Lecture 0
Introduction

Question 1 (Intersection theory)**.** Let X be a nonsingular algebraic variety over a field k. Let V and W be irreducible closed subvarieties of X. How can we describe the intersection $\mathrm{V} \cap \mathrm{W} \subset \mathrm{X}$?

Example 2 (Bézout)**.** Let $\mathrm{X} = \mathbf{P}^2_{\mathbf{C}}$ and V and W be smooth curves of degrees m and n, respectively. Suppose that V and W intersect properly (in a finite number of points). Then the number of intersection points is at most mn. This is an equality if and only if the intersection is transverse.

In the transverse case, we can give a "cohomological" reformulation of Bézout's theorem as follows.

Observation 3. Consider the Chow ring of algebraic cycles $\mathrm{CH}^*(\mathbf{P}^2_{\mathbf{C}}, \mathbf{Z})$. The curves V and W have fundamental classes $[\mathrm{V}], [\mathrm{W}] \in \mathrm{CH}^*(\mathbf{P}^2_{\mathbf{C}}, \mathbf{Z})$. If V and W intersect transversally, then Bézout's formula can be restated as the equality

$$[\mathrm{V}] \cdot [\mathrm{W}] = [\mathrm{V} \cap \mathrm{W}]$$

in $\mathrm{CH}^2(\mathbf{P}^2_{\mathbf{C}}, \mathbf{Z}) \approx \mathbf{Z}$, where $\cdot$ denotes the intersection product.

The equality above still holds in the non-transverse case as long as we "count with multiplicities". That is, intersection multiplicities need to be taken into account when defining the fundamental class $[\mathrm{V} \cap \mathrm{W}]$. In the above example, the correct multiplicity number can be recovered by considering the canonical scheme structure on $\mathrm{V} \cap \mathrm{W}$. In general, more than just the scheme structure needs to be taken into account (even in the case $\mathrm{X} = \mathbf{P}^3_{\mathbf{C}}$).

Theorem 4 (Serre)**.** *Let* X *be a nonsingular variety over a field* k*. Let* V *and* W *be irreducible closed subvarieties of* X *that intersect properly (so that* $\dim(\mathrm{V}) + \dim(\mathrm{W}) = \dim(\mathrm{X}) + \dim(\mathrm{V} \cap \mathrm{W})$*). Then for any generic point* $x \in \mathrm{V} \cap \mathrm{W}$*, the intersection multiplicity of* V *and* W *at* x *is given by the Euler characteristic*

$$\sum_i (-1)^i \cdot \mathrm{length}_{\mathcal{O}_{\mathrm{X},x}} \mathrm{Tor}_i^{\mathcal{O}_{\mathrm{X},x}}(\mathcal{O}_{\mathrm{V},x}, \mathcal{O}_{\mathrm{W},x}).$$

In the example above, we could have replaced the Chow ring with any reasonable cohomology theory (e.g. singular cohomology). It turns out that if we work with algebraic K-theory instead, intersection products are easy to define and are very closely related to Serre's formula for the intersection number.

Definition 5. Given a locally noetherian scheme X, let $\mathrm{K}_0(\mathrm{X}) = \mathrm{K}_0(\mathrm{Perf}(\mathrm{X}))$ denote the Grothendieck group of perfect complexes[1] on X. This is the free abelian group generated by (isomorphism classes of) perfect complexes on X, modulo the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ for any exact triangle $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}''$.

Similarly, let $\mathrm{G}_0(\mathrm{X}) = \mathrm{K}_0(\mathrm{Coh}(\mathrm{X}))$ denote the Grothendieck group of coherent sheaves on X. This is the free abelian group generated by (isomorphism classes of) coherent sheaves on X, modulo the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ for any exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$.

When X is regular (nonsingular), we can identify these two groups:

Theorem 6 (Poincaré duality)**.** *For any regular scheme* X*, there is a canonical isomorphism*

$$\mathrm{K}_0(\mathrm{X}) \xrightarrow{\sim} \mathrm{G}_0(\mathrm{X})$$

which is contravariantly functorial in X*. It is given by the assignment* $[\mathcal{E}] \mapsto \sum_i (-1)^i [\mathrm{H}_i(\mathcal{E})]$.

[1] A perfect complex is a bounded complex of vector bundles (locally and up to quasi-isomorphism). If X is quasi-projective, then it is possible to simply work with vector bundles instead of perfect complexes.

