Exercise sheet 12

1. Let $\mathcal{O}_K$ be the ring of integers in a number field K. Show that there is a canonical isomorphism between the group of Weil divisors modulo linear equivalence, and the ideal class group of $\mathcal{O}_K$.

2. Let A be a noetherian ring and $\mathfrak{p}$ a minimal prime ideal. Show that $[V(\mathfrak{p})]$ is nonzero in $CH_*(A)$.

The subgroup $R_d(A)$ consists of linear combinations of principal divisors $\mathrm{div}_{V(\mathfrak{q}_i)}(f_i)$, where $V(\mathfrak{q}_i)$ are integral of dimension $d+1$ and $f_i \notin \mathfrak{q}$:

$$\sum_i n_i \cdot \mathrm{div}_{V(\mathfrak{q}_i)}(f_i) = \sum_i n_i \cdot [(A/\mathfrak{q}_i)/f_i(A/\mathfrak{q}_i)]_d.$$

with coefficients $n_i \in \mathbf{Z}$. Expanding out the definition of $[-]_d$, the classes appearing on the right-hand side are all irreducible components of $V(\mathfrak{q}_i) \cap V(f_i)$. Thus if $V(\mathfrak{p})$ is an integral subset of dimension d such that $[V(\mathfrak{p})] \in R_d(A)$, then it must be an irreducible component of $V(\mathfrak{q}_i) \cap V(f_i)$. In particular, such a $V(\mathfrak{p})$ is contained inside $V(\mathfrak{q}_i)$, which is integral. But then $V(\mathfrak{p})$ cannot be an irreducible component of $|\mathrm{Spec}(A)|$, i.e., $\mathfrak{p}$ cannot be a minimal prime ideal of A.

3. Let A be an integral domain. Show that the set $\mathrm{Cart}^+(A)$ of effective Cartier divisors admits a canonical monoid structure, and there is a canonical injective homomorphism

$$\mathrm{Cart}^+(A) \to \mathrm{Cart}(A)$$

which exhibits $\mathrm{Cart}(A)$ as the group completion of $\mathrm{Cart}^+(A)$.

Since A is an integral domain, being a non-zero-divisor is equivalent to being non-zero.

Define the monoid structure by tensor product of invertible modules. Then the canonical map

$$(I, I \hookrightarrow A) \mapsto (I, I \hookrightarrow A \hookrightarrow \mathrm{Frac}(A))$$

is a monoid homomorphism $\mathrm{Cart}^+(A) \to \mathrm{Cart}(A)$ by construction. Injectivity is clear because if $I \subset A$ is an invertible ideal such that $(I, I \hookrightarrow A \hookrightarrow \mathrm{Frac}(A))$ is the neutral element of $\mathrm{Cart}(A)$, i.e., $I = A$ as sub-A-modules of $\mathrm{Frac}(A)$, then also $I = A$ as ideals of A.

To show that the map is a group completion, let $f : \mathrm{Cart}(A)^+ \to X$ be a monoid homomorphism with X a group. It will suffice to show that there is a unique extension $f : \mathrm{Cart}(A) \to X$. Let $(M, M \hookrightarrow \mathrm{Frac}(A)) \in \mathrm{Cart}(A)$. We can write it as a difference of effective Cartier divisors $(I, I \hookrightarrow A)$ and $(J, J \hookrightarrow A)$ (see §12.2). Then set

$$f(M) := f(I) - f(J)$$

(where we commit an abuse of notation to simplify the notation, and we denote the group operation of X additively). Suppose we can write it as another difference $\mathrm{M} = \mathrm{I}' - \mathrm{J}'$. Then we have to check

$$f(\mathrm{I}) - f(\mathrm{J}) = f(\mathrm{I}') - f(\mathrm{J}'),$$

or equivalently $f(\mathrm{I} + \mathrm{J}') = f(\mathrm{I}' + \mathrm{J})$. But $\mathrm{I} + \mathrm{J}' = \mathrm{I}' + \mathrm{J}$ in $\mathrm{Cart}^+(\mathrm{A})$ and f is a monoid homomorphism.

4. Let A be a noetherian ring of dimension d. Recall the homomorphism $\gamma : \mathrm{Z}_*(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$ defined in Sheet 9, Exercise 3. Note that γ sends $\mathrm{Z}_k(\mathrm{A})$ to $\mathrm{G}_0(\mathrm{A})_{\leqslant k}$, the subgroup generated by classes [M] such that $\dim(\mathrm{Supp}_\mathrm{A}(\mathrm{M})) \leqslant k$.

Let $\mathrm{M} \in \mathrm{Mod}_\mathrm{A}^{\mathrm{fg}}$ and suppose that $\mathrm{Supp}_\mathrm{A}(\mathrm{M})$ is of pure dimension n. Prove the formula

$$\gamma([\mathrm{M}]_n) = [\mathrm{M}]$$

in $\mathrm{G}_0(\mathrm{A})_{\leqslant n} / \mathrm{G}_0(\mathrm{A})_{\leqslant n-1}$.

Choose a filtration of M by submodules M_i such that the successive quotients $\mathrm{M}_i/\mathrm{M}_{i-1}$ are of the form $\mathrm{A}/\mathfrak{p}_i$. Then $\mathrm{Supp}_\mathrm{A}(\mathrm{M}) = \bigcup_i \mathrm{V}(\mathfrak{p}_i)$ and

$$[\mathrm{M}] = \sum_i [\mathrm{A}/\mathfrak{p}_i]$$

in $\mathrm{G}_0(\mathrm{A})$. Let $\{\mathfrak{q}_j\}_j$ be the subset of *minimal* elements of $\{\mathfrak{p}_i\}_i$, i.e., those which correspond to the irreducible components $\mathrm{V}(\mathfrak{q}_j)$ of $\mathrm{Supp}_\mathrm{A}(\mathrm{M})$. Let $\eta_j = [\mathrm{A} \to \kappa(\mathfrak{q}_j)]$ denote the generic point of $\mathrm{V}(\mathfrak{q}_j)$. Recall that the number of times $\mathrm{A}/\mathfrak{q}_j$ appears as a quotient $\mathrm{M}_i/\mathrm{M}_{i-1}$ is exactly the multiplicity $\mathrm{mult}_{\mathrm{A},\eta_j}(\mathrm{M})$ (see §9.2). Modulo $\mathrm{G}_0(\mathrm{A})_{\leqslant n-1}$ the class of $[\mathrm{A}/\mathfrak{p}_i]$ will die for every $\mathrm{V}(\mathfrak{p}_i)$ which is not an irreducible component (as it then has dimension $< n$), so we have

$$[\mathrm{M}] = \sum_i [\mathrm{A}/\mathfrak{p}_i] = \sum_j \mathrm{mult}_{\mathrm{A},\eta_j}(\mathrm{M}) \cdot [\mathrm{A}/\mathfrak{q}_j]$$

in $\mathrm{G}_0(\mathrm{A})_{\leqslant n} / \mathrm{G}_0(\mathrm{A})_{\leqslant n-1}$. This is precisely the image of $[\mathrm{M}]_n$ by γ, as by definition,

$$[\mathrm{M}]_n = \sum_j \mathrm{mult}_{\mathrm{A},\eta_j}(\mathrm{M}) \cdot [\mathrm{V}(\mathfrak{q}_j)]$$

in $\mathrm{Z}_n(\mathrm{A})$.