Lecture 4

Recall: * $DSch^{Aff} := SCRing^{op}$

* $DStk \subset Fun(SCRing, \underset{sSet}{\underset{\shortparallel}{Spc}})$ satisfying fpqc (hyper) descent.
(Derived Stacks)

We also defined open immersions of SCRing: $f: A \to B$ is open if flat, (homotopically) of finite presentation & $B \otimes_A B \to B$ is an equiv.

3.1. From Derived Stacks to Derived Schemes (Zariski covers).

$\mathcal{X} = Spec(R) \in DStk$, affine. $j: \mathcal{U} \to \mathcal{X}$ morphism in DStk

if $\mathcal{U} = Spec(A)$ also affine, then we say that j is an open immersion if $R \to A$ is

Def: open in the above sense.

If $\mathcal{U}$ is NOT affine, we say that j is a Zariski open immersion if

(1) j is a monomorphism in DStk

(2) $\exists$ a family $(\mathcal{U}_\alpha \to \mathcal{U})_\alpha$ such that

(2a) each $\mathcal{U}_\alpha$ is affine and $\overrightarrow{\mathcal{U}_\alpha \to \mathcal{U} \to \mathcal{X}}$ is open immersion of affine DSch

(2b) $\coprod_\alpha \mathcal{U}_\alpha \to \mathcal{U}$ is an effective epimorphism.

Recall: $\mathcal{Y} \to \mathcal{X}$ is effective epi $\iff$ $\varinjlim \check{C}(\mathcal{X}/\mathcal{Y})_m \to \mathcal{Y}$ is an equivalence.
(cfr with Toën: $\mathcal{Y} \to \mathcal{X}$ is epi $\iff$ $\pi_0(\mathcal{Y}) \to \pi_0(\mathcal{X})$ is epi of sheaves of sets).

In general, we define $j: \mathcal{U} \to \mathcal{X}$ to be an open immersion if $\forall\ Spec(R) \to \mathcal{X}$,
the base change $\mathcal{U} \times_{\mathcal{X}} Spec(R) \longrightarrow Spec(R)$ is an open immersion in the above sense.

Def: A derived stack $\mathcal{X}$ is called a derived scheme if $\exists$ family $(Spec(R_\alpha) \to \mathcal{X})_\alpha$
such that each $Spec(R_\alpha) \to \mathcal{X}$ is an open immersion and $\coprod Spec(R_\alpha) \to \mathcal{X}$
is an effective epi. The family $(Spec(R_\alpha) \to \mathcal{X})$ will be called
an Atlas for $\mathcal{X}$.

Given this definition, we can define what it means for a morphism $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ of
Derived schemes to be flat (and later, étale, smooth): we require the existence
of atlases $(Spec(A_i) \to \mathcal{X})$, $(Spec(B_j) \to \mathcal{Y})$ together with commutative squares

$$\begin{tikzcd} \mathcal{X} \arrow[r] & \mathcal{Y} \\ Spec(A_i) \arrow[u] \arrow[r] & Spec(B_j) \arrow[u] \end{tikzcd}$$

such that $Spec(A_i) \longrightarrow Spec(B_j)$ is flat
($\underset{def}{\iff}$ $B_j \to A_i$ is a flat morphism of SCRing).

Def: $\mathcal{X}$ Derived scheme is a classical scheme if it admits an atlas
$(Spec(A_i) \to \mathcal{X})$ where each A_i is discrete SCRing ($\rightsquigarrow$ classical ring).

Rmk: Note that in this case $\mathcal{X}$ is necessarily a discrete presheaf $\Rightarrow$ it is indeed
a classical scheme.

Notation:
$$\begin{tikzcd} \mathrm{Sch}^{\mathrm{aff}} \arrow[r, hook] \arrow[d] & \mathrm{DSch}^{\mathrm{aff}} \arrow[d] \\ \mathrm{Sch} \arrow[r, hook, "(*)"] & \mathrm{DSch} \end{tikzcd}$$

Note that the adjunction $\pi_0 : \mathrm{SCRing} \rightleftarrows \mathrm{CRing}$

Extends to $\mathcal{X} \in \mathrm{DSch} \mapsto \mathcal{X}_{cl}$, right adjoint to the inclusion $(*)$.

On the affine level: $\mathcal{X} = \mathrm{Spec}(R) \mapsto \mathcal{X}_{cl} = \mathrm{Spec}(\pi_0(R))$.