The group $K_0(X)$ admits a multiplication given by the *derived tensor product* of perfect complexes:

$$[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}].$$

If X is regular this induces a product on $G_0(X)$ which can be described as

$$[\mathcal{E}] \cdot [\mathcal{F}] = \sum_i (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})].$$

For example, given closed subschemes V and W, we can consider the classes of $\mathcal{O}_V$ and $\mathcal{O}_W$ in $G_0(X)$, and their product recovers Serre's formula (after passing to stalks).

Furthermore, K-theory is closely related to the Chow ring via a natural filtration.

Construction 7. The *coniveau filtration*[2] on $G_0(X)$ is defined as follows: for any integer $n \geqslant 0$, $F^n_{\mathrm{coniv}} G_0(X)$ is the subgroup of $G_0(X)$ generated by classes $[\mathcal{E}]$ where $\mathcal{E}$ is a coherent sheaf whose support is of codimension $\geqslant n$.

The following is one corollary of (one form of) the Grothendieck–Riemann–Roch theorem.

Theorem 8. *For any noetherian scheme* X*, there is a canonical map*

$$\mathrm{CH}^k(X) \to \mathrm{Gr}^k_{\mathrm{coniv}}(G_0(X))$$

given by the assignment $[Z] \mapsto [\mathcal{O}_Z]$*, which is surjective for each integer* k*. Here the right-hand side denotes the associated graded pieces of the coniveau filtration. Moreover, it induces an isomorphism*

$$\mathrm{CH}^k(X)_{\mathbf{Q}} \to \mathrm{Gr}^k_{\mathrm{coniv}}(G_0(X)_{\mathbf{Q}})$$

with rational coefficients.

We might hope to use this identification to understand intersection products at least rationally. For this, we would also need to know that the coniveau filtration is *multiplicative* (i.e. compatible with the multiplication in the evident way).

Theorem 9. *Let* X *be a smooth quasi-projective variety over a field* k*. Then the coniveau filtration is multiplicative.*

Grothendieck proved the above theorem in the quasi-projective case by using Chow's moving lemma. Over more general schemes (say regular noetherian), this is still an open conjecture. In order to circumvent this issue, Grothendieck defined another filtration on $K_0(X)$ which is tautologically multiplicative.

Construction 10. Let $\lambda^i : K_0(X) \to K_0(X)$ denote the λ-operations $[\mathcal{E}] \mapsto [\Lambda^i(\mathcal{E})]$. Let $\gamma^i : K_0(X) \to K_0(X)$ denote the operations $x \mapsto \lambda^n(x + (n-1)[\mathcal{O}_X])$.

Define the γ-*filtration* $F^*_\gamma(K_0(X))$ as follows:

(i) Define $F^1_\gamma(K_0(X))$ as the subgroup generated by classes of virtual rank zero (i.e. classes $[E] - [F]$ where E and F are locally free of the same rank).

(ii) For each $x \in F^1_\gamma(K_0(X))$, require that $\gamma^i(x) \in F^i_\gamma(K_0(X))$.

(iii) Require that the filtration is multiplicative, i.e. $F^i_\gamma \cdot F^j_\gamma \subset F^{i+j}_\gamma$.

One can define "Chern classes" $c_k(\mathcal{E}) = \gamma^k([\mathcal{E}] - \mathrm{rk}(\mathcal{E}))$ and therefore a Chern character map

$$\mathrm{ch} : K_0(X) \to \mathrm{Gr}^*_\gamma(K_0(X)).$$

[2] Also known as the *codimension* or *topological* filtration

Theorem 11. *Let* X *be a regular noetherian scheme. Then there is a canonical isomorphism*

$$\mathrm{Gr}^*_{\mathrm{coniv}}(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}) \xrightarrow{\sim} \mathrm{Gr}^*_\gamma(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}).$$

Furthermore, there is a canonical isomorphism

$$\mathrm{CH}^k(\mathrm{X})_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{Gr}^k_\gamma(\mathrm{K}_0(\mathrm{X}))$$

for each k*, given by the assignment* $[\mathrm{Z}] \mapsto \mathrm{ch}_k([\mathcal{O}_\mathrm{Y}])$*, where* ch_k *denotes the* k*th graded component of the Chern character.*

