Exercise sheet 3

1. Let A be a regular ring. Show that the polynomial ring $\mathrm{A}[t_1, \ldots, t_n]$ is regular for every $n \geqslant 0$.

(Hint: Show that you can reduce to the following: if A is regular local, then $\mathrm{A}[t]_{\mathfrak{p}}$ is regular local, where $\mathfrak{p} \subset \mathrm{A}[t]$ is a prime ideal containing the maximal ideal of A. Then use a resolution of the residue field of A to build a resolution for the residue field of $\mathrm{A}[t]_{\mathfrak{p}}$.)

2. (i) Let X be the commutative monoid with two elements $0, x$ with $x + x = x$ (and 0 is the neutral element). Show that its group completion X^{gp} is zero.

(ii) Let Y be the additive commutative monoid whose underlying set is $\mathbf{N} \cup \{\infty\}$ and where $\infty + \infty = \infty$ and $n + \infty = \infty$ for every $n \in \mathbf{N}$. Show that its group completion Y^{gp} is zero.

3. Let A be a nonzero commutative ring.

(i) Show that there is a canonical group homomorphism $\phi : \mathbf{Z} \to \mathrm{K}_0(\mathrm{A})$ sending $n \mapsto [\mathrm{A}^{\oplus n}]$ for $n \geqslant 0$.

(ii) Show that ϕ exhibits $\mathbf{Z}$ as a direct summand of $\mathrm{K}_0(\mathrm{A})$. (Hint: recall $\mathbf{Z} \simeq \mathrm{K}_0(k)$ for any field k. Since A is nonzero there exists at least one ring homomorphism $\mathrm{A} \to k$. Use this to construct a retraction of ϕ, i.e., a morphism $\psi : \mathrm{K}_0(\mathrm{A}) \to \mathbf{Z}$ such that $\psi \circ \phi = \mathrm{id}$.)

(iii) Show that ϕ is bijective iff every f.g. projective A-module is stably free (i.e., stably equivalent to a free module).

4. (i) If A is an integral domain, show that there is a well-defined homomorphism $\mathrm{G}_0(\mathrm{A}) \to \mathbf{Z}$ sending [M] to the *rank* $\mathrm{rk}_{\mathrm{A}}(\mathrm{M}) := \dim_{\mathrm{K}}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{K})$, where K is the field of fractions.

(ii) If A is a PID, use (i) to show that the canonical homomorphism $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$ is injective.

(iii) If A is a PID, show that the canonical map $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$ is also surjective by using the structure theory of f.g. modules over a PID.

(In the lecture, we will show that (ii) and (iii) hold for every regular ring A; this is a special case since PID's are regular.)