Lecture 4
K-theory of derived schemes I

In this lecture we will introduce the K-theory (K_0) of a stable ∞-category, and begin studying the example $K_0(\mathrm{Perf}(X))$ where X is a derived scheme. We start by briefly recalling the definition of a stable ∞-category.

Definition 1. Let $\mathbf{C}$ be an ∞-category. A *zero object* 0 is an object that is both *initial* and *final*, so that the spaces

$$\mathrm{Maps}_{\mathbf{C}}(0, x), \quad \mathrm{Maps}_{\mathbf{C}}(x, 0)$$

are (weakly) contractible for all objects $x \in \mathbf{C}$.

Lemma 2. *The space of zero objects in an ∞-category $\mathbf{C}$ is either empty or contractible.*

If $\mathbf{C}$ admits a zero object, it is called a *pointed* ∞-category. The zero object is then unique in the ∞-categorical sense, i.e., it is unique up to a contractible space of choices.

Remark 3. If $\mathbf{C}$ is pointed, then there is always a zero map $0 : x \to y$ between any two objects, which is by definition the composite of the two unique morphisms $x \to 0$ and $0 \to y$.

Definition 4. Let $\mathbf{C}$ be a pointed ∞-category. A *triangle* in $\mathbf{C}$ is a diagram $x' \xrightarrow{f} x \xrightarrow{g} x''$ together with a null-homotopy of $g \circ f$, i.e., an isomorphism $g \circ f \simeq 0$ in the ∞-groupoid $\mathrm{Maps}_{\mathbf{C}}(x', x'')$. Equivalently, it is the datum of a square

$$\begin{array}{ccc} x' & \xrightarrow{f} & x \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & x'' \end{array}$$

and a 2-simplex witnessing its commutativity.

Definition 5. A triangle is called a *fibre sequence* if the above square is (homotopy) cartesian. Dually, a *cofibre sequence* is a triangle such that the above square is (homotopy) cocartesian.

Suppose that $\mathbf{C}$ is pointed and admits finite limits. Then given any morphism $f : x \to y$, we can consider the pullback of the diagram

$$\begin{array}{ccc} & & x \\ & & \downarrow f \\ 0 & \longrightarrow & y \end{array}$$

and call this the (homotopy) *fibre* of f, denoted $\mathrm{Fib}(f)$. By construction we have a fibre sequence $\mathrm{Fib}(f) \to x \xrightarrow{f} y$ for any morphism f. Dually, we have a notion of *cofibre*, denoted $\mathrm{Cofib}(f)$, fitting in a cofibre sequence $x \xrightarrow{f} y \to \mathrm{Cofib}(f)$.

Example 6. For any object $x \in \mathbf{C}$, we write $x[1] = \mathrm{Cofib}(x \to 0)$ (when this cofibre exists). Dually we write $x[-1] = \mathrm{Fib}(0 \to x)$ (again when it exists).

Definition 7. Let $\mathbf{C}$ be a pointed ∞-category that admits finite limits and colimits. We say that it is *stable* if it satisfies one of the following equivalent conditions:

(a) The functors $x \mapsto x[1]$ and $x \mapsto x[-1]$ define mutually inverse auto-equivalences of $\mathbf{C}$.

(b) Any given triangle in $\mathbf{C}$ is a fibre sequence iff it is a cofibre sequence.

(c) Any given commutative square in $\mathbf{C}$ is cartesian iff it is cocartesian.

In a stable ∞-category, we will simply use the term *exact triangle* to refer to triangles that are fibre sequences, or equivalently cofibre sequences.

Exercise 8. Let $\mathbf{C}$ be a stable ∞-category. Then $\mathbf{C}$ is additive, i.e., the canonical morphisms $x \sqcup y \to x \times y$ are invertible for all objects $x, y \in \mathbf{C}$. Equivalently, the homotopy category $\mathrm{Ho}(\mathbf{C})$ is additive.

We will use the notation $x \oplus y$ for the object $x \sqcup y \simeq x \times y$ and call this the *direct sum*.