In particular, it follows that the coniveau filtration is multiplicative *rationally*. In any case, we can "do intersection theory" rationally, completely independently of the theory of algebraic cycles, instead working with K-theory and the γ-filtration. A crucial ingredient in all this is the following, which describes the (failure of) covariant functoriality of the Chern character:

Theorem 12 (Grothendieck–Riemann–Roch)**.** *Let* $i : \mathrm{Z} \to \mathrm{X}$ *be a regular closed immersion of quasi-projective noetherian schemes. Then the following square commutes:*

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{Z}) \arrow[r, "i_*"] \arrow[d, "\mathrm{ch}(-)\cdot\mathrm{Td}(-\mathcal{N}_i)"'] & \mathrm{K}_0(\mathrm{X}) \arrow[d, "\mathrm{ch}"] \\
\mathrm{Gr}_\gamma(\mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}}) \arrow[r, "i_*"] & \mathrm{Gr}_\gamma(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}})
\end{tikzcd}$$

Our goal in this course is to give a proof of the above theorem in the more general setting where the schemes are allowed to be *derived*. The motivation comes again from Serre's intersection formula: given a regular scheme X, say $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ for simplicity, and closed subvarieties $\mathrm{V} = \mathrm{Spec}(\mathrm{A}/\mathrm{I})$ and $\mathrm{W} = \mathrm{Spec}(\mathrm{A}/\mathrm{J})$, we would like to enhance $\mathrm{V} \cap \mathrm{W}$ into some more structured geometric object that encodes the data of the Tor groups $\mathrm{Tor}_i^{\mathrm{A}}(\mathrm{A}/\mathrm{I}, \mathrm{A}/\mathrm{J})$ (and hence the correct intersection multiplicity numbers). This means that we want to be able to make sense of the "Zariski spectrum" of the derived tensor product $\mathrm{A}/\mathrm{I} \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}/\mathrm{J}$. In particular, we want to view $\mathrm{A}/\mathrm{I} \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}/\mathrm{J}$ as some sort of generalized commutative ring (just as the usual tensor product $\mathrm{A}/\mathrm{I} \otimes_{\mathrm{A}} \mathrm{A}/\mathrm{J}$ is a commutative ring).

Quillen observed that the theory of derived functors works equally well in non-abelian settings (like the non-abelian category of commutative rings), if we work with simplicial objects instead of chain complexes. For the moment we will ignore the difference and think of simplicial commutative rings roughly as "chain complexes of abelian groups with multiplicative structures". The main point for this discussion is that the construction $\mathrm{A}/\mathrm{I} \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}/\mathrm{J}$ can be naturally viewed as a simplicial commutative ring.

Quasi-Definition 13. A *derived scheme* X is a pair $(\mathrm{X}_{\mathrm{cl}}, \mathcal{O}_\mathrm{X})$, where X_{cl} is a classical scheme, and $\mathcal{O}_\mathrm{X}$ is a sheaf of simplicial commutative rings on X_{cl} such that $\mathrm{H}_0(\mathcal{O}_\mathrm{X}) \approx \mathcal{O}_{\mathrm{X}_{\mathrm{cl}}}$.

We refer to X_{cl} as the "underlying classical scheme" of X. We think of X as an infinitesimal thickening of X_{cl} in the same way that X_{cl} is a thickening of its underlying reduced scheme $\mathrm{X}_{\mathrm{red}}$. Thanks to the work of Lurie and Toën–Vezzosi, most of the language from classical scheme theory also makes sense in the derived setting.

Example 14. Any classical scheme X is a derived scheme, with "discrete" structure sheaf.

Example 15. Given any simplicial commutative ring A, there is a naturally associated affine derived scheme $\mathrm{Spec}(\mathrm{A})$.

Example 16. Let X be a scheme and V, W closed subschemes. Then there is a derived scheme $\mathrm{V} \times^h_{\mathrm{X}} \mathrm{W}$, called the *derived fibred product*, whose underlying classical scheme is the usual fibred product $\mathrm{V} \times_{\mathrm{X}} \mathrm{W}$. Its structure sheaf encode the Tor groups appearing in Serre's intersection formula. In particular, it coincides with the usual fibred product if and only if the V and W intersect transversely.

