Exercise sheet 10

1. Show that for any noetherian ring A and every $n \geqslant 0$, inverse image along $\phi : \mathrm{A} \to \mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ induces an isomorphism

$$\phi^* : \mathrm{G}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n]).$$

(We showed this when A is a field and $n = 1$. For the case of an integral domain, use a noetherian induction argument and the localization sequence to reduce to the fraction field.)

The homomorphism ϕ^* exists since ϕ is flat. By induction we may assume $n = 1$.

Let $\rho : \mathrm{A} \twoheadrightarrow \mathrm{A}_{\mathrm{red}}$ be the reduction homomorphism. Since $\phi : \mathrm{A} \to \mathrm{A}[\mathrm{T}]$ is flat, the base change formula (§6.3) implies that the square

$$\begin{array}{ccc} \mathrm{G}_0(\mathrm{A}_{\mathrm{red}}) & \xrightarrow{\phi^*} & \mathrm{G}_0(\mathrm{A}_{\mathrm{red}}[\mathrm{T}]) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\ \mathrm{G}_0(\mathrm{A}) & \xrightarrow{\phi^*} & \mathrm{G}_0(\mathrm{A}[\mathrm{T}]) \end{array}$$

commutes. We have abused notation by writing ϕ also for $\mathrm{A}_{\mathrm{red}} \to \mathrm{A}_{\mathrm{red}}[\mathrm{T}]$, and ρ also for $\mathrm{A}[\mathrm{T}] \twoheadrightarrow (\mathrm{A}[\mathrm{T}])_{\mathrm{red}} \simeq \mathrm{A}_{\mathrm{red}}[\mathrm{T}]$. By nil-invariance (§6.4), the vertical arrows are invertible. Therefore, we may replace A by $\mathrm{A}_{\mathrm{red}}$ (which is still noetherian) and thereby assume that A is reduced.

Since A is noetherian, it has finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Let B denote the total ring of fractions of A, i.e., the localization at the set of zero divisors. Recall:

Lemma 1. *Let* A *be a reduced ring with finitely many minimal prime ideals* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. *If* B *denotes the total ring of fractions of* A, *then we have a canonical isomorphism*

$$\mathrm{B} \simeq \prod_i \mathrm{B}/\mathfrak{p}_i\mathrm{B}.$$

Moreover, each $\mathrm{B}/\mathfrak{p}_i\mathrm{B}$ *is the field of fractions of the integral domain* $\mathrm{A}/\mathfrak{p}_i$.

Therefore, we have

$$\mathrm{G}_0(\mathrm{B}) \simeq \bigoplus_i \mathrm{G}_0(\mathrm{K}_i)$$

where $\mathrm{K}_i = \mathrm{Frac}(\mathrm{A}/\mathfrak{p}_i)$, since it is easy to check that G_0 sends finite products of rings to direct sums.

We now want to apply the localization sequence (§7.3), except that we are not in the case of a localization at a single element. Nevertheless, it's easy to check that our proof generalizes to show:

Theorem 2. *Let* A *be a ring and* S *a multiplicative subset. Then there is a canonical exact sequence*

$$\varinjlim_{s\in\mathrm{S}} \mathrm{G}_0(\mathrm{A}/\langle s\rangle) \to \mathrm{G}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A}[\mathrm{S}^{-1}]) \to 0$$

(Implicit here is a filtered colimit argument,

$$\mathrm{K}_0(\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(\mathrm{S})) \simeq \varinjlim_{s\in\mathrm{S}} \mathrm{G}_0(\mathrm{A}/\langle s\rangle),$$

where $\mathrm{Mod}^{\mathrm{fg}}_{\mathrm{A}}(\mathrm{S})$ denotes the category of f.g. A-modules that are S-torsion, i.e., s-torsion for some element $s \in \mathrm{S}$.)

