**Exercise sheet 1**

**1.** Let A be a ring. Suppose that every A-module is flat. Show that every ideal I is idempotent, i.e., $I^2 = I$. Deduce that every prime ideal of A is maximal.

**2.** Let A be a ring and M a finitely generated A-module. Show that the following two conditions are equivalent:

(i) M is finitely presented and for every prime ideal $\mathfrak{p}$, the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free.

(ii) There exists a set of elements $\{f_i\}_i \subseteq A$ generating the unit ideal, i.e., $\langle f_i \rangle_i = A$, such that $M[f_i^{-1}]$ is a free $A[f_i^{-1}]$-module for each $i$.

If A is noetherian, then we can drop the "finitely presented" hypothesis in (i).

Hint: you can use (or prove) the following fact: If M is an A-module, then it is finitely generated iff there are elements $\{f_i\}_i \subseteq A$ generating the unit ideal such that $M[f_i^{-1}]$ are all finitely generated. More to the point, you can use this to deduce the analogue for finite presentation.

**3.** Let A be a ring and M a finitely presented A-module. Show that M is flat iff it is locally free, i.e., satisfies the conditions of Exercise 2.

*More generally, if* M *is only assumed finitely generated, then one can show:* M *is flat iff* $M_\mathfrak{p}$ *is free for all prime ideals* $\mathfrak{p}$. *See [Matsumura, CRT, Thm. 7.10].*

**4.** Let A be a ring and M a finitely generated A-module. Show that $d(\kappa) = \dim_\kappa(M \otimes_A \kappa)$ is finite for all $A \to \kappa$ with $\kappa$ a field. If A is an integral domain, show that $d(\kappa) \geqslant d(K)$ where $A \to K$ is the field of fractions.