Lecture 5

K-theory of derived schemes II

In this lecture we will introduce the *additive K-theory* of a derived scheme X, and compare it with the perfect complex K-theory in the affine case. In the second part we will discuss projective spaces over derived schemes in some more detail.

Definition 1. An ∞-category $\mathbf{C}$ is *additive* if for any objects x, y, the canonical map $x \sqcup y \to x \times y$ is invertible.

In this case we write $x \oplus y$ for the object $x \sqcup y \simeq x \times y$. Recall that any stable ∞-category is additive.

Example 2. For a simplicial commutative ring R, consider the full subcategory $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}} \subset \mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}}$ spanned by finitely generated projective R-modules. Then the ∞-category $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}}$ is additive but not stable.

Construction 3 (Additive K-theory)**.** Let $\mathbf{C}$ be an additive ∞-category. The abelian group $\mathrm{K}_0^{\oplus}(\mathbf{C})$ is the free abelian group generated by objects of $\mathbf{C}$, modulo the relation identifying $[x \oplus y] = [x] + [y]$ for any two objects x and y.

Remark 4. Note that the construction $\mathrm{K}_0^{\oplus}(\mathbf{C})$ only depends on the homotopy category $\mathrm{Ho}(\mathbf{C})$ (which is also additive).

Example 5. Let R be a simplicial commutative ring. Then the additive K-theory of R is defined as $\mathrm{K}_0^{\oplus}(\mathrm{R}) := \mathrm{K}_0^{\oplus}(\mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}})$.

Example 6. Let X be a derived scheme. A quasi-coherent sheaf $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$ is *locally free of finite rank* if there exists a Zariski covering $(\mathrm{X}_\alpha \hookrightarrow \mathrm{X})_\alpha$ such that there are isomorphisms $\mathcal{F}|_{\mathrm{X}_\alpha} \simeq \mathcal{O}_{\mathrm{X}_\alpha}^{\oplus n_\alpha}$ for some $n_\alpha \geqslant 0$. Let $\mathrm{Qcoh}(\mathrm{X})^{\mathrm{locfr}} \subset \mathrm{Qcoh}(\mathrm{X})$ denote the full subcategory of locally free sheaves of finite rank. The additive K-theory of X is defined as $\mathrm{K}_0^{\oplus}(\mathrm{X}) := \mathrm{K}_0(\mathrm{Qcoh}(\mathrm{X})^{\mathrm{locfr}})$. We have $\mathrm{K}_0^{\oplus}(\mathrm{Spec}(\mathrm{R})) \simeq \mathrm{K}_0^{\oplus}(\mathrm{R})$.

Theorem 7. *Let* R *be a simplicial commutative ring. Then there is a canonical isomorphism*

$$\iota : \mathrm{K}_0^{\oplus}(\mathrm{R}) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{R})$$

of abelian groups. Moreover, this isomorphism is (covariantly) functorial in R*.*

Proof. It is clear that the inclusion $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}} \hookrightarrow \mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}}$ induces a homomorphism $\iota : \mathrm{K}_0^{\oplus}(\mathrm{R}) \to \mathrm{K}_0(\mathrm{R})$. Since the conditions "perfect" and "projective finitely generated" are stable under extensions of scalars $\mathrm{M} \mapsto \mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}'$, the map ι is functorial. We will construct an inverse map χ. Let $[\mathrm{M}] \in \mathrm{K}_0(\mathrm{R})$ be the class of a perfect R-module M. Replacing M with some shift $\mathrm{M}[k]$, we can assume that M is connective (since $[\mathrm{M}[k]] = (-1)^k[\mathrm{M}]$ in $\mathrm{K}_0(\mathrm{R})$). Recall that M is of tor-amplitude $\leqslant n$ for some n. If $n = 0$, then we saw last time that M belongs to $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}}$, so we set $\chi[\mathrm{M}] := [\mathrm{M}]$. In general, we know that $\pi_0(\mathrm{M})$ is of finite presentation as a $\pi_0(\mathrm{R})$-module, so we can find a map $\phi : \mathrm{R}^{\oplus m} \to \mathrm{M}$ that is surjective on π_0. Then we have an exact triangle

