightarrow{\text{id}} \mathrm{A} \xrightarrow{0} \mathrm{A} \xrightarrow{\text{id}} \mathrm{A} \xrightarrow{0} \cdots\right)$$

for any commutative ring A. Since $\mathrm{H}_i(\mathrm{P}_\bullet) = 0$ for all i, this complex is acyclic. By exercise 1, it is perfect.

(ii) For example, the complex of $\mathbf{Z}$-modules

$$\mathrm{Q}_\bullet = \left(0 \to \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \to 0\right)$$

where the map is multiplication by 2. Certainly $\mathbf{Z}/4\mathbf{Z}$ is not a projective $\mathbf{Z}$-module, and the complex has non-vanishing H_0 and H_1 (both isomorphic to $\mathbf{Z}/2\mathbf{Z}$). But it is indeed perfect:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\binom{2}{0}} & \mathbf{Z}\oplus\mathbf{Z} & \xrightarrow{(0,2)} & \mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (2,1)} & & \downarrow{\scriptstyle 1} & & \\
 & & 0 & \longrightarrow & \mathbf{Z}/4\mathbf{Z} & \xrightarrow{2} & \mathbf{Z}/4\mathbf{Z} & \longrightarrow & 0
\end{array}$$

This diagram depicts a quasi-isomorphism between the upper row, a finite complex of f.g. free $\mathbf{Z}$-modules, and the lower row, $\mathrm{Q}_\bullet$.