Lecture 8
The blow-up formula

Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion. Then we may form the blow-up square

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{i_{\mathrm{E}}} & \tilde{\mathrm{X}} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X} \end{array}$$

where i_{E} is a virtual Cartier divisor, i.e., a quasi-smooth closed immersion of virtual codimension 1. In this lecture our goal is to compute the group $\mathrm{K}_0(\tilde{\mathrm{X}})$.

Theorem 1. *Let* X *be a qcqs derived scheme. Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a quasi-smooth closed immersion of virtual codimension* n*. Then there is a canonical isomorphism of abelian groups*

$$\mathrm{K}_0(\mathrm{X}) \oplus \bigoplus_{k=1}^{n-1} \mathrm{K}_0(\mathrm{Z}) \simeq \mathrm{K}_0(\tilde{\mathrm{X}}),$$

induced by the assignments $[\mathcal{F}_{\mathrm{X}}] \mapsto [p^*\mathcal{F}_{\mathrm{X}}]$ *and* $[\mathcal{F}_{\mathrm{Z}}] \mapsto [(i_{\mathrm{E}})_*(q^*(\mathcal{F}_{\mathrm{Z}}) \otimes \mathcal{O}(-k))]$ *for* $1 \leqslant k \leqslant n-1$*.*

This arises from a semi-orthogonal decomposition at the level of quasi-coherent sheaves:

Theorem 2. *The stable* ∞*-category* $\mathrm{Qcoh}(\tilde{\mathrm{X}})$ *admits a semi-orthogonal decomposition*

$$\mathrm{Qcoh}(\tilde{\mathrm{X}}) = \langle \mathbf{C}(0), \mathbf{C}(-1), \dots, \mathbf{C}(-n+1) \rangle,$$

where $\mathbf{C}(0)$ *denotes the essential image of* $p^* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\tilde{\mathrm{X}})$*, and* $\mathbf{C}(-k)$ *denotes the essential image of* $(i_{\mathrm{E}})_*(q^*(-) \otimes \mathcal{O}(-k)) : \mathrm{Qcoh}(\mathrm{Z}) \to \mathrm{Qcoh}(\tilde{\mathrm{X}})$*, for* $1 \leqslant k \leqslant n-1$*. Moreover, this restricts to a semi-orthogonal decomposition of* $\mathrm{Perf}(\tilde{\mathrm{X}})$*.*

Recall that a closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ is called quasi-smooth if it is locally the derived zero-locus of some functions $f_1, \dots, f_n \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$. In other words, there exists Zariski-locally on X a morphism $f : \mathrm{X} \to \mathbf{A}^n$ and a cartesian square of derived schemes

(0.1)
$$\begin{array}{ccc} \mathrm{Z} & \xrightarrow{i} & \mathrm{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \{0\} & \longrightarrow & \mathbf{A}^n. \end{array}$$

Equivalently, i is quasi-smooth if and only if the quasi-coherent sheaf $\mathcal{N}_{\mathrm{Z}/\mathrm{X}} := \mathcal{L}_{\mathrm{Z}/\mathrm{X}}[-1]$ is locally free of finite rank. The integer $n = \mathrm{rk}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$ is called the *virtual codimension* of i.

More generally, a morphism $\mathrm{Y} \to \mathrm{X}$ is called *quasi-smooth* if it factors, Zariski-locally on Y, through a quasi-smooth closed immersion followed by a smooth morphism.

Lemma 3. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a quasi-smooth closed immersion of virtual codimension* $n \geqslant 1$*. Let* $p : \tilde{\mathrm{X}} \to \mathrm{X}$ *be the blow-up. Then* p *factors, Zariski-locally on* X*, as a composite*

$$i : \tilde{\mathrm{X}} \xrightarrow{i'} \mathbf{P}^{n-1}_{\mathrm{X}} \xrightarrow{\pi} \mathrm{X},$$

where i' *is a quasi-smooth closed immersion, and* π *is the canonical projection. In particular,* p *is quasi-smooth and proper.*