$$\mathrm{F} \to \mathrm{R}^{\oplus m} \xrightarrow{\phi} \mathrm{M}$$

where F is the fibre of ϕ. We claim that F is of tor-amplitude $\leqslant n-1$. Indeed this follows immediately from the long exact sequence

$$\cdots \to \pi_{n+1}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{N}) \to \pi_n(\mathrm{F} \otimes_{\mathrm{A}} \mathrm{N}) \to \pi_n(\mathrm{R}^{\oplus m} \otimes_{\mathrm{R}} \mathrm{N}) \to \pi_n(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}) \to \cdots$$

associated to the exact triangle $\mathrm{F} \otimes_{\mathrm{R}} \mathrm{N} \to \mathrm{R}^{\oplus m} \otimes_{\mathrm{R}} \mathrm{N} \to \mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}$, where we note that, if N is discrete, then so is $\mathrm{R}^{\oplus m} \otimes_{\mathrm{R}} \mathrm{N} \simeq \mathrm{N}^{\oplus m}$. Now we have $[\mathrm{M}] = [\mathrm{R}^{\oplus m}] - [\mathrm{F}]$ in $\mathrm{K}_0(\mathrm{R})$, so we set

$$\chi[\mathrm{M}] = \chi[\mathrm{R}^{\oplus m}] - \chi[\mathrm{F}] = [\mathrm{R}^{\oplus m}] - \chi[\mathrm{F}],$$

where $\chi[\mathrm{F}]$ is defined by recursion. It is easy to check that this is independent of the chosen ϕ and m, that it indeed induces a well-defined map $\chi : \mathrm{K}_0(\mathrm{R}) \to \mathrm{K}_0^{\oplus}(\mathrm{R})$, and that the latter is inverse to ι. □

As an application of this comparison result, we can deduce the following "derived nil-invariance" property for K_0:

Theorem 8. *Let* R *be a simplicial commutative ring. Then the canonical homomorphism*

$$\mathrm{K}_0(\mathrm{R}) \to \mathrm{K}_0(\pi_0(\mathrm{R}))$$

is bijective.

Proof. By Theorem 7 we reduce to showing that

$$\mathrm{K}_0^{\oplus}(\mathrm{R}) \to \mathrm{K}_0^{\oplus}(\pi_0(\mathrm{R}))$$

is bijective, where the map is induced by the assignment $\mathrm{M} \mapsto \mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})$. Since every $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}}$ is flat, this is identified with $\mathrm{M} \mapsto \pi_0(\mathrm{M})$. Therefore the claim follows from the following fact, which we leave as an exercise. □

Exercise 9. The functor $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}} \to \mathrm{Mod}_{\pi_0(\mathrm{R})}^{\mathrm{proj}}$ induces an equivalence on homotopy categories.

Remark 10. We will not discuss them in this course, but the *higher* K-groups $\mathrm{K}_i(\mathrm{R})$ do see the difference between R and $\pi_0(\mathrm{R})$ (starting from $i \geqslant 2$). In fact, one can show that if $\mathrm{K}_i(\mathrm{R}) \to \mathrm{K}_i(\pi_0(\mathrm{R}))$ are bijective for all $i \geqslant 2$, then $\mathrm{R} \simeq \pi_0(\mathrm{R})$.

We will now switch topics. An important ingredient in the Grothendieck–Riemann–Roch theorem is the projective bundle formula, which describes the K-theory of a projective bundle. In order to prove it we will need a more detailed discussion of projective bundles over derived schemes.

Let X be a derived scheme and $\mathcal{E} \in \mathrm{Qcoh}(\mathrm{X})^{\mathrm{locfr}}$. Recall that the projective bundle $p : \mathbf{P}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ classifies pairs $(\mathcal{L}, u)$, where $\mathcal{L}$ is a locally free sheaf of rank one, and $u : p^*(\mathcal{E}) \to \mathcal{L}$ is surjective on π_0. The universal such pair is denoted $(\mathcal{O}(1), u_{\mathrm{univ}})$. We let $\mathcal{O}(m) := \mathcal{O}(1)^{\otimes m}$ for each integer $m \in \mathbf{Z}$.

