Lecture 7
The projective bundle formula

Our goal for this lecture is to prove the following:

Theorem 1. *Let* S *be a qcqs derived scheme. Let* $\mathcal{E}$ *be a locally free* $\mathcal{O}_{\mathrm{S}}$*-module of rank* $n+1$*, and* $p : \mathbf{P}_{\mathrm{S}}(\mathcal{E}) \to \mathrm{S}$ *the associated projective bundle. Then the canonical homomorphisms*

$$\mathrm{K}_0(\mathrm{S}) \to \mathrm{K}_0(\mathbf{P}_{\mathrm{S}}(\mathcal{E})), \quad [\mathcal{F}] \mapsto [p^*(\mathcal{F}) \otimes \mathcal{O}(-k)],$$

for $0 \leqslant k \leqslant n$*, induce a bijection*

$$\mathrm{K}_0(\mathbf{P}_{\mathrm{S}}(\mathcal{E})) \simeq \bigoplus_{i=0}^{n} \mathrm{K}_0(\mathrm{S}).$$

In particular, $\mathrm{K}_0(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$ *is freely generated as a* $\mathrm{K}_0(\mathrm{S})$*-module by the classes* $[\mathcal{O}(0)], \dots, [\mathcal{O}(-n)]$*.*

In order to prove Theorem 1 we will analyze the structure of the stable ∞-category $\mathrm{Qcoh}(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$.

Definition 2. Let $\mathbf{C}$ be a stable ∞-category and $\mathbf{D}$ a stable full subcategory. An object $x \in \mathbf{C}$ is *left orthogonal* (resp. *right orthogonal*) to $\mathbf{D}$ if the mapping space $\mathrm{Maps}_{\mathbf{C}}(x, d)$ (resp. $\mathrm{Maps}_{\mathbf{C}}(d, x)$) is contractible for all objects $d \in \mathbf{D}$. We let ${}^{\perp}\mathbf{D} \subset \mathbf{C}$ and $\mathbf{D}^{\perp} \subset \mathbf{C}$ denote the full subcategories of left orthogonal and right orthogonal objects, respectively.

Theorem 3. *Let* S *be a qcqs derived scheme. Let* $\mathcal{E}$ *be a locally free* $\mathcal{O}_{\mathrm{S}}$*-module of rank* $n+1$*, and* $p : \mathbf{P}_{\mathrm{S}}(\mathcal{E}) \to \mathrm{S}$ *the associated projective bundle. Then we have:*

(i) *For all integers* k*, the assignment* $\mathcal{F} \mapsto p^*(\mathcal{F}) \otimes \mathcal{O}(k)$ *defines a fully faithful functor* $\mathrm{Qcoh}(\mathrm{S}) \to \mathrm{Qcoh}(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$*.*

(ii) *If* $\mathbf{C}(k) \subset \mathrm{Qcoh}(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$ *denotes the essential image of the functor in (i), then we have* $\mathbf{C}(k) \subset {}^{\perp}\mathbf{C}(k-i)$ *for all integers* k *and* $1 \leqslant i \leqslant n$*.*

(iii) *For any integer* k*, the* ∞*-category* $\mathrm{Qcoh}(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$ *is generated by the subcategories* $\mathbf{C}(k), \dots, \mathbf{C}(k-n)$*. That is, every object* $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$ *belongs to the full subcategory* $\langle \mathbf{C}(k), \dots, \mathbf{C}(k-n) \rangle$ *they generate under finite colimits and limits.*

Theorem 3 can be summarized by saying that the collection of subcategories $\mathbf{C}(k), \dots, \mathbf{C}(k-n)$ forms a *semi-orthogonal decomposition* for each k.

Definition 4. Let $\mathbf{C}$ be a stable ∞-category and let $\mathbf{C}(0), \dots, \mathbf{C}(-n)$ be full stable subcategories. Suppose that the following conditions hold:

(i) $\mathbf{C}(i) \subset {}^{\perp}\mathbf{C}(j)$ for all integers $i > j$.

(ii) The ∞-category $\mathbf{C}$ is generated by the subcategories $\mathbf{C}(0), \dots, \mathbf{C}(-n)$.

