Exercise sheet 1

1. Show that if $\mathrm{R} = k$ is a field, then the derived category $\mathrm{D}(k)$ is equivalent to the category of $\mathbf{Z}$-graded k-modules (and in particular is abelian).

2. Show that $\mathrm{D}(\mathrm{R})$ does not admit colimits for general R.

3. Let I be the category with a single object, whose set of endomorphisms is the monoid of natural numbers. Show that the categories $\mathrm{Funct}(\mathrm{I}, \mathrm{D}(\mathrm{R}))$ and $\mathrm{D}^{\mathrm{I}}(\mathrm{R})$ are not equivalent.

4. Let $f : \mathrm{M}_\bullet \to \mathrm{N}_\bullet$ be a morphism of chain complexes of R-modules. Show that the mapping cone $\mathrm{Cone}(f)$ is a model for the homotopy cofibre.

5. Show that for any R-modules M and N, there are canonical isomorphisms

$$\mathrm{H}_n(\mathrm{Hom}_{\underline{\mathrm{D}}(\mathrm{R})}(\mathrm{M}, \mathrm{N})_\bullet) \approx \mathrm{Ext}^n_{\mathrm{R}}(\mathrm{M}_\bullet, \mathrm{N}_\bullet) \approx \mathrm{Hom}_{\mathrm{D}(\mathrm{R})}(\mathrm{M}, \mathrm{N}),$$

where we view M and N as chain complexes concentrated in degree zero.

6. Let F and G be functors $\mathbf{C} \rightrightarrows \mathbf{D}$ between two ordinary categories. Show that natural transformations $\mathrm{F} \Rightarrow \mathrm{G}$ are in bijection with morphisms of simplicial sets $\varphi : \Delta^1 \times \mathrm{N}(\mathbf{C}) \to \mathrm{N}(\mathbf{D})$ such that $d^0(\varphi) = \mathrm{F}$ and $d^1(\varphi) = \mathrm{G}$.

7. Let $\mathbf{C}$ be a simplicially enriched category. Suppose that for all objects $x, y \in \mathbf{C}$, the simplicial Hom-set $\mathrm{Hom}_{\mathbf{C}}(x, y)$ is a Kan complex. Then show that the simplicial nerve $\mathrm{N}_{\mathbf{\Delta}}(\mathbf{C})$ is a weak Kan complex.

8. Show that a simplicial set X is the nerve of a category (resp. nerve of a groupoid) if and only if it has the lifting property for inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, for $0 < i < n$ (resp. for $0 \leqslant i \leqslant n$).

9. Show that the polynomial rings $\mathbf{Z}[\mathrm{T}_0, \ldots, \mathrm{T}_n]$ are cofibrant as (constant) simplicial commutative rings.

10. Show that the assignment $\mathrm{X} \mapsto c(\mathrm{X})$, sending a set to the associated constant simplicial set, is fully faithful. Show that c admits a left adjoint, given by $\mathrm{X} \mapsto \pi_0(\mathrm{X}) := \mathrm{Coeq}(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)$, where the arrows are the face maps d^0, d^1.

Given a simplicial abelian group M, there is an associated chain complex whose nth term is M_n, and whose differentials are given by $d_n = \sum_{i=0}^n (-1)^i d^i_n$. There is a variant of this called the *normalized chain complex* $\mathrm{N}(\mathrm{M})_\bullet$ associated to M, whose nth term is the intersection of the abelian groups $\mathrm{Ker}(d^i_n)$, $0 \leqslant i < n$, and differentials given by $d_n = (-1)^n d^n_n$. The *Dold–Kan correspondence* asserts that the assignment $\mathrm{M} \mapsto \mathrm{N}(\mathrm{M})_\bullet$ determines an equivalence between the category of simplicial abelian groups and chain complexes of abelian groups.

11. Let A be a commutative ring. Let $f \in \mathrm{A}$ be an element determining a ring homomorphism $\mathbf{Z}[\mathrm{T}] \to \mathrm{A}$, $\mathrm{T} \mapsto f$. Consider the derived tensor product $\mathrm{A}/\!\!/(f) := \mathrm{A} \otimes^{\mathbf{L}}_{\mathbf{Z}[\mathrm{T}]} \mathbf{Z}[\mathrm{T}]/(\mathrm{T})$ (as a simplicial commutative ring). Show that the underlying chain complex of A is the Koszul complex $0 \to \mathrm{A} \xrightarrow{f} \mathrm{A} \to 0$. In particular, deduce that there is a canonical map $\mathrm{A}/\!\!/(f) \to \mathrm{A}/(f)$ which is an isomorphism if and only if f is a non-zero-divisor.

12. With the notation as in A, let $g \in \mathrm{A}$ be an element. Compute the space $\mathrm{Maps}_{\mathrm{A}/\!\!/(f)}(g, 0)$ of paths $g \approx 0$ in the underlying space of the simplicial commutative ring $\mathrm{A}/\!\!/(f)$.