Exercise sheet 3

1. Let A be a regular ring. Show that the polynomial ring $\mathrm{A}[t_1, \ldots, t_n]$ is regular for every $n \geqslant 0$.

(Hint: Show that you can reduce to the following: if A is regular local, then $\mathrm{A}[t]_{\mathfrak{p}}$ is regular local, where $\mathfrak{p} \subset \mathrm{A}[t]$ is a prime ideal containing the maximal ideal of A. Then use a resolution of the residue field of A to build a resolution for the residue field of $\mathrm{A}[t]_{\mathfrak{p}}$.)

By induction we may take $n = 1$. To show that $\mathrm{A}[t]$ is regular it will suffice to show that $(\mathrm{A}[t])_{\mathfrak{p}}$ is regular for every prime ideal $\mathfrak{p} \subset \mathrm{A}[t]$ (Lecture notes, §2.3). The preimage of $\mathfrak{p}$ under the canonical homomorphism $\mathrm{A} \to \mathrm{A}[t]$ is the prime ideal $\mathfrak{q} = \mathrm{A} \cap \mathfrak{p} \subset \mathrm{A}$. We would like to replace A by $\mathrm{A}_{\mathfrak{q}}$ to be able to reduce to the local case, so we need to relate $(\mathrm{A}[t])_{\mathfrak{p}}$ with $\mathrm{A}_{\mathfrak{q}}[t]$. Note that there is a canonical isomorphism $\mathrm{A}_{\mathfrak{q}}[t] \simeq (\mathrm{A}[t])[(\mathrm{A} \smallsetminus \mathfrak{q})^{-1}]$. Since $\mathrm{A} \smallsetminus \mathfrak{q} \subset \mathrm{A}[t] \smallsetminus \mathfrak{p}$, we see that $(\mathrm{A}[t])_{\mathfrak{p}}$ is a localization of $\mathrm{A}_{\mathfrak{q}}[t]$. Since we know that localizations of regular rings are regular (Lecture notes, §2.3), we would be done if the assertion was known for the local ring $\mathrm{A}_{\mathfrak{q}}$. Thus we may replace A by $\mathrm{A}_{\mathfrak{q}}$.

Now A is a regular local ring, with maximal ideal $\mathfrak{m}$, and our prime ideal $\mathfrak{p} \subset \mathrm{A}[t]$ contains $\mathfrak{m}$. We still want to show that $\mathrm{A}[t]_{\mathfrak{p}}$ is regular. It is enough to show that its residue field $\kappa(\mathfrak{p})$ is perfect as an $\mathrm{A}[t]_{\mathfrak{p}}$-module (Lecture notes, §2.2). Note that $\kappa(\mathfrak{p})$ is the localization of $\kappa[t] = \kappa \otimes_{\mathrm{A}} \mathrm{A}[t]$ at the multiplicative subset $\mathrm{A}[t] \smallsetminus \mathfrak{p}$, where $\kappa = \mathrm{A}/\mathfrak{m}$. Since A is regular, there exists a finite f.g.proj. resolution $\mathrm{P}_\bullet \to \kappa$ of A-modules. Tensoring with the flat A-module $\mathrm{A}[t]$, we get a finite f.g.proj. resolution $\mathrm{P}_\bullet \otimes_{\mathrm{A}} \mathrm{A}[t] \to \kappa[t]$ of $\mathrm{A}[t]$-modules. Localizing at the subset $\mathrm{A}[t] \smallsetminus \mathfrak{p}$ is also exact and thus yields a finite f.g.proj. resolution of $\kappa(\mathfrak{p})$.

2. (i) Let X be the commutative monoid with two elements $0, x$ with $x + x = x$ (and 0 is the neutral element). Show that its group completion X^{gp} is zero.

(ii) Let Y be the additive commutative monoid whose underlying set is $\mathbf{N} \cup \{\infty\}$ and where $\infty + \infty = \infty$ and $n + \infty = \infty$ for every $n \in \mathbf{N}$. Show that its group completion Y^{gp} is zero.

(i) We have $(x, 0) = (0, 0)$ in X^{gp} since $x + x = x$ in X, and the same for its inverse $-(x, 0) = (0, x)$, and for $(x, x) = (x, 0) + (0, x)$.

(ii) We have $(m, n) = (0, 0)$ for all $m, n \in \mathbf{N}$ since $m + \infty = n + \infty$. Also $(\infty, x) = (0, 0)$ and $(x, \infty) = (0, 0)$ since $\infty + \infty = x + \infty$ for all $x \in \mathrm{Y}$.

3. Let A be a nonzero commutative ring.

(i) Show that there is a canonical group homomorphism $\phi : \mathbf{Z} \to \mathrm{K}_0(\mathrm{A})$ sending $n \mapsto [\mathrm{A}^{\oplus n}]$ for $n \geqslant 0$.

(ii) Show that ϕ exhibits $\mathbf{Z}$ as a direct summand of $\mathrm{K}_0(\mathrm{A})$. (Hint: recall $\mathbf{Z} \simeq \mathrm{K}_0(k)$ for any field k. Since A is nonzero there exists at least one ring homomorphism $\mathrm{A} \to k$. Use this to construct a retraction of ϕ, i.e., a morphism $\psi : \mathrm{K}_0(\mathrm{A}) \to \mathbf{Z}$ such that $\psi \circ \phi = \mathrm{id}$.)

(iii) Show that ϕ is bijective iff every f.g. projective A-module is stably free (i.e., stably equivalent to a free module).

