Exercise sheet 2

1. Prove the Proposition from §1.3 in the lecture: every regular sequence in a ring A is Koszul-regular. (Hint: induction.)

Note that for a sequence of length one, it is obvious that it is regular iff Koszul-regular. We let $n > 1$ and assume that every regular sequence of length $n-1$, in any ring, is Koszul-regular. Let $(a_1, \ldots, a_n)$ be a regular sequence in a ring A. Write $\mathrm{K}' = \mathrm{Kosz}_{\mathrm{A}}(a_2, \ldots, a_n)$ and $\mathrm{K} = \mathrm{Kosz}_{\mathrm{A}}(a_1, \ldots, a_n)$. We have

$$\mathrm{K} = \mathrm{Kosz}_{\mathrm{A}}(a_1) \otimes_{\mathrm{A}} \mathrm{K}' \simeq \mathrm{A}/\langle a_1 \rangle \otimes_{\mathrm{A}} \mathrm{K}'$$

where the first equality is the definition and the second is a quasi-isomorphism because a_1 is a non-zerodivisor. But the formula $(\mathrm{M}_1 \otimes_{\mathrm{A}} \cdots \otimes_{\mathrm{A}} \mathrm{M}_k) \otimes_{\mathrm{A}} \mathrm{B} \simeq (\mathrm{M}_1 \otimes_{\mathrm{A}} \mathrm{B}) \otimes_{\mathrm{B}} \cdots \otimes_{\mathrm{B}} (\mathrm{M}_k \otimes_{\mathrm{A}} \mathrm{B})$, for A-modules M_i and B an A-algebra, shows that the right-hand side above is isomorphic to the Koszul complex of the image of $(a_2, \ldots, a_n)$ in $\mathrm{A}/\langle a_1 \rangle$:

$$\mathrm{K} \simeq \mathrm{Kosz}_{\mathrm{A}/\langle a_1 \rangle}(\overline{a}_2, \ldots, \overline{a}_n).$$

By the induction hypothesis, $(\overline{a}_2, \ldots, \overline{a}_n)$ is Koszul-regular. So K is acyclic in positive degrees.

2. Let k be a field, $\mathrm{A} = k[x]/\langle x^2 \rangle$. Show that k, viewed as an A-module, is not perfect.

By the Proposition in §2.1, it will suffice to show that k is of infinite Tor-amplitude as an A-module. We claim $\mathrm{Tor}_i^{\mathrm{A}}(k, k)$ are all nonzero for all $i \geqslant 0$. Let's build a free resolution of k. We start with the A-linear surjection $\mathrm{A} \twoheadrightarrow k$, whose kernel is the ideal $\langle x \rangle$. This is the image of the map $\mathrm{A} \to \mathrm{A}$ which is multiplication by x. So we have the resolution $\ldots \to \mathrm{A} \xrightarrow{x} \mathrm{A}$ so far. The kernel of the multiplication map $x : \mathrm{A} \to \mathrm{A}$ is $\mathrm{Ann}_{\mathrm{A}}(x)$. But this is again $\langle x \rangle$ so we end up with the infinite resolution

$$\cdots \to \mathrm{A} \xrightarrow{x} \mathrm{A} \xrightarrow{x} \mathrm{A}$$

Tensoring with k over A produces the infinite complex

$$\cdots \to k \xrightarrow{0} k \xrightarrow{0} k$$

which has nonzero homology in every degree.

3. Let $\phi : \mathrm{A} \to \mathrm{B}$ be a flat ring homomorphism (i.e., ϕ exhibits B as a flat A-module).

(i) Show that if a f.g. A-module M is of Tor-amplitude $\leqslant n$, then so is the B-module $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}$.

(ii) Suppose that ϕ is *faithfully* flat, i.e., that a sequence of A-modules $\mathrm{M}' \to \mathrm{M} \to \mathrm{M}''$ is exact iff $\mathrm{M}' \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{M} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{M}'' \otimes_{\mathrm{A}} \mathrm{B}$ is exact. Show that a f.g. A-module M is of Tor-amplitude $\leqslant n$ if and only if $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}$ is of Tor-amplitude $\leqslant n$.

(i) Since B is flat, $M \otimes_A B \simeq M \otimes^{\mathbf{L}}_A B$. Thus for every B-module N,

$$\mathrm{Tor}_i^B(M \otimes_A B, N) = H_i(M \otimes^{\mathbf{L}}_A B \otimes^{\mathbf{L}}_B N) \simeq H_i(M \otimes^{\mathbf{L}}_A N_{[A]}) = \mathrm{Tor}_i^A(M, N_{[A]}),$$

whence the claim. Alternatively, choose a finite fgproj resolution of M. The proof of the Proposition in §2.1 of the lecture actually shows that M admits such a resolution of length n. Applying the functor $? \otimes_A B$, which is exact since B is flat, will result in a complex which is still acyclic in positive degrees, hence a resolution of $M \otimes_A B$. Using this resolution to compute $\mathrm{Tor}_i^B(M \otimes_A B, N)$ shows that these groups will vanish if $i > n$.