Remark 9. Let $\mathbf{C}$ be a stable ∞-category. Then the homotopy category $\mathrm{Ho}(\mathbf{C})$ admits a structure of *triangulated category*:

- It is additive, by Exercise 8.
- The shift functor on $\mathrm{Ho}(\mathbf{C})$ is induced by the auto-equivalence $x \mapsto x[1]$ on $\mathbf{C}$.
- The distinguished triangles are those isomorphic to triangles in the image of the localization functor $\mathbf{C} \to \mathrm{Ho}(\mathbf{C})$.

Example 10. For any commutative ring R, consider the ∞-category $\mathrm{Mod}_{\mathrm{R}}$, by definition the dg-nerve of the dg-category $\underline{\mathrm{D}}(\mathrm{R})$. The ∞-category $\mathrm{Mod}_{\mathrm{R}}$ is stable.

Example 11. More generally, let R be a simplicial commutative ring. Then we defined $\mathrm{Mod}_{\mathrm{R}}$ as the dg-nerve of the dg-category of cofibrant dg-modules (over the normalized chain complex of R). The ∞-category $\mathrm{Mod}_{\mathrm{R}}$ is stable.

Definition 12. A functor $\mathbf{C} \to \mathbf{D}$ between stable ∞-categories is called *exact* if it commutes with finite limits, or equivalently with finite colimits.

Let $\mathbf{D}$ be a stable ∞-category. A *stable subcategory* $\mathbf{C} \subset \mathbf{D}$ is a full subcategory whose objects are closed under finite (co)limits (formed in $\mathbf{D}$).

Example 13. For any simplicial commutative ring R, the full subcategory $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}} \subset \mathrm{Mod}_{\mathrm{R}}$ of perfect R-modules is a *stable* subcategory.

Construction 14. Let $\mathbf{C}$ be an essentially small stable ∞-category. The abelian group $\mathrm{K}_0(\mathbf{C})$ is freely generated by the objects of $\mathbf{C}$, modulo the relations $[x] = [x'] + [x'']$ for all exact triangles $x' \to x \to x''$ in $\mathbf{C}$.

Remark 15. The group $\mathrm{K}_0(\mathbf{C})$ can be defined only using the homotopy category $\mathrm{Ho}(\mathbf{C})$ (equipped with its triangulated structure).

Example 16.

- Let $x \simeq y$ be an isomorphism in $\mathbf{C}$. Then the exact triangle $x \xrightarrow{\sim} y \to 0$ gives the relation $[x] = [y]$ in $\mathrm{K}_0(\mathbf{C})$.
- Let x be an object in $\mathbf{C}$. Then the exact triangle $x \to 0 \to x[1]$ gives $[x[1]] = -[x]$ in $\mathrm{K}_0(\mathbf{C})$.
- Let x, y be objects in $\mathbf{C}$. Then the exact triangle $x \to x \oplus y \to y$ gives $[x] + [y] = [x \oplus y]$ in $\mathrm{K}_0(\mathbf{C})$.

Remark 17. Note that any element of the group $\mathrm{K}_0(\mathbf{C})$ can be represented as the class $[x]$ of some object $x \in \mathbf{C}$. This is not true for the K-theory of abelian or exact categories, for example.

Example 18. Let R be a simplicial commutative ring. Then $\mathrm{K}_0(\mathrm{R})$ is by definition the abelian group $\mathrm{K}_0(\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}})$.

Example 19. More generally, let X be a derived scheme. Then we defined an ∞-category $\mathrm{Perf}(\mathrm{X})$ of perfect complexes on X, a full subcategory of the ∞-category $\mathrm{Qcoh}(\mathrm{X})$ of quasi-coherent sheaves. This is stable, and we set $\mathrm{K}_0(\mathrm{X}) = \mathrm{K}_0(\mathrm{Perf}(\mathrm{X}))$. By construction, $\mathrm{K}_0(\mathrm{Spec}(\mathrm{R})) \simeq \mathrm{K}_0(\mathrm{R})$.

Example 20 (Eilenberg swindle)**.** For any derived scheme X, the group $\mathrm{K}_0(\mathrm{Qcoh}(\mathrm{X}))$ is zero. More generally, let $\mathbf{C}$ be a stable ∞-category admitting *infinite* coproducts. Then for any object $x \in \mathbf{C}$, the isomorphism

$$x \oplus \bigoplus_{n \geqslant 1} x \simeq \bigoplus_{n \geqslant 0} x$$

gives the relation $[x]=0$ in $\mathrm{K}_0(\mathbf{C})$.

