# Lecture 6: derived Blow-ups and virtual Cartier divisors.

## §1. A quick review of classical Blow ups in algebraic geometry.

As warm-up, recall the following properties of Blow ups:

Setting: $X$ locally Noetherian scheme.
$\mathcal{I}$ quasi-coherent sheaf of ideals.
$\pi: \widetilde{X} = Bl_{V(\mathcal{I})}(X) \to X$ Blow up of $X$ with center in $V(\mathcal{I})$.

Then:

(1) $\pi$ is iso $\Leftrightarrow \mathcal{I}$ invertible sheaf on $X$

(2) $\pi$ is proper

(3) $Z \xrightarrow[\text{flat}]{} X$, $Z$ locally Noeth. $\widetilde{Z} = Bl_{V(\mathcal{I}\mathcal{O}_Z)}(Z) \to Z$

$\Rightarrow \widetilde{Z} = \widetilde{X} \times_X Z$ (stability under base change)

(4) $\pi$ induces an isomorphism $\pi^{-1}(X \setminus V(\mathcal{I})) \xrightarrow{\sim} X \setminus V(\mathcal{I})$.
if $X$ is integral, then $\widetilde{X}$ is integral and $\pi$ is birational ($\mathcal{I} \neq 0$).

Universal property:

Prop: $f: W \to X$ morphism between locally Noetherian schemes.
$\mathcal{I}$ quasi-coherent sheaf of ideals in $X$, $\mathcal{J} = (f^{-1}\mathcal{I})\mathcal{O}_W \hookrightarrow \mathcal{O}_W$.
Let $\pi: \widetilde{X} = Bl_{V(\mathcal{I})}(X) \to X$.
$\rho: \widetilde{W} = Bl_{V(\mathcal{J})}(W) \to W$.

$\Rightarrow \exists!\ \tilde{f}: \widetilde{W} \to \widetilde{X}$ such that
$$\begin{tikzcd} \widetilde{W} \arrow[r, "\tilde f"] \arrow[d, "\rho"'] & \widetilde{X} \arrow[d, "\pi"] \\ W \arrow[r, "f"'] & X \end{tikzcd}$$
commutes.

Corollary: if $(f^{-1}\mathcal{I})\mathcal{O}_W = \mathcal{J}$ is invertible
$\Rightarrow W \cong \widetilde{W}$ by (1) above, and so $\exists!\ g: W \to \widetilde{X}$ s.that
$$\begin{tikzcd} W \arrow[r, "g"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd}$$

Geometric rephrasing: $Z := V(\mathcal{I}) \hookrightarrow X$.
$f: W \to X$, ~~$f^{-1}(Z)$~~ if $f^{-1}(Z)$ is (scheme theoretically)
an effective Cartier divisor in $W$, then $\exists!\ g: W \to \widetilde{X}$ over $X$.

Our goal: generalize to the derived setting (in particular, allow an arbitrary morphism $f$ and prove a stronger universal property).

## §2. Quasi smooth immersions.

We begin with the classical definition:

Def: $A$ commutative ring; $f \in A$ a (non zero) element. — of course the def. makes sense even for $f=0$.

Define $K^A(f) := (\underset{1}{A} \xrightarrow{\cdot f} \underset{0}{A})$ (chain complex). This is called the Koszul complex of $f$.

Note: $H_0(K^A(f)) = A/f.A$ ; $H_1(K^A(f)) = Ann_A(f) = Ker(A \xrightarrow{\cdot f} A)$; $H_i(K^A(f)) = 0$, $i \neq 0,1$

The element $f$ is regular (in $A$) $\Leftrightarrow K^A_*(f) \simeq A/f\,[0] \Leftrightarrow H_i(K^A(f)) = 0$, $i \neq 0$.

Given elements $f_1, \dots, f_r \in A$, write $K^A(f_1, \dots, f_r)$ for the tensor product:

Explicitely: $K_p^A(f_1, \ldots, f_r)$ free $A$-module, iso to $\Lambda^p(A^{\oplus r})$.

<u>Prop</u>: If, for all $i$, $1 \leq i \leq r$, $f_i$ is not a zero divisor in $A/(f_1, \ldots, f_{i-1})$, then, $H_p(K^A(f_1, \ldots, f_r)) = 0 \ \forall \ p > 0$

$(\Longleftrightarrow K^A(f_1, \ldots, f_r) \simeq A/(f_1, \ldots, f_r)[0])$.