Rmk: one could define DSch as "locally ringed spaces" in the appropriate sense. Cfr Kerz-Strunk-Tamme

3.2. Quasi-coherent sheaves and quasi coherent algebras

$R \in \mathrm{SCRing} \rightsquigarrow \mathrm{Mod}_R$ (stable) ∞-cat of R-Modules.

weak-Kan-complex, given by $N_{dg}(T(R)\text{-Mod}_{cof})$

We define them $\mathrm{QCoh}(\mathrm{Spec}(R)) := \mathrm{Mod}_R$

This gives a presheaf of ∞-Cat:
$$\begin{array}{lll} (\mathrm{DSch}^{\mathrm{Aff}})^{op} \to \infty\text{-Cat} & & \\ \mathrm{Spec}(R) \mapsto \mathrm{QCoh}(\mathrm{Spec}(R)) & & M \\ f: R' \to R \quad \downarrow & & \downarrow \\ \mathrm{Spec}(R') \mapsto \mathrm{QCoh}(\mathrm{Spec}(R')) & & M \otimes_R R' \end{array}$$

If $\mathcal{X} \in$ ~~DSch~~ DStK $\Rightarrow$ define $\mathrm{QCoh}(\mathcal{X}) = \lim\limits_{\substack{\longleftarrow \\ S \to \mathcal{X} \\ S = \mathrm{Spec}(R) \text{ affine}}} \mathrm{QCoh}(S)$

$$\begin{tikzcd} ((\mathrm{DSch})^{\mathrm{Aff}})^{op} \arrow[r] \arrow[d, "\text{Yoneda}"'] & \infty\text{-cat} \\ (\mathrm{DStK})^{op} \arrow[ur, dashed] & \end{tikzcd}$$
Right Kan extension.

if $\mathcal{X} \in \mathrm{DSch} \subset \mathrm{DStK}$, there is a description involving maps which are open immersions.

Prop: $\mathcal{X} \in \mathrm{DSch}$. Then $\mathrm{QCoh}(\mathcal{X}) \xrightarrow{\sim} \lim\limits_{\substack{\longleftarrow \\ S = \mathrm{Spec}(R) \hookrightarrow \mathcal{X} \\ \text{Zar. open}}} \mathrm{QCoh}(S)$.

proof: choose an atlas $(\overbrace{\mathrm{Spec}(R_\alpha)}^{U_\alpha} \to \mathcal{X})_\alpha$. By definition, we have that the map $\lim\limits_{\substack{\longrightarrow \\ m \in \Delta}} \underbrace{\check{C}(U_\alpha / \mathcal{X})_m}_{} \to \mathcal{X}$ is an equivalence.

Čech nerve of $(\amalg U_\alpha \to \mathcal{X})$, effective epimorphism.

$\mathrm{QCoh}(-)$ is Right Kan extension $\Rightarrow$ sends colimits to limits.

Thus $\mathrm{QCoh}(\mathcal{X}) \cong \lim\limits_{\substack{\longleftarrow \\ m \in \Delta}} \mathrm{QCoh}(\check{C}(U_\alpha / \mathcal{X})_m)$.

For any open $V \to \mathcal{X}$, we similarly get $\mathrm{QCoh}(V) \cong \lim\limits_{\substack{\longleftarrow \\ m \in \Delta}} \mathrm{QCoh}(\check{C}(U_\alpha \times_{\mathcal{X}} V / V)_m)$

$\Rightarrow$ enough to show that (by commuting the 2 limits)
$$\mathrm{QCoh}(\underbrace{\check{C}(U_\alpha / \mathcal{X})_m}_{T}) \xrightarrow{\sim} \lim\limits_{\substack{\longleftarrow \\ \mathcal{U} \hookrightarrow \mathcal{X} \text{ open}}} \mathrm{QCoh}(\mathcal{U} \times_{\mathcal{X}} T)$$

But now T itself is of the form

Thus T is open in $\mathrm{Spec}(B) \Rightarrow$ it is "separated", in the sense that

$T_{cl} \hookrightarrow \mathrm{Spec}(\pi_0(B)) = \mathrm{Spec}(B)_{cl}$ open in an affine scheme ($\Rightarrow$ separated in the classical sense)

$\Rightarrow$ T itself can be covered by affine derived schemes.

Have some faith.

Thus, we are reduced to the case T affine $\Rightarrow$ done by def. $\square$

one uses the following FACT: $\mathcal{X} \in \mathrm{DSch}$. Then $\mathcal{X}$ is affine $\Leftrightarrow \mathcal{X}_{cl}$ is affine (cfr. Toën)

We now define the category of Quasi-coherent algebras in a similar manner.