In our case, we get the localization sequence

$$\varinjlim_{s} \mathrm{G}_0(\mathrm{A}/\langle s\rangle) \to \mathrm{G}_0(\mathrm{A}) \to \bigoplus_i \mathrm{G}_0(\mathrm{K}_i) \to 0.$$

where the colimit is taken over zero-divisors s. We also have

$$\mathrm{G}_0(\mathrm{B}[\mathrm{T}]) \simeq \bigoplus_i \mathrm{G}_0(\mathrm{K}_i[\mathrm{T}])$$

as $\mathrm{B}[\mathrm{T}] \simeq \prod_i \mathrm{K}_i[\mathrm{T}]$, so this fits into the commutative diagram with exact rows:

$$\begin{tikzcd}
\varinjlim_s \mathrm{G}_0(\mathrm{A}/\langle s\rangle) \arrow[r] \arrow[d, "\phi^*"] & \mathrm{G}_0(\mathrm{A}) \arrow[r] \arrow[d, "\phi^*"] & \bigoplus_i \mathrm{G}_0(\mathrm{K}_i) \arrow[r] \arrow[d, "\phi^*"] & 0 \\
\varinjlim_s \mathrm{G}_0(\mathrm{A}/\langle s\rangle[\mathrm{T}]) \arrow[r] & \mathrm{G}_0(\mathrm{A}[\mathrm{T}]) \arrow[r] & \bigoplus_i \mathrm{G}_0(\mathrm{K}_i[\mathrm{T}]) \arrow[r] & 0
\end{tikzcd}$$

where we again use ϕ to denote each of the ring homomorphisms $? \to ?[\mathrm{T}]$.

Since the K_i are fields, we already know that the right-hand vertical arrow is invertible (Sheet 7, Exercise 3). We also know that our middle vertical arrow is injective, by the existence of a morphism $\psi : \mathrm{A}[\mathrm{T}] \to \mathrm{A}$, which is of Tor-amplitude $\leqslant 1$ (use the Koszul complex $\mathrm{Kosz}_{\mathrm{A}[\mathrm{T}]}(\mathrm{T})$ to resolve A as an A-module), and is a retraction of ϕ (so that $\psi^*\phi^* = \mathrm{id}$). Thus it remains to show surjectivity. For that we could apply the five lemma if we only knew that the left-hand vertical arrow was invertible.

We can finish the proof by a noetherian induction argument. Call an ideal $\mathrm{I} \subseteq \mathrm{A}$ *good* if the inverse image map $\mathrm{G}_0(\mathrm{A}/\mathrm{I}) \to \mathrm{G}_0(\mathrm{A}/\mathrm{I}[\mathrm{T}])$ is invertible. The conclusion of the discussion above can be summarized as: if every nonzero ideal of A is good, then the zero ideal is also good. This holds for any reduced noetherian ring A.

Now let's show that every nonzero ideal of A is good. Suppose that isn't the case. Then since A is noetherian, we can choose a maximal one out of the bad ideals, say I, and then every nonzero ideal of A/I is good. Then the above argument yields that the zero ideal of A/I is good, i.e., $\mathrm{G}_0(\mathrm{A}/\mathrm{I}) \to \mathrm{G}_0(\mathrm{A}/\mathrm{I}[\mathrm{T}])$ is invertible. But that contradicts the assumption that I was bad. Thus, every nonzero ideal of A is good, so by above, the zero ideal is good, as desired.

2. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{X},\mathrm{Y},\mathrm{Z}]/\langle \mathrm{XZ}, \mathrm{Z}(\mathrm{Z}^2-\mathrm{Y}^3)\rangle$. Show that $|\mathrm{Spec}(\mathrm{A})|$ has two irreducible components, $\mathrm{Y}_1 = \mathrm{V}(\langle \mathrm{Z}\rangle)$ and $\mathrm{Y}_2 = \mathrm{V}(\langle \mathrm{X}, \mathrm{Z}^2-\mathrm{Y}^3\rangle)$, and is not of pure dimension.

Let $\mathrm{I}_1 = \langle \mathrm{Z}\rangle$ and $\mathrm{I}_2 = \langle \mathrm{X}, \mathrm{Z}^2-\mathrm{Y}^3\rangle$ as ideals of $\mathrm{B} = k[\mathrm{X},\mathrm{Y},\mathrm{Z}]$. Then $\mathrm{I}_1\mathrm{I}_2 = \langle \mathrm{ZX}, \mathrm{Z}(\mathrm{Z}^2-\mathrm{Y}^3)\rangle$, so $\mathrm{V}_\mathrm{B}(\mathrm{I}_1)\cup \mathrm{V}_\mathrm{B}(\mathrm{I}_2) = \mathrm{V}_\mathrm{B}(\mathrm{I}_1\mathrm{I}_2)$ as subsets of $|\mathrm{Spec}(\mathrm{B})|$. Under the identification $\mathrm{V}_\mathrm{B}(\mathrm{I}_1\mathrm{I}_2) \simeq |\mathrm{Spec}(\mathrm{A})|$, we get $\mathrm{Y}_1\cup\mathrm{Y}_2 = \mathrm{V}_\mathrm{A}(\mathrm{I}_1\mathrm{A})\cup\mathrm{V}_\mathrm{A}(\mathrm{I}_2\mathrm{A}) = |\mathrm{Spec}(\mathrm{A})|$. Moreover, Y_1 and Y_2 are integral since