Construction 21. Let $f:\mathrm{Y}\to\mathrm{X}$ be a morphism of derived schemes. Then the exact functor $f^*:\mathrm{Qcoh}(\mathrm{X})\to\mathrm{Qcoh}(\mathrm{Y})$ preserves perfect complexes, so we get an induced homomorphism

$$f^*:\mathrm{K}_0(\mathrm{X})\to\mathrm{K}_0(\mathrm{Y}).$$

Remark 22. If $f:\mathrm{Y}\to\mathrm{X}$ is proper, of finite presentation, and of finite tor-amplitude, then it is a theorem of Lurie that the functor $f_*:\mathrm{Qcoh}(\mathrm{Y})\to\mathrm{Qcoh}(\mathrm{X})$ also preserves perfect complexes. Therefore there is also covariant functoriality (Gysin maps)

$$f_*:\mathrm{K}_0(\mathrm{Y})\to\mathrm{K}_0(\mathrm{X}).$$

Our next goal will be to give a simpler description of the K-theory of X when X is affine, using vector bundles instead of perfect complexes. For this we will need to make a bit of a digression.

The *t-structure* on Mod_R is a useful organizational tool:

Proposition 23. *Let* R *be a simplicial commutative ring.*

(i) *Let* $(\mathrm{Mod}_\mathrm{R})_{\geqslant 0}$ *denote the full subcategory of* Mod_R *spanned by* connective R*-modules, satisfying the condition that* $\pi_i(\mathrm{M}):=\mathrm{H}^{-i}(\mathrm{M})=0$ *for* $i<0$*. The inclusion* $(\mathrm{Mod}_\mathrm{R})_{\geqslant 0}\hookrightarrow\mathrm{Mod}_\mathrm{R}$ *admits a right adjoint* $\mathrm{M}\mapsto\tau_{\geqslant 0}(\mathrm{M})$*.*

(ii) *Dually let* $(\mathrm{Mod}_\mathrm{R})_{\leqslant 0}$ *denote the full subcategory of* R*-modules such that* $\pi_i(\mathrm{M})=0$ *for* $i>0$*. The inclusion* $(\mathrm{Mod}_\mathrm{R})_{\leqslant 0}\hookrightarrow\mathrm{Mod}_\mathrm{R}$ *admits a left adjoint* $\mathrm{M}\mapsto\tau_{\leqslant 0}(\mathrm{M})$*.*

(iii) *These two subcategories define a canonical t-structure* $((\mathrm{Mod}_\mathrm{R})_{\geqslant 0},(\mathrm{Mod}_\mathrm{R})_{\leqslant 0})$ *on* Mod_R*.*

For any integer n, we will also write $(\mathrm{Mod}_\mathrm{R})_{\geqslant n}:=(\mathrm{Mod}_\mathrm{R})_{\geqslant 0}[n]$ and $(\mathrm{Mod}_\mathrm{R})_{\leqslant n}:=(\mathrm{Mod}_\mathrm{R})_{\leqslant 0}[n]$. We have functors $\tau_{\geqslant n}:\mathrm{Mod}_\mathrm{R}\to(\mathrm{Mod}_\mathrm{R})_{\geqslant n}$ and $\tau_{\leqslant n}:\mathrm{Mod}_\mathrm{R}\to(\mathrm{Mod}_\mathrm{R})_{\leqslant n}$, right and left adjoints to the respective inclusions.