$X = \mathrm{Spec}(R)$, $R \in \mathrm{SCRing} \rightsquigarrow \mathrm{SCRing}_R$ ∞-cat of R-algebras. (Lecture 1).

$$\mathrm{QCohAlg}(\mathrm{Spec}(R)) := \mathrm{SCRing}_R.$$

This gives another p-sheaf of (stable) ∞-Cat : $(\mathrm{DSch}^{\mathrm{Aff}})^{op} \to \infty\text{-Cat}$.

If $\mathcal{X}$ is a derived stack/scheme, $\mathrm{QCohAlg}(\mathcal{X}) = \varprojlim_{\substack{S \to \mathcal{X} \\ S = \mathrm{Spec}(R),\ \text{affine}}} \mathrm{QCohAlg}(S)$

again by Right Kan extension.

<u>RmK</u>: $\mathcal{X} = \mathrm{Spec}(R)$, $\mathcal{A} \in \mathrm{QCohAlg}(\mathcal{X}) \xrightarrow[\Gamma(\mathcal{X},-)]{\sim} \mathrm{SCRing}_R \Rightarrow A = \Gamma(\mathcal{X}, \mathcal{A})$ (Notation).

<u>Construction</u>: $\mathcal{A} \in \mathrm{QCohAlg}(\mathcal{X})$, $\mathcal{X} \in \mathrm{DSch}$.

$$\mathrm{Fun}((\mathrm{DSch}_{/\mathcal{X}})^{op}, \mathrm{Spc}) \simeq \mathrm{Fun}((\mathrm{DSch}^{\mathrm{Aff}})^{op}, \mathrm{Spc})_{/\mathcal{X}} = \text{Derived prestacks}/\mathcal{X}$$

$\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A}) := (S \xrightarrow{f} \mathcal{X}) \mapsto \mathrm{Map}_{\mathrm{QCohAlg}(S)}(f^*\mathcal{A}, \mathcal{O}_S)$ presheaf of spaces.

<u>Ex</u>: This is a derived stack, and it's schematic i.e. $\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A}) \in \mathrm{DSch}$.

Moreover, $\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A}) \to \mathcal{X}$ is affine, i.e. $\forall\, S \to \mathcal{X}$, $S = \mathrm{Spec}(R)$, $S \times_{\mathcal{X}} \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$ is affine.

<u>Example</u>: $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$. Suppose that $\mathcal{E}$ is locally free, i.e. $\mathcal{E}|_{\mathcal{U}_i} \cong \mathcal{O}_{\mathcal{U}_i}^{\oplus n}$

$(\mathcal{U}_i \to \mathcal{X})$; Atlas of $\mathcal{X} \in \mathrm{DSch}$. $\mathcal{U}_i = \mathrm{Spec}(R_i)$ $\qquad \mathcal{E} \otimes_{\mathcal{X}} \mathcal{U}_i \in \mathrm{QCoh}(\mathcal{U}_i)$

$\rightsquigarrow \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}) \in \mathrm{QCohAlg}(\mathcal{X})$

<u>Locally</u>: $M \in \mathrm{Mod}_R$, $\mathrm{Sym}_R(M)$ = symmetric algebra

$$\text{"} = \Big(\bigoplus_{n \geq 0} (\underbrace{M \otimes_R \otimes \cdots \otimes_R M}_{n})\Big) \Big/ (x \otimes y - y \otimes x) \text{"}$$

Two-sided ideal.

<u>Note</u>: if M connective

we can use explicit model of M as simplicial module $/R$

$\Rightarrow$ do the construction levelwise.

$\mathcal{E}$ locally free of finite rank is clearly connective $\Rightarrow$ easy to define.

Our next task will be to introduce $\mathbb{P}(\mathcal{E})$ for $\mathcal{E}$ locally free module.
Before that we need:

3.3. More on morphisms: separated and proper morphisms, and projective spaces.

The definition is easy:

<u>Def</u>: $p: \mathcal{X} \to \mathcal{Y}$ morphism of DSch. We say that p is proper if

i) $p_{cl}: \mathcal{X}_{cl} \to \mathcal{Y}_{cl}$ is of finite type (we say then that p is of finite type)

ii) $p_{cl}: \mathcal{X}_{cl} \to \mathcal{Y}_{cl}$ is separated ($\rightsquigarrow$ p is separated)

iii) p satisfies the valuative criterion of properness:

$R \in \mathrm{CRing}$ valuation ring, $R \hookrightarrow K = \mathrm{Frac}(R)$.

$$\rightsquigarrow \begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & \exists \nearrow & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & \mathcal{Y} \end{array}$$

($\Rightarrow \exists !$ by ii)).