► Warning: if ϕ is not flat, (i) is false. As the second argument shows, what is potentially problematic is that the extension of scalars of a resolution may not be a resolution again. For example, take $f \in A$ a non-zerodivisor, so that $M = A/f$ has a resolution by the Koszul complex $\mathrm{Kosz}_A(f) = [A \to A]$. Take $\phi : A \to B$ to be a map which sends f to a zerodivisor. Then $\mathrm{Kosz}_A(f) \otimes_A B = \mathrm{Kosz}_B(\phi(f))$ is not acyclic in degree 1. For an actual example, take e.g. $A = \mathbf{Z}$, $f = 2$, $B = \mathbf{Z}/4\mathbf{Z}$. Then $M = A/f = \mathbf{Z}/2\mathbf{Z}$ is of Tor-amplitude $\leqslant 1$ as a $\mathbf{Z}$-module, but $M \otimes_A B = \mathbf{Z}/2\mathbf{Z}$ is of infinite Tor-amplitude as a $\mathbf{Z}/4\mathbf{Z}$-module (which can be proven just like in Exercise 2).

(ii) Let M be a f.g. A-module such that $M \otimes_A B$ is of Tor-amplitude $\leqslant n$. The claim is that for every A-module N, $\mathrm{Tor}_i^A(M, N) = 0$ for $i > n$. Recall that since ϕ is faithfully flat, this can be checked after extending scalars. Since ϕ is flat, we have:

$$\begin{aligned}\mathrm{Tor}_i^A(M, N) \otimes_A B &= H_i(M \otimes^{\mathbf{L}}_A N) \otimes_A B \\ &\simeq H_i((M \otimes^{\mathbf{L}}_A N) \otimes_A B) \\ &\simeq H_i((M \otimes_A B) \otimes^{\mathbf{L}}_A (N \otimes_A B)) \\ &= \mathrm{Tor}_i^B(M \otimes_A B, N \otimes_A B) = 0\end{aligned}$$

implicitly using the fact that $? \otimes^{\mathbf{L}}_A B = ? \otimes_A B$ since ϕ is flat.

4. Let A be a noetherian ring and M a finitely generated A-module. Show that M is of finite length iff $M_{\mathfrak{p}} = 0$ for all non-maximal prime ideals $\mathfrak{p}$. (Use the Proposition in §1.3 of the lecture.)

The length *of an* A*-module* M *is the maximal length of a composition series (a filtration where the successive quotients are all simple, i.e., are nonzero and have no non-trivial, non-proper submodules). For example,* A *has length 1 iff* A *is a field. For a field, length coincides with dimension of vector spaces.*

By the Proposition in §1.3, M admits a finite increasing filtration $(M_i)_i$ where the quotients M_{i+1}/M_i are of the form $A/\mathfrak{p}_i$, $\mathfrak{p}_i$ being prime ideals. Localizing at a prime $\mathfrak{p}_i$, we get

$$(M_{i+1})_{\mathfrak{p}_i}/(M_i)_{\mathfrak{p}_i} \simeq (M_{i+1}/M_i)_{\mathfrak{p}_i} \simeq (A_{\mathfrak{p}_i})/\mathfrak{p}_i A_{\mathfrak{p}_i} = \kappa(\mathfrak{p}_i)$$

for all i. In particular these are nonzero, so $(M_{i+1})_{\mathfrak{p}_i}$ are nonzero.

Suppose $M_{\mathfrak{p}} = 0$ for all non-maximal primes $\mathfrak{p}$. Then all the primes $\mathfrak{p}_i$ must be maximal by above. Thus the successive quotients $M_{i+1}/M_i \simeq A/\mathfrak{p}_i$ are simple, so the filtration $(M_i)_i$ is a composition series for M. In particular, M is of finite length.

Conversely if M is of finite length, choose a composition series $(M_i)_i$. Let $\mathfrak{p}$ be a non-maximal prime. We have $M_1 = A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, hence $(M_1)_{\mathfrak{p}} \simeq A_{\mathfrak{p}}/\mathfrak{m}A_{\mathfrak{p}} = 0$ since $\mathfrak{m}$ is not contained in $\mathfrak{p}$. Similarly $M_2/M_1 \simeq A/\mathfrak{m}$ for some (possibly different) maximal ideal $\mathfrak{m}$, hence $(M_2)_{\mathfrak{p}} = (M_2)_{\mathfrak{p}}/(M_1)_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{m}A_{\mathfrak{p}} = 0$ by the same argument. Repeating this we eventually get $M_{\mathfrak{p}} = 0$.