Exercise 24. Let $(\mathrm{Mod}_\mathrm{R})^\heartsuit$ denote the heart of the t-structure, defined as the intersection of the two categories $(\mathrm{Mod}_\mathrm{R})_{\geqslant 0}$ and $(\mathrm{Mod}_\mathrm{R})_{\leqslant 0}$. The assignment $\mathrm{M}\mapsto\pi_0(\mathrm{M})$ defines a functor $\mathrm{Mod}_\mathrm{R}\to(\mathrm{Mod}_{\pi_0(\mathrm{R})})^\heartsuit$, and induces an equivalence

$$(\mathrm{Mod}_\mathrm{R})^\heartsuit\simeq(\mathrm{Mod}_{\pi_0\mathrm{R}})^\heartsuit.$$

Proposition 25. *The t-structure on* Mod_R *is left- and right-complete. In particular, for any* R*-module* M *we have functorial isomorphisms*

$$\mathrm{M}\xrightarrow{\sim}\varprojlim_n\tau_{\leqslant n}(\mathrm{M}),$$

$$\varinjlim_n\tau_{\geqslant n}(\mathrm{M})\xrightarrow{\sim}\mathrm{M}.$$

Recall the following definitions:

Definition 26. An R-module M is *finitely generated projective* if it is a direct summand of a free module $\mathrm{R}^{\oplus n}$.

We let $\mathrm{Mod}_\mathrm{R}^{\mathrm{proj}}\subset\mathrm{Mod}_\mathrm{R}$ denote the full subcategory of finitely generated projective R-modules.

Exercise 27. An R-module M is finitely generated projective iff it is *locally free of finite rank*; that is, if there exists a Zariski covering $(\mathrm{R}\to\mathrm{R}_\alpha)_\alpha$ such that each $\mathrm{M}\otimes_\mathrm{R}\mathrm{R}_\alpha$ is isomorphic to $\mathrm{R}^{\oplus n_\alpha}$ for some n_α.

We have seen that any finitely generated projective R-module M gives rise to a vector bundle $\mathrm{Spec}(\mathrm{Sym}_\mathrm{R}(\mathrm{M}))$ over $\mathrm{Spec}(\mathrm{R})$. In order to relate the K-theory of perfect modules with that of locally free modules, we would like to define a filtration on $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}}$ whose first piece is the subcategory of locally frees. We begin by discussing finiteness conditions on R-modules in more detail.

Proposition 28. *Let* $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}}$. *Then we have:*

(i) M *is* n*-connective, i.e.* $\mathrm{M} \in (\mathrm{Mod}_{\mathrm{R}})_{\geqslant n}$*, for some* n*. In other words,* M *is bounded below.*

(ii) *Let* $\pi_n(\mathrm{M})$ *be the lowest nonvanishing homotopy group. Then* $\pi_n(\mathrm{M})$ *is of finite presentation as a* $\pi_0(\mathrm{R})$*-module.*

Proof.
(i) Recall that the perfect R-modules coincide with the compact objects of $\mathrm{Mod}_{\mathrm{R}}$. Therefore, writing M as a filtered colimit of its n-connective covers $\tau_{\geqslant n}(\mathrm{M})$, we have:

$$\mathrm{Maps}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M},\mathrm{M}) \simeq \varinjlim_{n} \mathrm{Maps}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M}, \tau_{\geqslant n}(\mathrm{M})).$$

It follows that the identity morphism $\mathrm{M} \to \mathrm{M}$ factors through $\tau_{\geqslant n}(\mathrm{M})$ for some M, which means that M is a direct summand of $\tau_{\geqslant n}(\mathrm{M})$. This clearly implies that $\pi_i(\mathrm{M}) = 0$ for $i < n$.

(ii) By (i), we can replace M by some $\mathrm{M}[n]$ to make it connective. The assertion that $\pi_0(\mathrm{M})$ is of finite presentation as a $\pi_0(\mathrm{R})$-module is equivalent to the assertion that $\pi_0(\mathrm{M})$ is compact in $(\mathrm{Mod}_{\pi_0(\mathrm{R})})^{\heartsuit}$, i.e., that the assignment $\mathrm{N} \mapsto \mathrm{Maps}_{(\mathrm{Mod}_{\pi_0(\mathrm{R})})^{\heartsuit}}(\pi_0(\mathrm{M}),\mathrm{N})$ preserves filtered colimits when viewed as a functor $(\mathrm{Mod}_{\pi_0(\mathrm{R})})^{\heartsuit} \to \mathrm{Set}$. But we have functorial equivalences

$$\mathrm{Maps}_{(\mathrm{Mod}_{\pi_0(\mathrm{R})})^{\heartsuit}}(\pi_0(\mathrm{M}),\mathrm{N}) \simeq \mathrm{Maps}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M},\mathrm{N}),$$

where N is viewed as an R-module via restriction of scalars along $\mathrm{R} \to \pi_0(\mathrm{R})$. Thus the claim follows by compactness of M in $\mathrm{Mod}_{\mathrm{R}}$. ☐

We next give another equivalent characterization of locally free modules in $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}}$.