(i) There is a unique monoid homomorphism $\mathbf{N} \to \mathcal{M}(\mathrm{A})$ that sends $1 \mapsto [\mathrm{A}]$. Here $\mathcal{M}(\mathrm{A})$ is the monoid of isomorphism classes of objects of $\mathrm{Mod}_{\mathrm{A}}^{\mathrm{fgproj}}$, and the operation on $\mathbf{N}$ is addition (this is the free commutative monoid on one generator). Passing to group completions, we get an induced homomorphism $\mathbf{Z} \to \mathrm{K}_0(\mathrm{A})$ (group completion is functorial).

Alternatively, one can show that the unique ring homomorphism $\mathbf{Z} \to \mathrm{A}$ induces a homomorphism $\mathbf{Z} \simeq \mathrm{K}_0(\mathbf{Z}) \to \mathrm{K}_0(\mathrm{A})$ by extension of scalars (which preserves f.g. projectives, §1.2).

(ii) Since A is nonempty there exists a ring homomorphism $\mathrm{A} \to k$ with k a field. Extension of scalars defines an induced monoid homomorphism $\mathcal{M}(\mathrm{A}) \to \mathcal{M}(k)$. The homomorphism

$$\mathbf{N} \to \mathcal{M}(\mathrm{A}) \to \mathcal{M}(k)$$

sends $1 \mapsto [\mathrm{A}] \to [k]$, and is bijective. Hence so is the induced map on group completions:

$$\mathbf{Z} \xrightarrow{\phi} \mathrm{K}_0(\mathrm{A}) \to \mathrm{K}_0(k).$$

It follows that ϕ is a split monomorphism, so by the splitting lemma, $\mathbf{Z}$ is a direct summand of $\mathrm{K}_0(\mathrm{A})$.

(iii) By the Lemma in §3.1 of the Lecture, every $x \in \mathrm{K}_0(\mathrm{A})$ can be written as $[\mathrm{M}] - n \cdot [\mathrm{A}]$ where $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fgproj}}$ and $n \geqslant 0$ (note $n \cdot [\mathrm{A}] = [\mathrm{A}^{\oplus n}]$). Thus ϕ is surjective iff for every such M and n, there exists an integer $m \in \mathbf{Z}$ such that $[\mathrm{M}] - n \cdot [\mathrm{A}] = m \cdot [\mathrm{A}]$ in $\mathrm{K}_0(\mathrm{A})$. Adding some multiple of [A] to both sides, this is equivalent to $[\mathrm{M} \oplus \mathrm{A}^{\oplus k}] = (m+n) \cdot [\mathrm{A}] = [\mathrm{A}^{\oplus m+n}]$ for some integer $m \geqslant 0$. Then by the second part of the Lemma in §3.1, this is equivalent to $\mathrm{M} \oplus \mathrm{A}^{\oplus k}$ being stably free, which is equivalent to M being stably free.

4. (i) If A is an integral domain, show that there is a well-defined homomorphism $\mathrm{G}_0(\mathrm{A}) \to \mathbf{Z}$ sending [M] to the *rank* $\mathrm{rk}_{\mathrm{A}}(\mathrm{M}) := \dim_{\mathrm{K}}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{K})$, where K is the field of fractions.

(ii) If A is a PID, use (i) to show that the canonical homomorphism $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$ is injective.

(iii) If A is a PID, show that the canonical map $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$ is also surjective by using the structure theory of f.g. modules over a PID.

(In the lecture, we will show that (ii) and (iii) hold for every regular ring A; this is a special case since PID's are regular.)

(i) If $0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}'' \to 0$ is an exact sequence of f.g. A-modules, then one has $\mathrm{rk}_{\mathrm{A}}(\mathrm{M}) = \mathrm{rk}_{\mathrm{A}}(\mathrm{M}') + \mathrm{rk}_{\mathrm{A}}(\mathrm{M}'')$. This follows from the fact that K is flat as an A-module (since it is a localization), and for K-vector spaces, dimension is "additive".

Alternatively, take the composite

$$\mathrm{G}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{K}) \simeq \mathbf{Z},$$

where the first map is induced by $[\mathrm{M}] \mapsto [\mathrm{M} \otimes_{\mathrm{A}} \mathrm{K}]$. The fact that this is well-defined again follows from the flatness of K.

(ii) We know that for a PID, $\mathrm{K}_0(\mathrm{A}) \simeq \mathbf{Z}$ since every f.g. projective A-module is free. By (i), we know that this extends to a map $\mathrm{G}_0(\mathrm{A}) \to \mathbf{Z}$ making the diagram

$$\begin{array}{ccc} \mathrm{K}_0(\mathrm{A}) & \xrightarrow{\sim} & \mathbf{Z} \\ \downarrow & \nearrow & \\ \mathrm{G}_0(\mathrm{A}) & & \end{array}$$

commute. So $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$ is a split monomorphism.

(iii) Recall that $\mathrm{K}_0(\mathrm{A}) \simeq \mathbf{Z}$ so the main thing is to compute $\mathrm{G}_0(\mathrm{A})$. Let M be a f.g. A-module. Then since A is a PID, M is a direct sum of a free A-module (say of rank r) and finitely many cyclic modules of the form $\mathrm{A}/x\mathrm{A}$, where $x \in \mathrm{A}$ is nonzero. Since A is a domain, the sequence

$$0 \to \mathrm{A} \xrightarrow{x} \mathrm{A} \to \mathrm{A}/x\mathrm{A} \to 0$$

is exact and induces a relation $[\mathrm{A}] = [\mathrm{A}] + [\mathrm{A}/x\mathrm{A}]$ in $\mathrm{G}_0(\mathrm{A})$, hence $[\mathrm{A}/x\mathrm{A}] = 0$. Thus $[\mathrm{M}] = r \cdot [\mathrm{A}]$. It follows that $\mathrm{G}_0(\mathrm{A})$ is generated by $[\mathrm{A}]$. In particular, $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$ is surjective.