Exercise sheet 2

1. Show that the functor of ∞-categories $c : \mathrm{CRing} \to \mathrm{SCRing}$ is fully faithful. Hint: show that it admits a left adjoint $\pi_0 : \mathrm{SCRing} \to \mathrm{CRing}$ such that $\pi_0 \circ c \simeq \mathrm{id}$ (so that CRing is a left localization of SCRing).

2. Show that the category CRing is the free completion of the category of polynomial rings $\mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$, $n \geqslant 0$, with respect to sifted colimits.

3. Show that the ∞-category SCRing is the free completion of the category of polynomial rings $\mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$, $n \geqslant 0$, with respect to sifted homotopy colimits. Deduce another proof that $c : \mathrm{CRing} \hookrightarrow \mathrm{SCRing}$ is fully faithful.

4. [For those familiar with $\mathcal{E}_\infty$-ring spectra] Show that there is a unique functor from SCRing into the ∞-category of $\mathcal{E}_\infty$-ring spectra, which commutes with colimits and sends a polynomial ring $\mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ to its Eilenberg–MacLane spectrum. Show that this functor is *not* fully faithful, even though its restriction to CRing is.

5. Let S be a derived scheme and let $\mathcal{E}$ be a locally free sheaf on S. Show that the associated vector bundle $\mathbf{V}_\mathrm{S}(\mathcal{E}) = \mathrm{Spec}_\mathrm{S}(\mathrm{Sym}_{\mathcal{O}_\mathrm{S}}(\mathcal{E}))$ is a derived scheme, and that $\mathbf{V}_\mathrm{S}(\mathcal{E}) \to \mathrm{S}$ is flat.

6. If $\mathcal{E}$ is a connective perfect complex that is not locally free, show that $\mathbf{V}_\mathrm{S}(\mathcal{E}) \to \mathrm{S}$ need not be flat.

7. Let S be a derived scheme and let $\pi : \mathbf{P}^n_\mathrm{S} \to \mathrm{S}$ be projective space of dimension n over S. Show that $(\mathbf{P}^n_\mathrm{S})_\mathrm{cl} \approx \mathbf{P}^n_{\mathrm{S}_\mathrm{cl}}$ (by comparison of universal properties). Deduce that π is proper.

8. Show that $\pi : \mathbf{P}^n_\mathrm{S} \to \mathrm{S}$ is flat.

9. Let $\mathcal{E}$ be a locally free sheaf on S. Show that $\mathbf{P}_\mathrm{S}(\mathcal{E}) \to \mathrm{S}$ is representable by a derived scheme.

10. Modify the construction of $\mathbf{P}^n_\mathrm{S}$ to define a derived version of the Grassmannian parametrizing "direct summands of $\mathcal{O}^{\oplus n}$ of rank k". Show that it is representable.

11. By definition the projective space $\mathbf{P}^1_\mathrm{S}$ classifies line bundles $\mathcal{L}$ equipped with a surjection $\mathcal{O}^2 \to \mathcal{L}$. Let $\mathcal{O}(1)$ denote the universal such, and write $\mathcal{O}(m) := \mathcal{O}(1)^{\otimes m}$ for $m \in \mathbf{Z}$. Show that there are canonical cocartesian squares

$$\begin{array}{ccc} \mathcal{O}(m) & \longrightarrow & \mathcal{O}(m+1) \\ \downarrow & & \downarrow \\ \mathcal{O}(m+1) & \longrightarrow & \mathcal{O}(m+2) \end{array}$$

for all m, in $\mathrm{Perf}(\mathbf{P}^1_\mathrm{S})$.

12. Show that our definition of "classical scheme" gives rise to a category that is equivalent to the classical definition of scheme.

13. Show that a derived scheme S is affine if and only if S_cl is affine.

14. Show that we have isomorphisms of ∞-groupoids $\mathrm{Maps}_{\mathrm{DSch}}(\mathrm{S}, \mathbf{A}^n_\mathbf{Z}) \approx \Gamma(\mathrm{S}, \mathcal{O}_\mathrm{S})^{\times n}$ for any derived scheme S.

15. Show that a morphism of derived schemes $f : \mathrm{X} \to \mathrm{Y}$ is separated (in the sense that f_cl is separated) iff the diagonal is a closed immersion.

16. For a derived scheme S, let $\mathrm{Qcoh}(\mathrm{S})^\mathrm{locfr} \subset \mathrm{Qcoh}(\mathrm{S})$ denote the full sub-∞-category of locally frees of finite rank. Show that there is an equivalence of categories $\mathrm{Ho}(\mathrm{Qcoh}^\mathrm{locfr}(\mathrm{S})) \xrightarrow{\sim} \mathrm{Qcoh}^\mathrm{locfr}(\mathrm{S}_\mathrm{cl})$.