<u>Rmk</u> (Serre, local Algebra or Berthelot in SGA 6). The converse to the above statement holds if $A$ is Noetherian and the $f_i$'s belong to the radical of $A$.

Following SGA 6, we say that the sequence is regular if $K^A(f_1, \ldots, f_r)$ is acyclic in positive degrees.

We also need to recall the following:

<u>Prop</u> (Serre, local algebra, IV.2). Suppose that $A$ is Noeth., that the $f_i$ are in the radical of $A$ and that the sequence $(f_1, \ldots, f_r)$ is regular.

Then $K^A(f_1, \ldots, f_r)$ is a free resolution of $A/(f_1, \ldots, f_r)$, and thus, $\forall$ $A$-Module $M$ (not necessarily finitely generated) we have:

$$\mathrm{Tor}_i^A(A/(f_1, \ldots, f_r), M) \simeq H_i(K^A(f_1, \ldots, f_r) \otimes_A M).$$

In other words, $K^A(f_1, \ldots, f_r) \otimes_A M \simeq A/(f_1, \ldots, f_r) \otimes_A^{\mathbb{L}} M$.

In general, consider now a sequence $(f_1, \ldots, f_r) \in A$ (not nec. regular).

This determines a morphism $\mathbb{Z}[T_1, \ldots, T_r] \to A$, $T_i \mapsto f_i$.

The Koszul complex $K^A(f_1, \ldots, f_r)$ is then q.iso to

$$A \otimes^{\mathbb{L}}_{\mathbb{Z}[T_1, \ldots, T_r]} \mathbb{Z}[T_1, \ldots, T_r]/(T_1, \ldots, T_r).$$

(Note that $T_1, \ldots, T_r$ is a reg. sequence in $\mathbb{Z}[T_1, \ldots, T_r]$)

Global version: $Z \overset{i}{\hookrightarrow} X$ closed immersion of schemes.

<u>Def</u>: $i$ is <u>regular</u> if its ideal of definition $\mathcal{J} \subset \mathcal{O}_X$ is (Zariski)-locally generated by a regular sequence.

Consequence: $\mathcal{J}/\mathcal{J}^2$ $(=: N_{Z/X})$ locally free of rank $= n$ $(=: \mathrm{codim}_X(Z))$

We say that $D \overset{i}{\hookrightarrow} X$ is an effective Cartier divisor if $i$ is a regular immersion such that $\mathcal{J}_D/\mathcal{J}_D^2$ is locally free of rank $= 1$.

<u>Rmk</u>: for a regular immersion $Z \hookrightarrow X$, we have $\mathbb{L}_{Z/X} = N_{Z/X}[1]$.

We can reformulate the above discussion in the framework of derived algebraic geometry as follows:

<u>Prop</u>: $i : Z \hookrightarrow X$ is a regular immersion

Zariski locally on $X$

$$\Longleftrightarrow \exists f : \begin{tikzcd} Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^n \end{tikzcd}$$ is homotopy cartesian in DSch.

locally

$(f_1, \ldots, f_n) : \mathrm{Spec}(A) \to \mathbb{A}^n$

<u>Rmk</u> on regular immersions and Blow ups:

$X$ locally Noetherian ~~$\mathbb{Z}$~~ regular scheme. $Z \hookrightarrow X$ regular immersion. (can replace locally Noeth. by quasi compact and q. sep., provided that we use the strong version of regularity (SGA 6))

$\tilde{X} = Bl_Z(X)$. Then $\tilde{X}$ satisfies:

Let $Z' := \tilde{X} \times_X Z$. Then $Z' \cong \mathbb{P}(N_{Z/X})$.

(in particular, if $N_{Z/X}$ free of rank $d$, $Z' \cong \mathbb{P}^{d-1}_Z$).

We are now ready to extend the notion of regularity to the derived setting.

$A \in$ SCRing. Let $f_1, \dots, f_r$ be elements of $A$ (i.e. $f_i$ are points of the underlying sSet).

We set $A/\!/(f_1) := A \otimes^{\mathbb{L}}_{\mathbb{Z}[T]} \mathbb{Z}[T]/(T)$, as SCRing
$f_1 \mapsfrom T$

As underlying $A$-module, we have $A/\!/(f_1) = \mathrm{Cof}(A \xrightarrow{\cdot f} A)$
(= homotopy cofiber in $\mathrm{Mod}_A$).
(stable category)

$\rightsquigarrow \quad A \xrightarrow{f_1} A \to A/\!/(f_1)$ fiber sequence
(of spaces ...).