$$\mathrm{A}/\mathrm{I}_1\mathrm{A} = k[\mathrm{X},\mathrm{Y},\mathrm{Z}]/\langle \mathrm{Z}, \mathrm{XZ}, \mathrm{Z}(\mathrm{Z}^2-\mathrm{Y}^3)\rangle \simeq k[\mathrm{X},\mathrm{Y}]$$
$$\mathrm{A}/\mathrm{I}_2\mathrm{A} = k[\mathrm{X},\mathrm{Y},\mathrm{Z}]/\langle \mathrm{X}, \mathrm{Z}^2-\mathrm{Y}^3, \mathrm{XZ}, \mathrm{Z}(\mathrm{Z}^2-\mathrm{Y}^3)\rangle \simeq k[\mathrm{Y},\mathrm{Z}]/\langle \mathrm{Z}^2-\mathrm{Y}^3\rangle$$

are integral domains. (For the second, it suffices to show that the polynomial $\mathrm{Z}^2-\mathrm{Y}^3$ is irreducible, since $k[\mathrm{Y},\mathrm{Z}]$ is factorial. Regarding it as a polynomial in $k[\mathrm{Z}][\mathrm{Y}]$, it suffices by Gauss's lemma to show that it is irreducible in $k(\mathrm{Z})[\mathrm{Y}]$. But it is a degree 3 polynomial with no root in $k(\mathrm{Z})$, so it is irreducible.) In particular, Y_1 and Y_2 are the irreducible components of $|\mathrm{Spec}(\mathrm{A})|$. Moreover, the computations above show that

$$\dim(\mathrm{Y}_1) = \dim(\mathrm{A}/\mathrm{I}_1\mathrm{A}) = \dim(k[\mathrm{X},\mathrm{Y}]) = 2,$$
$$\dim(\mathrm{Y}_2) = \dim(\mathrm{A}/\mathrm{I}_2\mathrm{A}) = \dim(k[\mathrm{Y},\mathrm{Z}]/\langle \mathrm{Z}^2-\mathrm{Y}^3\rangle) = 1$$

where the last equality on the second line follows from Krull's principal ideal theorem and the fact that $\mathrm{Z}^2-\mathrm{Y}^3$ is a zero divisor in the integral domain $k[\mathrm{Y},\mathrm{Z}]$. (Since $\dim(\mathrm{A}) = \sup_\mathfrak{m} \dim(\mathrm{A}_\mathfrak{m})$ by Sheet 9, Exercise 3, we can localize at a maximal ideal and then apply Krull's theorem.)

3. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{X},\mathrm{Y}]$. Let $f, g \in \mathrm{A}$ be polynomials. In each of the following examples, determine whether $\mathrm{V}(f)$ and $\mathrm{V}(g)$ intersect properly, and compute the cycle $[\mathrm{A}/\langle f,g\rangle]_d \in \mathrm{Z}_d(\mathrm{A})$, where $d = \dim(\mathrm{V}(f)) + \dim(\mathrm{V}(g)) - \dim(\mathrm{A})$.

(a) $f = \mathrm{X}$, $g = \mathrm{Y}$

(b) $f = \mathrm{X}$, $g = \mathrm{X}$

(c) $f = \mathrm{Y}-\mathrm{X}^2$, $g = \mathrm{Y}$

(d) $f = \mathrm{XY}$, $g = \mathrm{Y}^2$

(Note: we only defined properness of intersection between *irreducible* subsets. However the same definition makes sense as long as both subsets are of pure dimension.)