Then we say that the subcategories $\mathbf{C}(0), \dots, \mathbf{C}(-n)$ form a *semi-orthogonal decomposition* of $\mathbf{C}$, and we write $\mathbf{C} = \langle \mathbf{C}(0), \dots, \mathbf{C}(-n) \rangle$.

In order to prove Theorem 3 we will need some preliminaries on derived projective geometry. Let $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ with $\mathrm{R} \in \mathrm{SCRing}$ and let $\mathcal{E} = \mathcal{O}_{\mathrm{S}}^{\oplus n+1}$, so that $p : \mathbf{P}_{\mathrm{S}}(\mathcal{E}) = \mathbf{P}^n_{\mathrm{R}} \to \mathrm{Spec}(\mathrm{R})$ is n-dimensional projective space over R. Recall from Lect. 5:

Theorem 5 (Serre)**.** *Given a tuple* $k = (k_0, \dots, k_n) \in \mathbf{Z}^{n+1}$ *with* $k_i \geqslant 0$ *for each* i*, set* $m = k_0 + \cdots + k_n$*. There is an associated* R*-linear map*

$$x^k : \mathrm{R} \to \varprojlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{R}[\mathrm{M}_{\mathrm{I}}(m)] \simeq \Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m)).$$

- *If* $m \geqslant 0$*, then* $\Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m))$ *is free of rank* $\binom{m+n}{n}$*, generated by the global sections* x^k*.*

- *If $m < 0$, then $\Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m))$ is a direct sum of $\binom{-m-1}{n}$ copies of* $\mathrm{R}[-n]$*. In particular, it is zero if $-1 \geqslant m \geqslant -n$.*

Corollary 6. *We have $p_*\mathcal{O}(0) \simeq \mathcal{O}_{\mathrm{S}}$ and $p_*\mathcal{O}(m) = 0$ for $-1 \geqslant m \geqslant -n$.*

For each $0 \leqslant i \leqslant n$ we have canonical map $x_i : \mathcal{O}(0) \to \mathcal{O}(1)$ (which is induced by the map x^k above with k the vector with $k_i = 1$ and $k_j = 0$ for $j \neq i$). These maps give rise, by taking tensor products, to a cubical diagram

$$\mathrm{P}([n]) \to \mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{R}}), \quad \mathrm{J} \mapsto \bigotimes_{j \in \mathrm{J}} \mathcal{O}(1) \simeq \mathcal{O}(\#\mathrm{J}),$$

where $\mathrm{P}([n])$ is the set of subsets of $[n]$, and $\#\mathrm{J}$ is the cardinality of a subset $\mathrm{J} \subset [n]$.

Lemma 7. *This is a colimit diagram. That is, there is a canonical isomorphism*

$$\varinjlim_{\mathrm{J} \subsetneq [n]} \mathcal{O}(\#\mathrm{J}) \xrightarrow{\sim} \mathcal{O}(n+1)$$

in $\mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{R}})$.

We can tensor this isomorphism with any $\mathcal{O}(k)$, $k \in \mathbf{Z}$. Since the ∞-category $\mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{R}})$ is stable, this lemma implies that all the sheaves $\mathcal{O}(m)$ can be built out of $\mathcal{O}(0), \ldots, \mathcal{O}(n)$ using finite colimits and limits. For example, for $n = 1$ we get squares

$$\begin{array}{ccc} \mathcal{O}(k) & \longrightarrow & \mathcal{O}(k+1) \\ \downarrow & & \downarrow \\ \mathcal{O}(k+1) & \longrightarrow & \mathcal{O}(k+2) \end{array}$$

for each $k \in \mathbf{Z}$, that are cocartesian and cartesian.