More generally, define the SCRing $A/\!/(f_1, -, f_r)$ by $A \otimes^{\mathbb{L}}_{\mathbb{Z}[T_1, -, T_r]} \dfrac{\mathbb{Z}[T_1 - T_r]}{(T_1, -, T_r)}$
(where the map $\mathbb{Z}[T_1, -, T_r]$ is given by $T_i \mapsto f_i$).

<u>Rmk/Examples</u>: 1) We have $\pi_0(A/\!/(f_1, -, f_r)) \cong \pi_0(A)/(f_1, -, f_r)$.

2) We have underlying module: $A/\!/_{(f_1, -, f_r)} \cong (A/\!/_{f_1}) \otimes^{\mathbb{L}}_A (A/\!/_{f_2}) \otimes^{\mathbb{L}}_A \cdots \otimes^{\mathbb{L}} (A/\!/_{f_r})$.

3) Suppose $f = 0$. Then $A/\!/(0) = \mathrm{Cof}(\underset{1}{A} \xrightarrow{0} \underset{0}{A}) \simeq A \oplus A[1]$
A discrete

4) Suppose $A$ discrete. Then $A/\!/(f_1, -, f_r) \underset{\text{as module}}{\simeq} K^A(f_1, -, f_r)$.
$(f_1, -, f_r) \in A$

$\Rightarrow \quad \pi_0(A/\!/(f_1, -, f_r)) = H_0(K^A(f_1 - , f_r)) = A/(f_1 -, f_r)$.

The sequence $(f_1, -, f_r)$ is regular $\iff A/\!/(f_1, -, f_r)$ is discrete, quasi iso to $A/(f_1, -, f_r)[0]$.

<u>Def</u>: $i: Z \hookrightarrow \mathcal{X}$ closed immersion of derived schemes. (<u>recall</u>: we have seen in lecture 4 that $Z \xrightarrow{i} \mathcal{X}$ morphism of derived schemes is a closed embedding $\iff Z_{cl} \underset{i}{\hookrightarrow} \mathcal{X}_{cl}$ is a closed embedding of classical schemes)

We say that $i$ is <u>quasi-smooth</u> if $\exists$, Zariski locally on $\mathcal{X}$ a map $f: \mathcal{X} \to \mathbb{A}^n$ in DSch

and a square

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^n \end{tikzcd}$$

which is homotopy cartesian in DSch.

Equivalently, locally on $\mathcal{X}$, $Z$ is of the form $A/\!/(f_1, -, f_r)$.

<u>Key remark</u>: if <u>both</u> $\mathcal{X}$ and $Z$ are classical, then $i: Z \hookrightarrow \mathcal{X}$ is q. smooth $\iff$ it is a regular embedding in the classical sense.

One can prove the following proposition:

<u>Prop</u>: $i: Z \hookrightarrow \mathcal{X}$ closed immersion of derived schemes. Then $i$ is quasi-smooth $\iff$ $\mathcal{L}_{Z/\mathcal{X}}[-1]$ (shifted cotangent complex) is a locally free $\mathcal{O}_Z$-module of finite rank.

<u>Sketch</u>: if $i: Z \to \mathcal{X}$ is quasi smooth, then $\mathcal{L}_{Z/\mathcal{X}} \simeq f^*\mathcal{L}_{\mathbb{A}^n/\{0\}}$ (Zariski-locally) (since the square is cartesian) $\Rightarrow$ look at $\mathcal{L}_{\mathbb{A}^n/\{0\}}[-1] \simeq N_{\{0\}/\mathbb{A}^n} \equiv \mathcal{I}/\mathcal{I}^2$ is (locally) free of rank $= n$.

Conversely, suppose $\mathcal{L}_{Z/\mathcal{X}}^{[-1]}$ loc. free of rank $n$. $\rightsquigarrow$ assume $\mathcal{X} = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(B)$, $\mathcal{L}_{Z/\mathcal{X}}^{[-1]} \cong B^{\oplus n}$. Look at $\pi_0(A) \twoheadrightarrow \pi_0(B)$.

$F := \mathrm{Fib}(A \to B) \rightsquigarrow$ get map $B \otimes^{\mathbb{L}}_A F[1] \to L_{B/A}$ (Thm. 7.4.3.1 Lurie H.A.)