Let $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ and $\mathcal{E} = \mathcal{O}_{\mathrm{X}}^{\oplus n+1}$. In this case we can give an explicit combinatorial description of $\mathbf{P}_{\mathrm{X}}(\mathcal{E}) = \mathbf{P}_{\mathrm{R}}^n$.

Construction 11. Let $[n]$ denote the set $\{0, 1, \ldots, n\}$. For each subset $\mathrm{I} \subset [n]$, consider the additive commutative monoid $\mathrm{M}_{\mathrm{I}} \subset \mathbf{Z}^{n+1}$ of tuples $(k_0, \ldots, k_n)$ with $k_0 + \cdots + k_n = 0$ and $k_i \geqslant 0$ for $i \notin \mathrm{I}$. The associated monoid algebra $\mathrm{R}[\mathrm{M}_{\mathrm{I}}]$ is the subalgebra of $\mathrm{R}[\mathbf{Z}^{n+1}] = \mathrm{R}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by x_j/x_i for $i \in \mathrm{I}$, $j \in [n]$.

As I varies, we get a diagram $\mathrm{I} \mapsto \mathrm{R}[\mathrm{M}_{\mathrm{I}}]$. For any inclusion $\mathrm{I} \subset \mathrm{J}$ with I nonempty, the transition map $\mathrm{R}[\mathrm{M}_{\mathrm{I}}] \to \mathrm{R}[\mathrm{M}_{\mathrm{J}}]$ is a localization at x_j/x_i for $j \in \mathrm{J}$ and $i \in \mathrm{I}$. In particular, the morphisms $\mathrm{Spec}(\mathrm{R}[\mathrm{M}_{\mathrm{J}}]) \to \mathrm{Spec}(\mathrm{R}[\mathrm{M}_{\mathrm{I}}])$ are open immersions.

Theorem 12. *There is an isomorphism*

$$\varinjlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{Spec}(\mathrm{R}[\mathrm{M}_{\mathrm{I}}]) \to \mathbf{P}_{\mathrm{R}}^n$$

in the ∞-category of derived stacks.

This gives the following combinatorial description of the category of quasi-coherent sheaves:

Corollary 13. *There is an equivalence of ∞-categories*

$$\mathrm{Qcoh}(\mathbf{P}_{\mathrm{R}}^n) \xrightarrow{\sim} \varprojlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{Mod}_{\mathrm{R}[\mathrm{M}_{\mathrm{I}}]}.$$

In terms of this equivalence, the line bundles $\mathcal{O}(m)$ can be described as follows.

Construction 14. Fix an integer $m \in \mathbf{Z}$. For each subset $\mathrm{I} \subset [n]$, let $\mathrm{M_I}(m) \subset \mathbf{Z}^{n+1}$ denote the submonoid of tuples $(k_0, \dots, k_n)$ such that $k_0 + \cdots + k_n = m$ and $k_i \geqslant 0$ for $i \notin \mathrm{I}$. Then the monoid algebra $\mathrm{R}[\mathrm{M_I}(m)]$ is a free $\mathrm{R}[\mathrm{M_I}]$-module of rank one, and we have:

$$\Gamma(\mathrm{Spec}(\mathrm{R}[\mathrm{M_I}]), \mathcal{O}(m)) \simeq \mathrm{R}[\mathrm{M_I}(m)]$$

for each nonempty subset $\mathrm{I} \subset [n]$.

We'll end today's lecture by calculating the space of global sections $\Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m))$ explicitly.

Construction 15. Given a tuple $k = (k_0, \dots, k_n) \in \mathbf{Z}^{n+1}$ with $k_i \geqslant 0$ for each i, set $m = k_0 + \cdots + k_n$. Then we can view k as an element of $\mathrm{M_I}(m)$ for any subset $\mathrm{I} \subset [n]$. This gives rise to R-linear maps $\mathrm{R} \to \mathrm{R}[\mathrm{M_I}(m)]$, compatible as I varies, and hence an R-linear map

$$x^k : \mathrm{R} \to \varprojlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{R}[\mathrm{M_I}(m)] \simeq \Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m)).$$

We can view x^k as a global section of the line bundle $\mathcal{O}(m)$.