Example 17. Let $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ be an affine scheme and $f \in \mathrm{A}$ an element. Then the Koszul complex $0 \to \mathrm{A} \xrightarrow{f} \mathrm{A} \to 0$ can be viewed as a simplicial commutative ring, which we denote $\mathrm{A}/\!\!/(f)$. In particular there is a morphism of derived schemes $\mathrm{Spec}(\mathrm{A}/\!\!/(f)) \to \mathrm{Spec}(\mathrm{A})$, which can be viewed as a thickening of the closed immersion $\mathrm{Spec}(\mathrm{A}/f) \to \mathrm{Spec}(\mathrm{A})$. Recall that if f is a non-zero-divisor, then the Koszul complex is just a resolution of A/f; in other words, there is a quasi-isomorphism $\mathrm{A}/\!\!/(f) \approx \mathrm{A}/f$, and in particular there is no difference in this case between $\mathrm{Spec}(\mathrm{A}/\!\!/(f))$ and $\mathrm{Spec}(\mathrm{A}/f)$.

However, if f is a non-zero-divisor, then the morphism $\mathrm{Spec}(\mathrm{A}/\!\!/(f)) \to \mathrm{Spec}(\mathrm{A})$ has more structure than the morphism of underlying classical schemes $\mathrm{Spec}(\mathrm{A}/f) \to \mathrm{Spec}(\mathrm{A})$. For example, the former "remembers" that it is cut out by a single equation.

In fact, there is a version of the construction $\mathrm{A}/\!\!/(f)$ with many elements, that we denote $\mathrm{A}/\!\!/(f_1, \ldots, f_n)$. If $(f_1, \ldots, f_n)$ form a regular sequence, then this is quasi-isomorphic to the usual quotient $\mathrm{A}/(f_1, \ldots, f_n)$. Moreover, this construction also makes sense when A is itself a simplicial commutative ring (and $f_1, \ldots, f_n$ are elements in the ordinary commutative ring $\mathrm{H}_0(\mathrm{A})$). This leads to the following derived generalization of the notion of "regular closed immersion":

Definition 18. Let $i : \mathrm{Z} \to \mathrm{X}$ be a closed immersion of derived schemes. We say that i is *quasi-smooth* (of virtual codimension n) if it is locally of the form $\mathrm{Spec}(\mathrm{A}/\!\!/(f_1, \ldots, f_n)) \to \mathrm{Spec}(\mathrm{A})$.

A general principle that we wish to illustrate throughout the course is that in many ways, quasi-smooth closed immersions are just as well-behaved as regular closed immersions, and furthermore that it is often necessary to take quasi-smooth morphisms into account in order to obtain a complete picture of a given situation.

Here is an example. Let $\mathrm{Z} \to \mathrm{X}$ be a regular closed immersion of schemes, and let $\pi : \tilde{\mathrm{X}} \to \mathrm{X}$ denote the blow-up of X in Z. Recall that π induces an isomorphism away from Z, and over Z the fibre $\pi^{-1}(\mathrm{Z})$ is the projectivized normal bundle $\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$. Recall also that π has the following characterization: for any morphism of schemes $f : \mathrm{S} \to \mathrm{X}$ such that the schematic fibre $f^{-1}(\mathrm{Z}) \to \mathrm{S}$ is an effective Cartier divisor (= regular immersion of codimension 1), there exists a unique morphism $\tilde{f} : \mathrm{S} \to \tilde{\mathrm{X}}$ lifting f. If we take quasi-smooth morphisms into account, we get the following more complete characterization:

Theorem 19. *With notation as above, there is a bijective correspondence between:*

(1) *Morphisms* $\tilde{f} : \mathrm{S} \to \tilde{\mathrm{X}}$ *lifting* f.

(2) *Commutative squares*

$$\begin{array}{ccc} \mathrm{D} & \longrightarrow & \mathrm{S} \\ \big\downarrow & & \big\downarrow f \\ \mathrm{Z} & \longrightarrow & \mathrm{X} \end{array}$$

where $\mathrm{D} \to \mathrm{S}$ *is a quasi-smooth closed immersion of virtual codimension* 1*, with* D *a derived thickening of* $f^{-1}(\mathrm{Z})$ *(i.e.* $\mathrm{D}_{\mathrm{cl}} \approx f^{-1}(\mathrm{Z})$*), such that the induced map on conormal sheaves* $\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \to \mathcal{N}_{\mathrm{D}/\mathrm{S}}$ *is surjective.*