Proof of Lemma 7. Using the canonical equivalence

$$\mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{R}}) \xrightarrow{\sim} \varprojlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{Mod}_{\mathrm{R}[\mathrm{M}_{\mathrm{I}}]},$$

we see that it suffices to check the isomorphism in question after restriction along each of the open immersions $j_{\mathrm{I}} : \mathrm{Spec}(\mathrm{R}[\mathrm{M}_{\mathrm{I}}]) \to \mathbf{P}^n_{\mathrm{R}}$. But $(j_{\mathrm{I}})^*(x_i) : (j_{\mathrm{I}})^*\mathcal{O}(0) \to (j_{\mathrm{I}})^*\mathcal{O}(1)$ is invertible by construction whenever $i \in \mathrm{I}$, so the claim is clear. □

Lemma 8. *Let $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{R}})$ be a connective quasi-coherent sheaf. Then there exists a map*

$$\mu : \bigoplus_{\alpha} \mathcal{O}(d_\alpha) \to \mathcal{F},$$

with $d_\alpha \in \mathbf{Z}$, which is surjective on π_0.

Proof. For each $\mathrm{I} \subset [n]$, let $\mathrm{M}^+_{\mathrm{I}} \subset \mathbf{Z}^{n+1}$ be the submonoid of tuples $(k_0, \ldots, k_n)$ such that $k_i \geqslant 0$ for $i \notin \mathrm{I}$. Then $\mathrm{M}_{\mathrm{I}} \subset \mathrm{M}^+_{\mathrm{I}}$ is the subset of tuples satisfying the further condition that $k_0 + \cdots + k_n = 0$. We have a canonical map

$$q : \mathrm{U} = \varinjlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{Spec}(\mathrm{R}[\mathrm{M}^+_{\mathrm{I}}]) \to \varinjlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{Spec}(\mathrm{R}[\mathrm{M}_{\mathrm{I}}]) \simeq \mathbf{P}^n_{\mathrm{R}}.$$

Note that $q_*\mathcal{O}_{\mathrm{U}} \simeq \bigoplus_{m \in \mathbf{Z}} \mathcal{O}(m)$ and that q is faithfully flat. Note that U can also be described as the vanishing locus $\mathrm{D}(x_0, \ldots, x_n)$ in $\mathrm{X} = \mathrm{Spec}(\mathrm{R}[x_0, \ldots, x_n])$. Let $j : \mathrm{U} \hookrightarrow \mathrm{X}$ denote the open immersion.

Since X is affine, we can construct a map

$$\bigoplus_{\alpha} \mathcal{O}_{\mathrm{X}} \to j_* q^*(\mathcal{F})$$

that is surjective on π_0 (for example, take the sum of the maps $\mathcal{O}_\mathrm{X} \to j_* q^*(\mathcal{F})$ corresponding to each element of π_0 of the target). Applying j^* this induces a map

$$\rho_\mathrm{U} : \bigoplus_\alpha \mathcal{O}_\mathrm{U} \to q^*(\mathcal{F})$$

which is still surjective on π_0. By adjunction this corresponds to a map $\bigoplus_\alpha \mathcal{O}(0) \to q_* q^*(\mathcal{F})$, where by the projection formula the target is identified with $q_*(\mathcal{O}_\mathrm{U}) \otimes \mathcal{F} \simeq \bigoplus_{m\in\mathbf{Z}} \mathcal{F} \otimes \mathcal{O}(m)$. Thus we get a collection of maps

$$\rho_\alpha : \mathcal{O}(0) \to \bigoplus_{m\in\mathbf{Z}} \mathcal{F} \otimes \mathcal{O}(m)$$

for each α. Since $\mathcal{O}(0)$ is a compact object (as it is a perfect complex on the quasi-compact quasi-separated derived scheme $\mathbf{P}^n_\mathrm{R}$), it follows that each ρ_α factors through a map

$$\rho_\alpha : \mathcal{O}(0) \to \bigoplus_{|m|\leqslant c_\alpha} \mathcal{F} \otimes \mathcal{O}(m),$$

for some integer $c_\alpha \geqslant 0$. Equivalently, we get maps $\rho_{\alpha,m} : \mathcal{O}(-m) \to \mathcal{F}$ for each α and all $|m| \leqslant c_\alpha$. Let μ denote the induced map

$$\mu : \mathcal{G} = \bigoplus_{\alpha,|m|\leqslant c_\alpha} \mathcal{O}(-m) \to \mathcal{F}.$$

It suffices to show that μ is surjective on π_0. Since q is faithfully flat it suffices to show this after applying q^*. This follows from the fact that the map $\rho_\mathrm{U} : \bigoplus_\alpha \mathcal{O}_\mathrm{U} \to q^*(\mathcal{F})$ is surjective on π_0 and factors through $q^*(\mu)$ by construction. □

In order to get statements about general projective bundles $\mathbf{P}_\mathrm{S}(\mathcal{E})$, we will use fpqc descent for quasi-coherent sheaves. The important point for us is that we can check invertibility of a given morphism in Qcoh(S) fpqc-locally.