(a) We have $\mathrm{V}(\mathrm{X})\cap\mathrm{V}(\mathrm{Y}) = \mathrm{V}(\langle \mathrm{X},\mathrm{Y}\rangle)$ which is a closed point x, since $\langle \mathrm{X},\mathrm{Y}\rangle$ is maximal. In particular it is of dimension 0 which is the same as $d = 1+1-2$. So the intersection is proper.

We compute $[\mathrm{A}/\langle f,g\rangle]_0 = [k[\mathrm{X},\mathrm{Y}]/\langle \mathrm{X},\mathrm{Y}\rangle]_0 = [k]_0$. The support of k is $\mathrm{V}(\langle \mathrm{X},\mathrm{Y}\rangle) = \{x\}$ which has a single irreducible component corresponding to the generic point x. The multiplicity at x is 1 because k is a simple $\mathrm{A}_{\mathfrak{p}(x)}$-module. So $[\mathrm{A}/\langle f,g\rangle]_0 = [\mathrm{V}(\langle \mathrm{X},\mathrm{Y}\rangle)]$.

(b) We have $\mathrm{V(X)} \cap \mathrm{V(X)} = \mathrm{V(X)}$ which is of dimension 1, while $d = 0$. So the intersection is not proper.

We have $[\mathrm{A}/\langle f,g\rangle]_0 = [k[\mathrm{X},\mathrm{Y}]/\langle \mathrm{X}\rangle]_0 = 0$ because $\mathrm{V(X)}$ has no irreducible component of dimension 0.

(c) We have $\mathrm{V}(\mathrm{Y}-\mathrm{X}^2) \cap \mathrm{V(Y)} = \mathrm{V}(\langle \mathrm{Y}, \mathrm{Y}-\mathrm{X}^2\rangle) = \mathrm{V}(\langle \mathrm{Y},\mathrm{X}^2\rangle)$. Since the radical of $\langle \mathrm{X}^2,\mathrm{Y}\rangle$ is $\langle \mathrm{X},\mathrm{Y}\rangle$, this is again the closed point $\{x\}$ of (a). The intersection is proper as $d = 1+1-2$ again.

We have $[\mathrm{A}/\langle f,g\rangle]_0 = [k[\mathrm{X},\mathrm{Y}]/\langle \mathrm{X}^2,\mathrm{Y}\rangle]_0 = [k[\mathrm{X}]/\langle \mathrm{X}^2\rangle]_0$. Since $\mathrm{V}_{k[\mathrm{X}]}(\mathrm{X}^2) = \mathrm{V}_{k[\mathrm{X}]}(\mathrm{X})$ is a closed point, it has a single irreducible component of dimension 0. The multiplicity at that point is given by the length of $k[\mathrm{X}]_{\langle \mathrm{X}\rangle}/\langle \mathrm{X}^2\rangle$ as a module over $k[\mathrm{X}]_{\langle \mathrm{X}\rangle}$, which is 2. So we get $[\mathrm{A}/\langle f,g\rangle]_0 = 2\cdot[\mathrm{V}(\langle \mathrm{X},\mathrm{Y}\rangle)]$. Note that this is double the cycle in (a), which reflects that the parabola $\mathrm{Y} = \mathrm{X}^2$ intersects the line $\mathrm{Y} = 0$ in a double point at $(0,0)$.

(d) We have $\mathrm{V(XY)} \cap \mathrm{V}(\mathrm{Y}^2) = \mathrm{V(XY)} \cap \mathrm{V(Y)} = \mathrm{V(Y)}$ which is of dimension 1 (note $\mathrm{V(Y)} \subset \mathrm{V(XY)}$). But $d = 1+1-2$ (note $\mathrm{V(XY)} = \mathrm{V(X)} \cup \mathrm{V(Y)}$ has two irreducible components both of dimension 1). So the intersection is not proper.

We have $[\mathrm{A}/\langle f,g\rangle]_0 = [k[\mathrm{X},\mathrm{Y}]/\langle \mathrm{XY},\mathrm{Y}^2\rangle]_0 = 0$ because $\mathrm{V(Y)}$ has no irreducible component of dimension 0.

4. Let A be a noetherian ring and $f \in \mathrm{A}$ an element. Let $\phi : \mathrm{A} \to \mathrm{A}[f^{-1}]$ and $\psi : \mathrm{A} \to \mathrm{A}/\langle f\rangle$. Show that there is an exact sequence

$$\mathrm{CH}_n(\mathrm{A}/\langle f\rangle) \xrightarrow{\psi_*} \mathrm{CH}_n(\mathrm{A}) \xrightarrow{\phi^*} \mathrm{CH}_n(\mathrm{A}[f^{-1}]) \to 0$$

for every n.