Proof. The claim being local, it suffices to assume that X is affine and there exists a cartesian square (0.1). Then we also have a cartesian square

$$\begin{array}{ccc} \tilde{\mathrm{X}} & \longrightarrow & \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_0} \\ \mathrm{X} & \xrightarrow{f} & \mathbf{A}^n \end{array}$$

since blow-ups are stable under derived base change. It is well-known that p_0 is projective, and it is quasi-smooth since $\mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ and $\mathbf{A}^n$ are both smooth over $\mathrm{Spec}(\mathbf{Z})$. It follows that p is projective and quasi-smooth. For the more precise statement, see [2, Prop. 1.8]. □

Recall that a *virtual (effective) Cartier divisor* is a quasi-smooth closed immersion of virtual codimension 1. We can think of a virtual divisor as a "generalized (effective) Cartier divisor", cut out as the zero-locus of a section of a line bundle, where no regularity condition is imposed on the section. More precisely:

Proposition 4. *Let* X *be a derived scheme. There is a bijection between the set of isomorphism classes of virtual Cartier divisors on* X*, and the set of equivalence classes of pairs* $(\mathcal{L}, s)$*, where* $\mathcal{L}$ *is a locally free* $\mathcal{O}_\mathrm{X}$*-module of rank* 1*, and* $s : \mathcal{O}_\mathrm{X} \to \mathcal{L}^\vee$ *is a section of* $\mathcal{L}^\vee$*.*

Proof. Given a pair $(\mathcal{L}, s)$, let D be the derived zero-locus of s, fitting into the cartesian square

$$\begin{array}{ccc} \mathrm{D} & \xrightarrow{i} & \mathrm{X} \\ \downarrow & & \downarrow{\scriptstyle s} \\ \mathrm{X} & \xrightarrow{z} & \mathbf{V}_\mathrm{X}(\mathcal{L}), \end{array}$$

where $\mathbf{V}_\mathrm{X}(\mathcal{L}) = \mathrm{Spec}_\mathrm{X}(\mathrm{Sym}_{\mathcal{O}_\mathrm{X}}(\mathcal{L}))$ is the vector bundle associated to $\mathcal{L}$, and z is the zero section. Then i is a virtual divisor with conormal sheaf $\mathcal{N}_{\mathrm{D}/\mathrm{X}} \simeq \mathcal{L}|\mathrm{D}$. In the other direction, let $i : \mathrm{D} \hookrightarrow \mathrm{X}$ be a virtual Cartier divisor, and write $\mathcal{L}$ for the fibre of the canonical map $\mathcal{O}_\mathrm{X} \to i_*(\mathcal{O}_\mathrm{D})$, so that there is an exact triangle $\mathcal{L} \to \mathcal{O}_\mathrm{X} \to i_*(\mathcal{O}_\mathrm{D})$. Locally, we have $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ and $\mathrm{D} = \mathrm{Spec}(\mathrm{R}/\!\!/(f))$ for some $f \in \pi_0(\mathrm{R})$, and this triangle takes the form $\mathrm{R} \xrightarrow{f} \mathrm{R} \to \mathrm{R}/\!\!/(f)$. In particular, $\mathcal{L}$ is locally free of rank one, and we assign to $i : \mathrm{D} \hookrightarrow \mathrm{X}$ the pair $(\mathcal{L}, s)$, where s is the dual of $\mathcal{L} \to \mathcal{O}_\mathrm{X}$. One verifies that these two assignments are mutually inverse. □

Definition 5. Given a virtual Cartier divisor $i : \mathrm{D} \hookrightarrow \mathrm{X}$, we denote the fibre of $\mathcal{O}_\mathrm{X} \to i_*(\mathcal{O}_\mathrm{D})$ by $\mathcal{O}_\mathrm{X}(-\mathrm{D})$, and call this the *line bundle associated to* D. By construction there is a tautological exact sequence

$$\mathcal{O}_\mathrm{X}(-\mathrm{D}) \to \mathcal{O}_\mathrm{X} \to i_*(\mathcal{O}_\mathrm{D}). \tag{0.2}$$

Moreover, there is a canonical isomorphism $i^*(\mathcal{O}_\mathrm{X}(-\mathrm{D})) \simeq \mathcal{N}_{\mathrm{D}/\mathrm{X}}$.