Definition 29. A connective R-module $\mathrm{M} \in (\mathrm{Mod}_{\mathrm{R}})_{\geqslant 0}$ is *flat* if it satisfies one of the following equivalent conditions:

- *(i) The* $\pi_0(\mathrm{R})$*-module* $\pi_0(\mathrm{M})$ *is flat, and* $\pi_i(\mathrm{M}) \simeq \pi_i(\mathrm{R}) \otimes_{\pi_0(\mathrm{R})} \pi_0(\mathrm{M})$ *for all* i*.*
- (ii) The functor $\mathrm{N} \mapsto \mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}$ preserves discrete R-modules.
- *(iii) The functor* $\mathrm{N} \mapsto \mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}$ *is left t-exact; that is, it sends* $(\mathrm{Mod}_{\mathrm{R}})_{\leqslant 0}$ *into* $(\mathrm{Mod}_{\mathrm{R}})_{\leqslant 0}$*.*

Proposition 30. *Let* M *be a connective perfect* R*-module. Then* M *is flat iff* M *is finitely generated projective.*

Proof. Suppose that M is finitely generated free. Then it is clearly flat, since if N is discrete, then so is $\mathrm{R}^{\oplus n} \otimes_{\mathrm{R}} \mathrm{N} \simeq \mathrm{N}^{\oplus n}$. In general, if M is finitely generated projective, we can write $\mathrm{M} \oplus \mathrm{P} \simeq \mathrm{R}^{\oplus n}$ for some $\mathrm{P} \in \mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}}$ and integer n. Then for any discrete N we have $(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}) \oplus (\mathrm{P} \otimes_{\mathrm{R}} \mathrm{N}) \simeq \mathrm{N}^{\oplus n}$, which shows that $\pi_i(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{N})$ is a direct summand of zero for $i > 0$.

In the other direction, suppose that M is perfect and flat. By perfectness, we know that $\pi_0(\mathrm{M})$ is of finite presentation as a $\pi_0(\mathrm{R})$-module. Therefore we can find a morphism $\phi : \mathrm{R}^{\oplus n} \to \mathrm{M}$ that is surjective on π_0. By flatness of M, $\pi_0(\mathrm{M})$ is also flat, and hence projective, so that ϕ admits a splitting on π_0. Hence the claim follows from the following exercise. ☐

Exercise 31. Let M be a flat R-module. Then the following conditions are equivalent:

(i) M is *projective* in the sense that for any map of connective R-modules $\mathrm{N}_1 \to \mathrm{N}_2$ that is surjective on π_0, any map $\mathrm{M} \to \mathrm{N}_2$ lifts to N_1 (up to homotopy).

(ii) $\pi_0(\mathrm{M})$ is projective as a $\pi_0(\mathrm{R})$-module.

We now filter the category $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{perf}}$ by tor-amplitude:

Definition 32. An R-module $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}$ has *tor-amplitude* $\leqslant n$ if for all discrete R-modules $(\mathrm{Mod}_{\mathrm{R}})^{\heartsuit}$, we have $\pi_i(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{N}) = 0$ for $i > n$. We say that M is of *finite tor-amplitude* if it is of tor-amplitude $\leqslant n$ for some $n \geqslant 0$.

Example 33. If M is connective, then it is flat iff it is of tor-amplitude $\leqslant 0$.

Exercise 34.

(a) Show that the condition "of finite tor-amplitude" is stable under finite colimits and direct summands in $\mathrm{Mod}_{\mathrm{R}}$.

(b) Deduce that any perfect R-module is of finite tor-amplitude.

Next time we will see that every perfect R-module can be built out of finite colimits and direct summands from objects of $\mathrm{Mod}_{\mathrm{R}}^{\mathrm{proj}}$.