Theorem 9. *The assignments* $\mathrm{S} \mapsto \mathrm{Qcoh(S)}$ *and* $\mathrm{S} \mapsto \mathrm{Perf(S)}$ *satisfy fpqc descent (as presheaves on the* ∞*-category of qcqs derived schemes). In particular, for any fpqc covering family* $(f_\alpha : \mathrm{S}_\alpha \to \mathrm{S})_\alpha$*, the family of functors* $(f_\alpha^*)_\alpha$ *is conservative.*

We are now ready to prove Theorem 3.

Proof of Theorem 3. Since $\mathcal{O}(k)$ are invertible, the functors $-\otimes \mathcal{O}(k)$ are invertible for all $k \in \mathbf{Z}$. Therefore we can assume $k = 0$ in all the statements.

For claim (i) we want to show that the unit map $\mathcal{F} \to p_* p^*(\mathcal{F})$ is invertible for all $\mathcal{F} \in \mathrm{Qcoh(S)}$. Using Theorem 9 and base change for p_*, we can reduce to the case where $\mathrm{S} = \mathrm{Spec(R)}$ is affine and $\mathcal{E} = \mathcal{O}_\mathrm{S}^{n+1}$ is free (by choosing an affine cover of S such that the restrictions of $\mathcal{E}$ are all free). Now both functors p^* and p_* are exact and moreover commute with arbitrary colimits (the latter since p is quasi-compact), and $\mathrm{Qcoh(S)} \simeq \mathrm{Mod_R}$ is generated by $\mathcal{O}_\mathrm{S}$ under colimits and finite limits, so that we may assume $\mathcal{F} = \mathcal{O}_\mathrm{S}$. Then it suffices to show that the canonical map in $\mathrm{Mod_R}$

$$\Gamma(\mathrm{S}, \mathcal{O}_\mathrm{S}) \to \Gamma(\mathrm{S}, p_*\mathcal{O}(0))$$

is invertible. This follows from Serre's theorem (Corollary 6).

For claim (ii), let $\mathcal{F}, \mathcal{G} \in \mathrm{Qcoh(S)}$ and consider the mapping space

$$\mathrm{Maps}(p^*(\mathcal{F}), p^*(\mathcal{G}) \otimes \mathcal{O}(-i))$$

for $1 \leqslant i \leqslant n$. By adjunction and the projection formula, this is identified with the space $\mathrm{Maps}(\mathcal{F}, p_*(\mathcal{O}(-i)) \otimes \mathcal{G})$. It suffices to show that $p_*(\mathcal{O}(-i)) \simeq 0$. This follows from Serre's theorem (Corollary 6), after using Zariski descent (Theorem 9) and base change for p_*.

For claim (iii), let $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$. We define a sequence $\mathcal{G}_0, \dots, \mathcal{G}_n$ of objects of $\mathrm{Qcoh}(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$, depending on $\mathcal{F}$, such that each $\mathcal{G}_m$ is right orthogonal to each of the subcategories $\mathbf{C}(0), \dots, \mathbf{C}(m)$, for each $m \geqslant 0$, and such that we have exact triangles

$$p^* p_*(\mathcal{G}_m \otimes \mathcal{O}(1)) \xrightarrow{\text{counit}} \mathcal{G}_m \otimes \mathcal{O}(1) \to \mathcal{G}_{m+1}. \tag{0.1}$$

For $m = 0$ we define $\mathcal{G}_0$ so that it fits into an exact triangle

$$p^* p_*(\mathcal{F}) \xrightarrow{\text{counit}} \mathcal{F} \to \mathcal{G}_0.$$

Since $p^* p_*(\mathcal{F}) \in \mathbf{C}(0)$ it follows that the cofibre $\mathcal{G}_0$ is right orthogonal to $\mathbf{C}(0)$.