We will need the following computation:

Lemma 6. *Let* $i : \mathrm{D} \hookrightarrow \mathrm{X}$ *be a virtual divisor. Then there is a canonical isomorphism*

$$i^* i_*(\mathcal{O}_\mathrm{D}) \simeq \mathcal{O}_\mathrm{D} \oplus \mathcal{N}_{\mathrm{D}/\mathrm{X}}[1].$$

Proof. Applying i^* to the exact triangle (0.2), we get the triangle

$$\mathcal{N}_{\mathrm{D}/\mathrm{X}} \to \mathcal{O}_\mathrm{D} \to i^* i_*(\mathcal{O}_\mathrm{D}).$$

The map $\mathcal{O}_\mathrm{D} \to i^* i_*(\mathcal{O}_\mathrm{D})$ is induced by the natural transformation $i^*(\eta) : i^* \to i^* i_* i^*$ (where η is the unit), so by the triangle identities it has a retraction given by the co-unit map $i^* i_*(\mathcal{O}_\mathrm{D}) \to \mathcal{O}_\mathrm{D}$. In other words, the triangle splits. □

Finally, we need Grothendieck duality for virtual divisors. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of qcqs derived schemes. The functor i_* admits a right adjoint $i^!$, which for formal reasons can be computed by the formula

$$i^!(-) \simeq i^*(-) \otimes \omega_{\mathrm{D/X}},$$

where $\omega_{\mathrm{D/X}} := i^!(\mathcal{O}_{\mathrm{X}})$ is called the *relative dualizing sheaf*. When $i : \mathrm{D} \hookrightarrow \mathrm{X}$ is a virtual divisor, $\omega_{\mathrm{D/X}}$ can be computed as follows:

Proposition 7 (Grothendieck duality)**.** *Let* X *be a qcqs derived scheme. Then for any virtual Cartier divisor* $i : \mathrm{D} \hookrightarrow \mathrm{X}$*, there is a canonical isomorphism*

$$\mathcal{N}^{\vee}_{\mathrm{D/X}}[-1] \xrightarrow{\sim} \omega_{\mathrm{D/X}}$$

in Qcoh(D)*.*

Proof. Write $\mathcal{L} := \mathcal{O}_{\mathrm{X}}(-\mathrm{D})$ and recall the exact triangle $\mathcal{L} \to \mathcal{O}_{\mathrm{X}} \to i_*(\mathcal{O}_{\mathrm{D}})$. For any $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$, this gives rise after tensoring with $\mathcal{F}$ to an exact triangle $\mathcal{F} \otimes \mathcal{L} \to \mathcal{F} \to i_*i^*(\mathcal{F})$ (by the projection formula). That is, we have an exact triangle of natural transformations $\mathrm{id} \otimes \mathcal{L} \to \mathrm{id} \to i_*i^*$, and passing to right adjoints, an exact triangle $i_*i^! \to \mathrm{id} \to \mathrm{id} \otimes \mathcal{L}^{\vee}$. In particular we get the exact triangle

$$i_*i^!(\mathcal{O}_{\mathrm{X}}) \to \mathcal{O}_{\mathrm{X}} \to \mathcal{L}^{\vee}. \tag{0.3}$$

By adjunction, we obtain a canonical morphism

$$\mathcal{N}^{\vee}_{\mathrm{D/X}}[-1] \simeq i^*(\mathcal{L}^{\vee})[-1] \to i^!(\mathcal{O}_{\mathrm{X}}).$$