Note that this condition depends only on $\mathcal{X}_{cl} \to \mathcal{Y}_{cl}$ by adjunction.

<u>Lemma</u>: $p: \mathcal{X} \to \mathcal{Y}$ is proper $\iff$ p_{cl} is proper.

~~Thm~~ ~~[Construction: $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$ locally free of finite rank, m.~~

~~$\mathbb{P}_{\mathcal{X}}(\mathcal{E})$ is the derived (pre)stack corresponding to the presheaf on $\mathrm{DSch}/\mathcal{X}$~~

~~Informally: $(S \to \mathcal{X}) \mapsto$ "space of pairs" $(\mathcal{L}, u)$, $\mathcal{L}$ locally free sheaf of rank 1 on S~~

~~$u: f^*(\mathcal{E}) \to \mathcal{L}$ map in $\mathrm{QCoh}(S)$ s.t. u is onto on π_0.~~

~~More precise construction:~~ We start from the projective space $\mathbb{P}^n$:

$A \in \mathrm{SCRing}$. Fix $n \geq 0$. We define $X^n(A)$ to be the subcategory of the category $(\mathrm{Mod}_A)_{/A^{n+1}}$ (comma category over $A^{n+1} \in \mathrm{Mod}_A$)

Morphisms in $X^n(A)$ = equivalences.

Objects: maps $f: L \to A^{n+1}$ such that

1) f has a left (homotopy) inverse ($\iff$ L is a summand of A^{n+1})

2) L is locally free of rank 1.

(equivalently, one could consider the functor which assigns to A the cat having objects $K \to A^{n+1}$, K loc. free of rk n)

$$\rightsquigarrow \quad X^n(-): \mathrm{SCRing} = (\mathrm{DSch}^{\mathrm{Aff}})^{op} \longrightarrow \mathrm{Spc}.$$

<u>Prop</u>: $X^n(-)$ is representable by a Derived scheme $\rightsquigarrow$ $\mathbb{P}^n_{\mathrm{Spec}(\mathbb{Z})}$

<u>Rmk</u>: this is what Lurie calls "smooth projective space".

To prove representability, one can use the standard cover by affine spaces

$\mathcal{E} \in QCoh(\mathcal{X})$ locally free of finite rank.

Side remarks: 1) One can show that $\mathbb{P}^n_{\mathcal{X}}$ (and, in general, $\mathbb{P}_{\mathcal{X}}(\mathcal{E})$) is smooth over $\mathcal{X}$. We can't do it now as we first need to introduce the cotangent complex for this. ↑ see below

2) Smoothness of 1) implies $\mathbb{P}^n_{\mathcal{X}} \to \mathcal{X}$ is flat (we know what this means!)
$\Rightarrow (\mathbb{P}^n_{\mathcal{X}})_{cl} \cong \mathbb{P}^n_{\mathcal{X}_{cl}} \cong \mathbb{P}^n_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{X}_{cl}$.
(this can be seen directly, by looking at the definition of $\mathbb{P}^n_{\mathcal{X}}$ as functor of points, and then noting that $\pi_0(X^n(A)) = \mathbb{P}^n(\mathrm{Spec}(A))$ in classical sense)
In particular, $(\mathbb{P}^n_{\mathcal{X}})_{cl} \to \mathcal{X}_{cl}$ is $\mathbb{P}^n_{\mathcal{X}_{cl}} \to \mathcal{X}_{cl} \Rightarrow \mathbb{P}^n_{\mathcal{X}}$ is proper$/\mathcal{X}$.

3) More general construction of $\mathbb{P}_{\mathcal{X}}(\mathcal{E})$ for $\mathcal{E} \in QCoh(\mathcal{X})$ loc. free of rank n, $\mathcal{X} \in DSch$.

First, let $Perf(\mathcal{X}) \subset QCoh(\mathcal{X})$ full subcategory spanned by perfect complexes
($M \in Mod_A$ perfect $\iff$ M is compact $\iff$ $(N \mapsto M \otimes N)$ commutes with limits)
By def, $\mathcal{F} \in QCoh(\mathcal{X})$ perfect iff $\forall\ \mathrm{Spec}(R) \xrightarrow{f} \mathcal{X}$, $\Gamma(f^*\mathcal{F}) \in Mod_R$ is perfect.