Now suppose that we have defined $\mathcal{G}_m$, $0 \leqslant m < n$, so that the exact triangle (0.1) defines $\mathcal{G}_{m+1}$ inductively. We need to show that $\mathcal{G}_{m+1}$ is right orthogonal to all the subcategories $\mathbf{C}(0), \dots, \mathbf{C}(m+1)$. For $\mathbf{C}(0)$, this follows from the fact that $p^* p_*(\mathcal{G}_m \otimes \mathcal{O}(1))$ is contained in $\mathbf{C}(0)$. For $\mathbf{C}(i)$, $0 < i \leqslant m+1$, we observe that the left-hand and middle terms of the triangle (0.1) are both right orthogonal to $\mathbf{C}(i)$. Indeed, we have $\mathbf{C}(0) \subset \mathbf{C}(i)^{\perp}$ by (ii), so this takes care of the left-hand term. For the middle term $\mathcal{G}_m \otimes \mathcal{O}(1)$ the claim follows by the induction hypothesis.

Now we claim that $\mathcal{G}_n$ is zero. Using descent again (Theorem 9), we may assume that $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ and $\mathcal{E} = \mathcal{O}_{\mathrm{S}}^{\oplus n+1}$ (observe that the sequence $(\mathcal{G}_0, \dots, \mathcal{G}_n)$ is stable under base change). Then using Lemma 7 we deduce that $\mathcal{G}_n$ is right orthogonal to all $\mathbf{C}(i)$, $i \in \mathbf{Z}$ (not just for $0 \leqslant i \leqslant n$). Using Lemma 8 we can build a map

$$\bigoplus_{\alpha} \mathcal{O}(m_\alpha)[k_\alpha] \to \mathcal{G}_n$$

which is surjective on all homotopy groups. But the source of this map belongs to the stable subcategory generated by the $\mathbf{C}(i)$'s, $i \in \mathbf{Z}$, so this map must be null-homotopic. It follows that $\mathcal{G}_n \simeq 0$. Working backwords, we deduce that $\mathcal{G}_{n-1} \in \mathbf{C}(-1)$, ..., $\mathcal{G}_0 \in \langle \mathbf{C}(-1), \dots, \mathbf{C}(-n) \rangle$, and finally that $\mathcal{F} \in \langle \mathbf{C}(0), \mathbf{C}(-1), \dots, \mathbf{C}(-n) \rangle$ as claimed. □

Using the same ingredients (Theorem 5 and Lemma 8) one can show the following:

Exercise 10. The functor $p_* : \mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{S}}) \to \mathrm{Qcoh}(\mathrm{S})$ preserves almost perfect, resp. perfect complexes.

This implies that we have a direct image homomorphism

$$p_* : \mathrm{K}_0(\mathbf{P}^n_{\mathrm{S}}) \to \mathrm{K}_0(\mathrm{S}).$$

We can now prove Theorem 1:

Proof of Theorem 1. Let $[\mathcal{F}_i] \in \mathrm{K}_0(\mathrm{S})$ such that $\sum_{i=0}^n [p^*(\mathcal{F}_i) \otimes \mathcal{O}(-i)] = 0$. Applying p_* and using the projection formula, we get

$$0 = \sum_{i=0}^n [\mathcal{F}_i \otimes p_* \mathcal{O}(-i)] = [\mathcal{F}_0],$$

by Corollary 6. Similarly, applying $x \mapsto p_*([\mathcal{O}(1)] \cdot x)$ we deduce that $[\mathcal{F}_1] = 0$. Continuing in this way we find that $[\mathcal{F}_i] = 0$ for all i, which shows that the map $\varphi : \bigoplus_{i=0}^n \mathrm{K}_0(\mathrm{S}) \to \mathrm{K}_0(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$ is injective. For surjectivity, use the sequence $\mathcal{G}_0, \dots, \mathcal{G}_n$ and the exact triangles constructed in the proof of Theorem 3, to write any $[\mathcal{F}] \in \mathrm{K}_0(\mathbf{P}_{\mathrm{S}}(\mathcal{E}))$ as a sum of elements in the image of φ. □