Choose an affine Zariski covering $(\mathrm{X}_\alpha \hookrightarrow \mathrm{X})_\alpha$, and let $(j_\alpha : \mathrm{D}_\alpha \hookrightarrow \mathrm{D})_\alpha$ denote the induced covering of D. The family of functors j^*_α is conservative by Zariski descent. Since the functor $i^!$ commutes with f^* for any morphism f (for essentially formal reasons, and independently of Grothendieck duality), we may replace X by any X_α to assume that X is affine. In this case the functor i_* is conservative, so it will suffice to show that the canonical map

$$i_*(\mathcal{N}^{\vee}_{\mathrm{D/X}})[-1] \to i_*i^!(\mathcal{O}_{\mathrm{X}})$$

is invertible. Consider again the triangle $\mathcal{F} \otimes \mathcal{L} \to \mathcal{F} \to i_*i^*(\mathcal{F})$ above. Taking $\mathcal{F} = \mathcal{L}^{\vee}$ we get the exact triangle

$$\mathcal{O}_{\mathrm{X}} \to \mathcal{L}^{\vee} \to i_*i^*(\mathcal{L}^{\vee}) \simeq i_*(\mathcal{N}^{\vee}_{\mathrm{D/X}}),$$

since $\mathcal{L}$ is invertible. Comparing with (0.3) yields the claim. ☐

We are now ready to prove Theorem 2.

Claim 8. *The functor* $p^* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\tilde{\mathrm{X}})$ *is fully faithful.*

Proof. The claim is that for any $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$, the unit map $\mathcal{F} \to p_*p^*(\mathcal{F})$ is invertible. By Zariski descent we may reduce to the case where X is affine and i fits in a cartesian square of the form (0.1). Since Qcoh(X) is then generated under colimits and finite limits by $\mathcal{O}_{\mathrm{X}}$, we may assume that $\mathcal{F} = \mathcal{O}_{\mathrm{X}}$. In other words, the claim is that the canonical map $\mathcal{O}_{\mathrm{X}} \to p_*(\mathcal{O}_{\tilde{\mathrm{X}}})$ is invertible. The square

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{i_{\mathrm{E}}} & \tilde{\mathrm{X}} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X} \end{array}$$

is a derived base change of the square

$$\begin{array}{ccc} \mathbf{P}^{n-1} & \longrightarrow & \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \\ \downarrow & & \downarrow{\scriptstyle p_0} \\ \{0\} & \xrightarrow{i_0} & \mathbf{A}^n, \end{array}$$

and the map $\mathcal{O}_{\mathrm{X}} \to p_*(\mathcal{O}_{\tilde{\mathrm{X}}})$ is the pullback along $f : \mathrm{X} \to \mathbf{A}^n$ of the canonical map $\mathcal{O}_{\mathbf{A}^n} \to (p_0)_*(\mathcal{O}_{\mathrm{Bl}_{\{0\}/\mathbf{A}^n}})$. Thus we reduce to the case where i is the immersion $\{0\} \hookrightarrow \mathbf{A}^n$, which is a classical computation (use the standard affine cover of $\mathrm{Bl}_{\{0\}/\mathbf{A}^n}$, or see [2, Exp. 7]). □

Claim 9. *The functor $(i_{\mathrm{E}})_* q^* : \mathrm{Qcoh}(\mathrm{Z}) \to \mathrm{Qcoh}(\tilde{\mathrm{X}})$ is fully faithful.*

Proof. Again it suffices to show the unit map $\mathcal{F} \to q_*(i_{\mathrm{E}})^!(i_{\mathrm{E}})_* q^*(\mathcal{F})$ is invertible for all $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{Z})$. As in the previous claim we may assume X is affine and that $\mathcal{F} = \mathcal{O}_{\mathrm{Z}}$. Using Proposition 7 and the canonical identification $\mathcal{N}_{\mathrm{E}/\tilde{\mathrm{X}}} \simeq \mathcal{O}_{\mathrm{E}}(1)$, the unit map is identified with

$$\mathcal{O}_{\mathrm{Z}} \to q_*((i_{\mathrm{E}})^*(i_{\mathrm{E}})_*(\mathcal{O}_{\mathrm{E}}) \otimes \mathcal{O}_{\mathrm{E}}(-1))[-1].$$