Theorem 16 (Serre)**.** *Let* $\mathrm{R} \in \mathrm{SCRing}$*. For each* $n \geqslant 0$ *and each* $m \in \mathbf{Z}$*, the* R*-module* $\Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m))$ *can be described as follows.*

- *If* $m \geqslant 0$*, then* $\Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m))$ *is free of rank* $\binom{m+n}{n}$*, generated by the global sections* x^k*.*
- *If* $m < 0$*, then* $\Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m))$ *is a direct sum of* $\binom{-m-1}{n}$ *copies of* $\mathrm{R}[-n]$*. In particular, it is zero if* $-1 \geqslant m \geqslant -n$*.*

Proof (Lurie). We have equivalences

$$\begin{aligned}\Gamma(\mathbf{P}^n_{\mathrm{R}}, \mathcal{O}(m)) &\simeq \varprojlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{R}[\mathrm{M_I}(m)] \\ &\simeq \varprojlim_{\varnothing \neq \mathrm{I} \subset [n]} \bigoplus_{k \in \mathrm{M}_{[n]}(m)} \mathrm{R}[\mathrm{M_I}(k)] \\ &\simeq \bigoplus_{k \in \mathrm{M}_{[n]}(m)} \varprojlim_{\varnothing \neq \mathrm{I} \subset [n]} \mathrm{R}[\mathrm{M_I}(k)]\end{aligned}$$

where we have written $\mathrm{M_I}(k) := \mathrm{M_I} \cap \{k\}$; in other words, $\mathrm{M_I}(k)$ is either empty (if $k_i < 0$ for some $i \notin \mathrm{I}$), or the singleton $\{k\}$. For each $k \in \mathrm{M}_{[n]}(m)$, let λ_k denote the functor $\mathrm{I} \mapsto \mathrm{R}[\mathrm{M_I}(k)]$ (on the poset P of nonempty subsets of $[n]$), so that it suffices to compute $\varprojlim(\lambda_k)$ for any fixed k. Consider the canonical exact triangle

$$\lambda_k \xrightarrow{u} \mathrm{R} \to \mathrm{Cofib}(u)$$

of functors on P (where R is viewed as the constant diagram valued in R). When we restrict to the subset $\mathrm{Q} \subset \mathrm{P}$ of subsets $\mathrm{I} \subset [n]$ such that $\mathrm{M_I}(k) = \varnothing$, this takes the form

$$0 \xrightarrow{u|_{\mathrm{Q}}} \mathrm{R} \xrightarrow{\sim} \mathrm{Cofib}(u)|_{\mathrm{Q}}.$$

But $\mathrm{Cofib}(u)$ is clearly a right Kan extension of its restriction to Q, so that

$$\varprojlim_{\mathrm{I} \in \mathrm{P}} \mathrm{Cofib}(u)(\mathrm{I}) \simeq \varprojlim_{\mathrm{I} \in \mathrm{Q}} \mathrm{R}.$$

Thus we get:

$$\varprojlim \lambda_k \simeq \mathrm{Fib}(\varprojlim(\mathrm{R}) \to \varprojlim \mathrm{Cofib}(u)) \simeq \mathrm{Fib}(\mathrm{R} \to \varprojlim_{\mathrm{I} \in \mathrm{Q}} \mathrm{R}).$$

We therefore need to understand how the shape of (the nerve of) Q varies depending on the value of k.

- Suppose that $k_i \geqslant 0$ for all i. Then Q is empty, so $\varprojlim(\lambda_k) = \mathrm{R}$.

- Suppose that $k_i < 0$ for some but not all i. Then one can show that the simplicial set $\mathrm{N(Q)}$ is (weakly) contractible, so that $\varprojlim(\lambda_k) \simeq 0$.
- Suppose that $k_i < 0$ for all i. In this case one can show that $\mathrm{N(Q)}$ is weakly equivalent to $\partial\Delta^n$ so that $\varprojlim(\lambda_k) \simeq \mathrm{R}[-n]$.

It remains to count the possible contributions depending on the value of m. For example, if $m \geqslant 0$ then no k satisfies the third case, there is no contribution from the second case, and from the first case we get copies of R indexed by the set $\mathrm{M}_{\varnothing}(m)$. □