$\rightsquigarrow$ get $\pi_0(B \otimes^{\mathbb{L}}_A F) \simeq \pi_1(L_{B/A})$, iso of $\pi_0(B)$ modules. (Prop. 25.3.6.1 in SAG).

lift $f_1, \dots, f_n \in A \leftsquigarrow df_1, \dots, df_n$ generators (free by assumption)

$\Rightarrow$ Can look at $A/\!/(f_1, \dots, f_n) \xrightarrow{\varphi} B$, inducing iso on $\pi_0$.

By H.A. Cor 7.4.3.4, enough to show $L\varphi \cong 0$.

This follows from: $\underbrace{L(A/\!/(f_1, \dots, f_n)) \otimes^{\mathbb{L}}_A B}_{\text{free of rank} = n} \longrightarrow \overset{\text{free of rank} = n}{L_{B/A}} \longrightarrow L\varphi \longrightarrow +$

$\Rightarrow L\varphi \cong 0$.

Given the previous proposition, we can make the following definition:

<u>Def</u>: We define $N_{Z/\mathcal{X}} := \mathcal{L}_{Z/\mathcal{X}}[-1]$ for any quasi-smooth embedding $Z \hookrightarrow \mathcal{X}$.
It is locally free of finite rank $= n =:$ <u>virtual codimension</u> of $Z$ in $\mathcal{X}$.
$N_{Z/\mathcal{X}}$ is defined to be the <u>conormal sheaf</u> of $Z$ in $\mathcal{X}$.

<u>Rmk</u>: (1) In classical algebraic geometry, the typical example of regular embedding is the following:

$f: X \to Y$ morphism of finite type of regular schemes.
Then $f$ is a local complete intersection, i.e. $f$ can be factored, Zariski locally on $X$ as $X \underset{i}{\hookrightarrow} Z \underset{g}{\to} Y$, where $i$ is a regular immersion and $g$ is smooth.

$\rightarrow$ Indeed: locally $f$ can be factored as $X \hookrightarrow \mathbb{A}^n_Y \longrightarrow Y$. Assume $X, Y$ locally Noetherian.
Both $X$, $\mathbb{A}^n_Y$ are regular by assumption $\Rightarrow$ use the following Basic Comm. algebra

<u>Lemma</u>: $(A, m)$ regular local ring, Noetherian.
$I \subsetneq A$ ideal. Suppose $A/I$ is regular. Then $I$ is generated by $r$ elements of a system of parameters for $A$, where $r = \dim A - \dim A/I$.

<u>Consequence</u>: any closed immersion between regular schemes is a regular immersion.

(2) Back to DSch. Suppose $\mathcal{X}$, $Z$ are smooth over some base $S$.
(recall: $A$, $B$ SCRing, $B$ $A$-algebra of finite presentation. We say that $B$ is smooth over $A$ if $L_{B/A}$ is finitely gen & projective).

So, if $\mathcal{X}$ and $Z$ are smooth over some $S$, look at:

$$i^*\mathbb{L}_{\mathcal{X}/S} \to \mathbb{L}_{Z/S} \to \mathbb{L}_{Z/\mathcal{X}} \quad \Rightarrow \quad \mathbb{L}_{Z/\mathcal{X}}[-1] \text{ is locally free of finite rank.}$$
$$\Rightarrow Z \hookrightarrow \mathcal{X} \text{ quasi smooth.}$$

(3) Important remark: the virtual codimension is stable under arbitrary derived pullbacks (this follows from the stability of $\mathbb{L}_{Z/\mathcal{X}}$ under derived pullbacks). Note that this is NOT the case for the usual notion of codimension.

Def: $\mathcal{X} \in DSch$. We call $D \xhookrightarrow{i_D} \mathcal{X}$ a Virtual Cartier divisor (effective) if $i_D$ is a quasi smooth embedding of virtual codimension $= 1$.

$\to$ $\mathrm{codim.vir}(Z, \mathcal{X}) \geq \mathrm{codim}(Z^{cl}, \mathcal{X}^{cl})$ (if $\mathcal{X}$ locally Noetherian)

Ex: any effective Cartier divisor is a virtual Cartier divisor.

## § 3. Blow ups revisited.

We begin from the notion of Cartier divisor lying over a quasi smooth embedding.

$Z \xhookrightarrow{i} \mathcal{X}$ q. smooth. $\rightsquigarrow$ Def: A virtual Cartier divisor $D$ on $S$ over $(Z, \mathcal{X})$ is:

$f: S \to \mathcal{X}$ any

$$Q = \begin{tikzcd} D \arrow[r, hook] \arrow[d, "g"'] & S \arrow[d, "f"] \\ Z \arrow[r, hook, "i"'] & \mathcal{X} \end{tikzcd}$$

commutative diagram

such that:

1) $D \hookrightarrow S$ is Virtual Cartier divisor on $S$
2) The underlying square $Q^{cl}$ is Cartesian
3) The canonical map $g^* N_{Z/\mathcal{X}} \to N_{D/S}$ is surjective on $\pi_0$.