By Lemma 6 it is further identified with

$$\mathcal{O}_{\mathrm{Z}} \to q_*(\mathcal{O}_{\mathrm{E}}(-1)) \oplus q_*(\mathcal{O}_{\mathrm{E}}).$$

Recall that q is identified with the projection $\mathrm{E} \simeq \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \to \mathrm{Z}$. Thus we have identifications $q_*(\mathcal{O}_{\mathrm{E}}(-1)) \simeq 0$, $q_*(\mathcal{O}_{\mathrm{E}}) \simeq \mathcal{O}_{\mathrm{Z}}$ (Lecture 7), under which the map in question is the identity. □

Claim 10. *For each $1 \leqslant k \leqslant n-1$, the full stable subcategory $\mathbf{C}(-k) \subset \mathrm{Qcoh}(\tilde{\mathrm{X}})$ is right orthogonal to $\mathbf{C}(0), \ldots, \mathbf{C}(-k+1)$.*

Proof. For $\mathcal{F}_{\mathrm{X}} \in \mathrm{Qcoh}(\mathrm{X})$ and $\mathcal{F}_{\mathrm{Z}} \in \mathrm{Qcoh}(\mathrm{Z})$, we have for each $1 \leqslant k \leqslant n-1$,

$$\mathrm{Maps}(p^*(\mathcal{F}_{\mathrm{X}}), (i_{\mathrm{E}})_*(q^*(\mathcal{F}_{\mathrm{Z}}) \otimes \mathcal{O}(-k))) \simeq \mathrm{Maps}(q^* i^*(\mathcal{F}_{\mathrm{X}}), q^*(\mathcal{F}_{\mathrm{Z}}) \otimes \mathcal{O}(-k)) \simeq \mathrm{pt}$$

by Lecture 7 (since $\mathrm{E} \simeq \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$ and $\mathcal{N}_{\mathrm{Z}/\mathrm{X}}$ is of rank n). This shows that $\mathbf{C}(0)$ is left orthogonal to each $\mathbf{C}(-k)$, $1 \leqslant k \leqslant n-1$.

For $1 \leqslant k \leqslant k' \leqslant n-1$, we need to show that the mapping space

$$\mathrm{Maps}((i_{\mathrm{E}})_*(q^*(\mathcal{F}_{\mathrm{Z}}) \otimes \mathcal{O}(-k)), (i_{\mathrm{E}})_*(q^*(\mathcal{F}'_{\mathrm{Z}}) \otimes \mathcal{O}(-k')))$$

is contractible, for all $\mathcal{F}_{\mathrm{Z}}, \mathcal{F}'_{\mathrm{Z}} \in \mathrm{Qcoh}(\mathrm{Z})$. Choose an affine Zariski covering of X, and let $(j_\alpha^* : \tilde{\mathrm{X}}_\alpha \hookrightarrow \tilde{\mathrm{X}})_\alpha$ denote the induced covering of $\tilde{\mathrm{X}}$. Using Zariski descent for the presheaf of ∞-categories $\mathrm{Qcoh}(-)$, and base change for $(i_{\mathrm{E}})_*$ along j_α^*, we may replace X by any n-fold intersection $\mathrm{X}_{\alpha_0} \times_{\mathrm{X}} \cdots \times_{\mathrm{X}} \mathrm{X}_{\alpha_n}$, and Z, $\tilde{\mathrm{X}}$, and E by their respective base changes. Thus we may assume that X is separated. Repeating the same argument again, we may assume furthermore that X is affine. Since $\mathrm{Qcoh}(\mathrm{Z})$ is then generated under colimits and finite limits by $\mathcal{O}_{\mathrm{Z}}$, we may assume that $\mathcal{F}_{\mathrm{Z}} = \mathcal{F}'_{\mathrm{Z}} = \mathcal{O}_{\mathrm{Z}}$. Then we have