We have $\phi^*\psi_* = 0$ since for any prime ideal $\mathfrak{p}$ of $\mathrm{A}/\langle f\rangle$,

$$\phi^*\psi_*[\mathrm{V}_{\mathrm{A}/\langle f\rangle}(\mathfrak{p})] = \phi^*[\mathrm{V}_{\mathrm{A}}(\mathfrak{q})] = [\mathrm{A}[f^{-1}]/\mathfrak{q}\mathrm{A}[f^{-1}]]_n = 0$$

where $\mathfrak{q} = \psi^{-1}(\mathfrak{p})$ (since $f \in \mathfrak{q}$).

For surjectivity of ϕ^*, let $\mathrm{V}(\mathfrak{p})$ be an integral subset of $\left|\mathrm{Spec}(\mathrm{A}[f^{-1}])\right|$ of dimension n. The contraction of the prime ideal $\mathfrak{p}$ is a prime ideal $\mathfrak{q} = \phi^{-1}(\mathfrak{p}) \subset \mathrm{A}$ not containing f, and whose extension $\mathfrak{q}\mathrm{A}[f^{-1}]$ recovers $\mathfrak{p}$. Thus $\phi^*[\mathrm{V}(\mathfrak{q})] = [\mathrm{A}[f^{-1}]/\mathfrak{q}\mathrm{A}[f^{-1}]]_n = [\mathrm{A}[f^{-1}]/\mathfrak{p}]_n = [\mathrm{V}(\mathfrak{p})]$.

Finally let α be an n-cycle on A such that $\phi^*(\alpha) \in \mathrm{Z}_n(\mathrm{A}[f^{-1}])$ is rationally equivalent to zero. We need to show that there exists a cycle $\tilde{\alpha} \in \mathrm{Z}_n(\mathrm{A}/\langle f\rangle)$ such that $\psi_*(\tilde{\alpha}) - \alpha$ is rationally equivalent to zero. Write α as a linear combination

$$\alpha = \sum_i n_i[\mathrm{V}(\mathfrak{p}_i)]$$

where the $\mathfrak{p}_i$ are prime ideals of A. By assumption, we have

$$\sum_i n_i[\mathrm{V}(\mathfrak{p}_i)] = \sum_j \mathrm{div}_{\mathrm{V}(\mathfrak{q}_j)}(g_j)$$

in $\mathrm{Z}_n(\mathrm{A}[f^{-1}])$, where $\mathrm{V}(\mathfrak{q}_j)$ are integral subsets of $\left|\mathrm{Spec}(\mathrm{A}[f^{-1}])\right|$ of dimension $n+1$ and g_j are elements of $\mathrm{A}[f^{-1}]$ with $g_j \notin \mathfrak{q}_j$. Each $\mathfrak{q}_j$ is an extension of a prime of A (which we denote by $\mathfrak{q}_j$ again). We can also assume that g_j come from A by clearing denominators (multiply by a large enough power of f). Then the difference

$$\beta = \sum_i n_i[\mathrm{V}(\mathfrak{p}_i)] - \sum_j \mathrm{div}_{\mathrm{V}(\mathfrak{q}_j)}(g_j)$$

may be viewed as an element of $\mathrm{Z}_n(\mathrm{A})$ which goes to zero in $\mathrm{Z}_k(\mathrm{A}[f^{-1}])$. Since the latter is a free abelian group, this means that β can have nonzero multiplicity at an integral subset $\mathrm{V}(\mathfrak{p}) \subseteq \left|\mathrm{Spec}(\mathrm{A})\right|$ only if $\phi^*[\mathrm{V}(\mathfrak{p})] = 0 \in \mathrm{Z}_n(\mathrm{A}[f^{-1}])$, hence $\mathfrak{p}\mathrm{A}[f^{-1}] = \mathrm{A}[f^{-1}]$, hence $f \in \mathfrak{p}$. In particular, β lifts to an element $\tilde{\alpha} \in \mathrm{Z}_n(\mathrm{A}/\langle f \rangle)$ such that $\psi_*(\tilde{\alpha}) = \beta$. The claim follows.