Ex: if $\mathcal{X}, Z, S$ are classical and if $f^{-1}(Z)$ is a Cartier divisor $D$ on $S$

1) then $D \hookrightarrow S$ is also a virtual Cartier divisor lying over $(Z, \mathcal{X})$ in the above sense.
2) if $Q$ is Cartesian, then $g^* N_{Z/\mathcal{X}} \xrightarrow{\sim} N_{D/S}$ (by construction). This forces the virtual codim. of $Z$ in $\mathcal{X}$ to be already $= 1$ ($Z$ is a Cartier divisor in $\mathcal{X}$).

Equivalent description: Set $S_Z := S \times^{\mathbb{R}}_{\mathcal{X}} Z$.

Then $D \hookrightarrow S$ is equivalently the datum of a map $D \xrightarrow{h} S_Z$ such that $D \hookrightarrow S_Z \hookrightarrow S$ exhibits $D$ as Virtual Cartier divisor on $S$ s.that $D^{cl} \cong S_Z^{cl}$ and satisfying $h^* N_{S_Z/S} \to N_{D/S}$ onto (on $\pi_0$).

$\Leftrightarrow$ $\mathbb{L}_{D/S_Z}$ is 1-connected ($\pi_i(\mathbb{L}_{D/S_Z}) = 0 \quad i \leq 1$).

We now define a space of Cartier divisors lying over $(Z, \mathcal{X})$, functorial in $S$:

Step-by-step construction:

$$DSch/_{S_Z} = \infty\text{-Cat of DSchemes } W \to S_Z$$

Inside $DSch^{\simeq}/_{S_Z}$, consider subspace $\left\{ \begin{matrix} D \xrightarrow{h} S_Z \\ D^{cl} \simeq S_Z^{cl} \end{matrix} \;\middle|\; \begin{matrix} \mathrm{Cofib}(h^* N_{S_Z/S} \to N_{D/S}) \\ \text{is 1-connected} \end{matrix} \right\}$

$\Longleftrightarrow$ let's work on affine charts: $\mathcal{X} = \mathrm{Spec}(A)$, $\mathcal{Z} = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(R)$.

$\mathrm{VCart}_{B/A}(R)$ = subcategory of $(\mathrm{Mod}_R)/_{(R \otimes_A^{\mathbb{L}} B)}$ ($R \otimes_A^{\mathbb{L}} B$ seen as $R$-module) such that:

Morphisms in $\mathrm{VCart}_{B/A}(R)$ = equivalences.

Objects: maps $h: L \longrightarrow R \otimes_A^{\mathbb{L}} B$ such that

i) $\pi_0(L) \simeq \pi_0(R \otimes_A^{\mathbb{L}} B)$

ii) $\mathbb{L}(L/_R)^{[-1]}$ free of rank $= 1$

iii) $\mathbb{L}(R \otimes_A^{\mathbb{L}} B/R)^{[-1]} \longrightarrow \mathbb{L}(L/R)^{[-1]}$ has 1-connected cofiber.

Note that all conditions are stable under base change

$\Rightarrow$ Defines a subsheaf of $DSch/_{S_Z}$. Call such guy $Bl_{Z/\mathcal{X}}$. It is a derived stack.

$Bl_{Z/\mathcal{X}}: DSch/_{\mathcal{X}} \longrightarrow \mathrm{Spc}$, $(S \to \mathcal{X}) \longmapsto Bl_{Z/\mathcal{X}}(S \to \mathcal{X})$.

The S-points are precisely the Virtual Cartier divisors lying over $(Z, \mathcal{X})$.

<u>Thm</u>: (i) $Bl_{Z/\mathcal{X}}$ is schematic (i.e. it is a derived scheme)

(ii) $Bl_{Z/\mathcal{X}} \to \mathcal{X}$ is stable under derived base change (cfr with univ. property in classical sense)

(iii) There is a canonical closed immersion $\mathbb{P}_Z(N_{Z/\mathcal{X}}) \hookrightarrow Bl_{Z/\mathcal{X}}$

(cfr blow ups along regular immersions).

The scheme $\mathbb{P}_Z(N_{Z/\mathcal{X}})$ is the universal virtual Cartier divisor lying over $(Z, \mathcal{X})$.