$$\begin{aligned}\mathrm{Maps}((i_{\mathrm{E}})_*(\mathcal{O}(-k)), (i_{\mathrm{E}})_*(\mathcal{O}(-k'))) &\simeq \mathrm{Maps}((i_{\mathrm{E}})^*(i_{\mathrm{E}})_*(\mathcal{O}(-k)), \mathcal{O}(-k')) \\ &\simeq \mathrm{Maps}(\mathcal{O}(-k) \oplus \mathcal{O}(-k+1)[1], \mathcal{O}(-k')) \\ &\simeq \mathrm{Maps}(\mathcal{O}(-k), \mathcal{O}(-k')) \times \mathrm{Maps}(\mathcal{O}(-k+1)[1], \mathcal{O}(-k')) \\ &\simeq \Gamma(\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}), \mathcal{O}(k-k')) \times \Gamma(\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}), \mathcal{O}(k-k'-1)[-1])\end{aligned}$$

which is contractible by Serre's computation in Lecture 7. The isomorphism $(i_{\mathrm{E}})^*(i_{\mathrm{E}})_*(\mathcal{O}(-k)) \simeq \mathcal{O}(-k) \oplus \mathcal{O}(-k+1)[1]$ follows from Lemma 6 and the projection formula. □

Claim 11. *The stable ∞-category $\mathrm{Qcoh}(\tilde{\mathrm{X}})$ is generated by the full stable subcategories $\mathbf{C}(0)$, $\mathbf{C}(-1)$, …, $\mathbf{C}(-n+1)$.*

Proof. Denote by $\mathbf{C}(*)$ the full stable subcategory of $\mathrm{Qcoh}(\tilde{\mathrm{X}})$ by $\mathbf{C}(0)$, $\mathbf{C}(-1)$, …, $\mathbf{C}(-n+1)$. The claim is that the inclusion $\mathbf{C}(*) \subseteq \mathrm{Qcoh}(\tilde{\mathrm{X}})$ is an equality. Note that $\mathcal{O}_{\tilde{\mathrm{X}}} \in \mathbf{C}(0) \subset \mathbf{C}(*)$ and $(i_{\mathrm{E}})_*(\mathcal{O}_{\mathrm{E}}(-k)) \in \mathbf{C}(-k) \subset \mathbf{C}(*)$ for $1 \leqslant k \leqslant n-1$. Consider the exact triangle $\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E}) \to \mathcal{O}_{\tilde{\mathrm{X}}} \to (i_{\mathrm{E}})_*(\mathcal{O}_{\mathrm{E}})$ and recall that $\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E}) \simeq \mathcal{O}_{\tilde{\mathrm{X}}}(1)$. Tensoring with $\mathcal{O}(-k)$ and using the projection formula, we get the exact triangle

$$\mathcal{O}_{\tilde{\mathrm{X}}}(-k+1) \to \mathcal{O}_{\tilde{\mathrm{X}}}(-k) \to (i_{\mathrm{E}})_*(\mathcal{O}_{\mathrm{E}}(-k))$$

for each $1 \leqslant k \leqslant n-1$. Taking $k=1$ we deduce $\mathcal{O}_{\tilde{\mathrm{X}}}(-1) \in \mathbf{C}(*)$. Continuing recursively we find that $\mathcal{O}_{\tilde{\mathrm{X}}}(-k) \in \mathbf{C}(*)$ for all $1 \leqslant k \leqslant n-1$.

Now let $\mathcal{F} \in \mathrm{Qcoh}(\tilde{\mathrm{X}})$. Let $\mathcal{G}_0$ denote the cofibre of the co-unit $p^*p_*(\mathcal{F}) \to \mathcal{F} \to \mathcal{G}_0$. Note that $\mathcal{G}_0$ is right orthogonal to $\mathbf{C}(0)$. For $1 \leqslant k \leqslant n-1$ define $\mathcal{G}_k$ recursively so that we have exact triangles

$$(i_{\mathrm{E}})_*(q^*q_*((i_{\mathrm{E}})^!(\mathcal{G}_{k-1}) \otimes \mathcal{O}(k)) \otimes \mathcal{O}(-k)) \to \mathcal{G}_{k-1} \to \mathcal{G}_k.$$