If $K \in sSet$, define a functor $Perf_K \in Fun((DSch)^{Aff,op}, \underset{sSet}{Spc})$
by $Perf_K(\mathrm{Spec}(R)) = Fun(K, \underset{Mod_R}{\underset{\cap}{Perf(R)}})$

We also define, for $\mathcal{X}$ fixed, $Perf_{K,\mathcal{X}} \in Fun((DSch^{Aff})^{op}, Spc)/\mathcal{X}$ by "restricting to $\mathrm{Spec}(R) \to \mathcal{X}$". Similarly, $Perf_{\mathcal{X}}(\mathcal{E})$ is the functor $Perf_{\Delta^0, \mathcal{X}}$. We have, for each $\mathcal{E} \in Perf(\mathcal{X})$, a "classifying map"
$\mathcal{X} \longrightarrow Perf_{\Delta^0,\mathcal{X}}$ ($\mathcal{X}(\mathrm{Spec}(R)) \to Perf_{\Delta^0,\mathcal{X}}(\mathrm{Spec}(R))$ is the datum of $f^*\mathcal{E}$ for $f: \mathrm{Spec}(R) \to \mathcal{X}$)

Then, given $\mathcal{E}$ locally free of rank n ($\Rightarrow \mathcal{E}$ perfect), define the <u>subfunctor</u> of $Perf_{\Delta^1,\mathcal{X}} \times_{\overset{(*)\partial_1}{\to} Perf_{\Delta^0,\mathcal{X}} \swarrow \mathcal{E}} \mathcal{X}$ given by:

$$\underbrace{(Perf_{\Delta^1,\mathcal{X}} \times_{Perf_{\Delta^0,\mathcal{X}}} \mathcal{X})}_{\mathbb{P}_{\mathcal{X}}(\mathcal{E})}(\underset{(\mathrm{Spec}(R) \xrightarrow{f} \mathcal{X})}{\mathrm{Spec}(R)}) = \{ u : \mathcal{L} \longrightarrow f^*\mathcal{E} \mid \mathrm{cofib}(u) \text{ has rank } n-1 \text{ in } Mod_R \ (\iff \mathcal{L} \text{ has rank } 1) \}$$

This is representable by a Derived scheme
(choose an Atlas $U_i = \mathrm{Spec}(R_i) \to \mathcal{X}$ such that $\mathcal{E}|_{U_i} \cong R_i^n$)
$\Rightarrow$ this reduces to the previous case.

(*) $Perf_{\Delta^1,\mathcal{X}} \xrightarrow{\partial_1} Perf_{\Delta^0,\mathcal{X}}$ is given by $(u: M \to N) \mapsto N$.

4) We can use the functor of points approach to define "derived version" of the usual Grassmanians, classifying rank K direct summands of $\mathcal{O}_{\mathcal{X}}^{\oplus n}$.

3.4 Closed immersions.

We start from the following basic definition.

Def: let $Z \xrightarrow[i]{} \mathcal{X}$ be a morphism of derived schemes.

1) if $\mathcal{X}, Z$ affine, we say that i is a closed immersion if $i : \mathrm{Spec}(B) \to \mathrm{Spec}(A) \iff A \to B$ induces a surjection on π_0
$$(\iff \mathrm{Spec}(\pi_0(B)) \hookrightarrow \mathrm{Spec}(\pi_0(A))).$$

2) in general, i is a closed immersion if $\forall$ $\mathrm{Spec}(R) \to \mathcal{X}$, $\mathrm{Spec}(R) \times_{\mathcal{X}} Z$ is affine & $\mathrm{Spec}(R) \times_{\mathcal{X}} Z \to \mathrm{Spec}(R)$ is a closed immersion as in 1).

Prop: $i : Z \to \mathcal{X}$ is a closed immersion iff $Z_{cl} \xrightarrow[i_{cl}]{} \mathcal{X}_{cl}$ is a closed immersion.

Special example: $\mathcal{X}_{cl} \to \mathcal{X}$ is a closed immersion.

In general, we say that a closed immersion $\mathcal{X} \hookrightarrow \mathcal{X}'$ is a nil-immersion if it induces an isomorphism $\mathcal{X}_{cl} \xrightarrow{\sim} \mathcal{X}'_{cl}$.