(iv) $\pi: Bl_{Z/\mathcal{X}} \to \mathcal{X}$ is proper, and $Bl_{Z/\mathcal{X}} - \mathbb{P}_Z(N_{Z/\mathcal{X}}) \cong \mathcal{X} - Z$.

(v) if $Z, \mathcal{X}$ are classical, then $Bl_{Z/\mathcal{X}}$ is classical and coincides with $Bl^{cl}_{Z/\mathcal{X}}$ ($= Bl_{Z^{cl}/\mathcal{X}^{cl}}$ = classical Blow up).

<u>Cor</u>: $Z \to X$ regular closed immersion of Classical schemes.

$\forall\, S \to X$ classical, the sets $\mathrm{Hom}_{Sch/_X}(S, Bl^{cl}_{Z/X}) \simeq \{$virtual Cartier divisors lying over $(Z,X)\}$ are in bijection.

"Universal property for the classical blow up".

<u>Ex</u>: $X = \mathrm{Spec}(A)$ classical (Noetherian) scheme.

$Z = \mathrm{Spec}(A/_I)$, $I = (f_1, \dots, f_r)$ ideal. $\leadsto$ $A/\!/(f_1, \dots, f_r) = A \otimes^{\mathbb{L}}_{\mathbb{Z}[T_i]} \mathbb{Z}[T_i \dots]/_{(T_i \dots)}$

$\mathrm{Spec}(A/\!/(f_1, \dots, f_r)) \hookrightarrow \mathrm{Spec}(A)$ quasi smooth for <u>any</u> $I$.

$Bl_{(A/\!/f_1, \dots, f_r)/(A)} \overset{\cong}{\uparrow} \mathrm{Spec}(A) \times^{\mathbb{R}}_{\mathbb{A}^n} (Bl_{\{0\}/\mathbb{A}^n})$

This is the construction

The proof of the existence of $Bl_{Z/X}$ goes by glueing local charts.

<u>Special case</u>: $Y := Bl_{\{0\}/\mathbb{A}^n}$.

Affine chart: $1 \leq k \leq n$, $A_k = \mathbb{Z}[T_1/T_k, \ldots, T_n/T_k, T_k]$

$$\leadsto \begin{tikzcd} D_k = \mathrm{Spec}(A_k/(T_k)) \arrow[r, hook] \arrow[d] & \widetilde{Y}_k = \mathrm{Spec}(A_k) \arrow[d] \\ \{0\} = \mathrm{Spec}(\mathbb{Z}[T_1, \ldots, T_n]/(T_1, \ldots, T_n)) \arrow[r, hook] & \mathbb{A}^n \end{tikzcd}$$

$D_k$ is a virtual Cartier divisor lying over $(\mathbb{A}^n, \{0\})$.

By univ. property of $Y = Bl_{\{0\}/\mathbb{A}^n}$, $\exists$ map $\widetilde{Y}_K \longrightarrow Y$ corresponding to $D_K$

<u>Def</u>: $Y_K$ = derived substack of $Y$ given by "the image of $\widetilde{Y}_K$":

More precisely, consider the subsheaf of $Y_K$ given by:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circledast"] & S \arrow[d] \\ D_K = \mathrm{Spec}(A_K/T_K) \arrow[r] \arrow[d] & \mathrm{Spec}(A_K) \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

i.e. subsheaf given by morphisms factoring though $\mathrm{Spec}(A_K) \to \mathbb{A}^n$ such that $\circledast$ is homotopy Cartesian.

<u>Rmk</u>: If $S = \mathrm{Spec}(R)$, $f : S \longrightarrow \mathbb{A}^n \leftrightarrow (f_1, \ldots, f_n) \in R$

$S \to \mathrm{Spec}(A_K) \to \mathrm{Spec}(\mathbb{Z}[T_1, \ldots, T_n])$

$(f_1, \ldots, f_n) \quad \leftrightarrow \quad \mathbb{Z}[T_1, \ldots, T_n] \to \mathbb{Z}[T_1/T_K, \ldots, T_K] \to R$

$T_i \longmapsto f_i$

Then $D = \mathrm{Spec}(R/\!/(f_K))$.

<u>Claim</u>: $Y_K \cong \mathrm{Spec}(A_k)$.

<u>Lemma</u>: The family $(Y_K \hookrightarrow Y)_k$ defines a Zariski atlas for the derived stack $Y$ (i.e. $\coprod Y_K \to Y$ is an effective epi of sheaves of spaces).