Note that each $\mathcal{G}_k$ is right orthogonal to each of the subcategories $\mathbf{C}(0), \ldots, \mathbf{C}(k-1)$. For example, $\mathcal{G}_1$ is right orthogonal to $\mathbf{C}(-1)$ by construction. It is right orthogonal to $\mathbf{C}(0)$ since the middle term $\mathcal{G}_0$ in the above sequence is, and the left-hand term is because $\mathbf{C}(-1) \subseteq \mathbf{C}(0)^{\perp}$ by Claim 10. Proceeding inductively, the claim follows for each k. We now claim that $\mathcal{G}_{n-1}$ is zero. Working backwards, it will follow that $\mathcal{F}$ belongs to $\mathbf{C}(*)$.

Since the objects $\mathcal{G}_k$ are stable under base change, we may use Zariski descent and base change to assume that X is affine. Moreover we may assume that $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ fits in a cartesian square of the form (0.1). By Lemma 3, $p : \tilde{\mathrm{X}} \to \mathrm{X}$ factors through a quasi-smooth closed immersion $i' : \tilde{\mathrm{X}} \hookrightarrow \mathbf{P}^{n-1}_{\mathrm{X}}$. Recall from Lecture 7 that there is a canonical isomorphism $\varinjlim_{\mathrm{J} \subsetneq [n-1]} \mathcal{O}(\#\mathrm{J}) \simeq \mathcal{O}(n)$ in $\mathrm{Qcoh}(\mathbf{P}^{n-1}_{\mathrm{X}})$. Applying $(i')^*$, we get $\varinjlim_{\mathrm{J} \subsetneq [n-1]} \mathcal{O}_{\tilde{\mathrm{X}}}(\#\mathrm{J}) \simeq \mathcal{O}_{\tilde{\mathrm{X}}}(n)$ in $\mathrm{Qcoh}(\tilde{\mathrm{X}})$. In particular, every $\mathcal{O}_{\tilde{\mathrm{X}}}(k)$ belongs to $\mathbf{C}(*)$ for all $k \in \mathbf{Z}$. Recall also that we may find a map $\bigoplus_\alpha \mathcal{O}(d_\alpha)[n_\alpha] \to i'_*(\mathcal{G}_{n-1})$ which is surjective on all homotopy groups. By adjunction this corresponds to a map $\bigoplus_\alpha \mathcal{O}(d_\alpha)[n_\alpha] \to \mathcal{G}_{n-1}$ (which is also surjective on homotopy groups). But the source belongs to $\mathbf{C}(*)$, and the target is right orthogonal to $\mathbf{C}(*)$, so this map is null-homotopic. Thus $\mathcal{G}_{n-1}$ is zero. □

Proof of Theorem 2. Combine Claims 8, 9, 10, and 11. For the statement at the level of perfect complexes, simply note that all functors involved preserve perfect complexes. □

Proof of Theorem 1. Surjectivity follows from Claim 11. For injectivity, suppose given $\mathcal{F}_{\mathrm{X}} \in \mathrm{Perf}(\mathrm{X})$, $\mathcal{F}^k_{\mathrm{Z}} \in \mathrm{Perf}(\mathrm{Z})$ for $1 \leqslant k \leqslant n-1$, such that

$$0 = [p^*\mathcal{F}_{\mathrm{X}}] + \sum_{k=1}^{n-1} [(i_{\mathrm{E}})_*(q^*(\mathcal{F}^k_{\mathrm{Z}}) \otimes \mathcal{O}(-k))].$$

Applying p_* (which preserves perfect complexes), we deduce that $[\mathcal{F}_{\mathrm{X}}] = 0$ (using Claim 8 and Claim 10). Applying $q_*((i_{\mathrm{E}})^!(-) \otimes \mathcal{O}(k))$ (q_* preserves perfect complexes, and $(i_{\mathrm{E}})^!$ preserves perfect complexes by Grothendieck duality), we deduce that $[\mathcal{F}^k_{\mathrm{Z}}] = 0$ for each k (using Claim 9 and Claim 10). □

References

[1] A. A. Khan, *Virtual Cartier divisors and blow-ups.*
[2] SGA 6.